ISSN 2304-0122 Ufa Mathematical Journal. Vol. 9. No 2 (2017). P. 17-24.

doi:10.13108/2017-9-2-17 UDC 517.957

INTEGRATION OF EQUATION OF TODA PERIODIC CHAIN KIND

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Abstract. In this work we apply the method of the inverse spectral problem to integrating an equation of Toda periodic chain kind. For the one-band case we write out explicit formulae for the solutions to an analogue of Dubrovin system of equations and thus, for our problem. These solutions are expressed in term of Jacobi elliptic functions.

Keywords: Toda chain, discrete Hill operator, inverse spectral problem, trace formulae

Mathematics Subject Classification: 34K29, 37K15, 39A10

1. Introduction

The Toda chain [1]

$$\frac{\partial^2 u_n}{\partial t^2} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \quad n \in \mathbb{Z},$$

describing the dynamics of particles on a line with an exponential interaction in Flaschka variables [2] is of the form

$$\begin{cases} \dot{a}_n = a_n(b_n - b_{n+1})\,, \\ \dot{b}_n = 2(a_{n-1}^2 - a_n^2), \qquad n \in \mathbb{Z}. \end{cases}$$

In works [2]–[4], there was shown the integrability of Toda chain by the inverse scattering problem method in a fast decay case. The periodic Toda chain was considered in works [5]–[13].

In this work we consider a N-periodic equation of Toda chain kind

$$\begin{cases} \dot{a}_n = a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2) \\ \dot{b}_n = 2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1}) \\ a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad a_n > 0, \quad n \in \mathbb{Z}, \end{cases} \tag{1}$$

subject to the initial conditions

$$a_n(0) = a_n^0, \;\; b_n(0) = b_n^0, \quad n \in \mathbb{Z}, \tag{2}$$

with given N-periodic sequences a_n^0, b_n^0, $n \in \mathbb{Z}$. In system (1), $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$ are uknown functions.

By straightforward calculations one can check that system of equations (1) is equivalent to the following operator equation

$$\frac{dL}{dt} = BL - LB,$$

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Submitted May 12, 2016.

where

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1},$$
$$(B(t)y)_n \equiv a_n a_{n+1} y_{n+2} + a_n(b_n + b_{n+1})y_{n+1} - a_{n-1}(b_n + b_{n-1})y_{n-1} - a_{n-1}a_{n-2}y_{n-2},$$

that is, L and B are the Lax pairs for system (1). Therefore, system of equations (1) is integrable and hence, it has infinitely many symmetries [14], [15]. For instance, the Toda chain is the symmetry of system (1):

$$\begin{cases} \dfrac{\partial a_n}{\partial \tau} = a_n(b_n - b_{n+1})\,, \\ \dfrac{\partial b_n}{\partial \tau} = 2(a_{n-1}^2 - a_n^2), \qquad n \in \mathbb{Z}, \end{cases}$$

where $a_n = a_n(t,\tau), b_n = b_n(t,\tau)$. We note that similar to [16], [17], considered system (1) can be employed in certain models of special types of power lines.

In this work we obtain a representation for the solutions of problem (1)–(2) in terms of the inverse spectral problem for the discrete Hill equation:

$$(L(t)y)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{3}$$

namely, we find an analogue of Dubrovin system of equations for the spectral parameters of the discrete operator $L(t)$.

It is well known that the solvability of the Dubrovin system of equations for the periodic Toda chian in terms of theta-functions was studied in works [6], [7]. The matter of these works is that solving of the inverse problem by spectral data of equation (3) was reduced to the Jacobi problem on inverting Abel integrals and explicit formulae for the coefficients of the discrete operator $L(t)$ were obtained in terms of the theta-functions, and in this way the solution for Toda chain was found. Similar results hold for the problem considered in this work. This can be seen by applying the main theorem of the work in a particular, one-band case provided in the end of the paper. For the one-band case we write out explicit formulae for the solution to the analogue of the Dubrovin system of equations and in this way, for problem (1), (2). These solutions are expressed in terms of Jacobi elliptic functions.

2. Preliminary information on the direct and inverse spectral problem for the discrete Hill equation

In this section we provide some information on the direct and inverse spectral problem for the discrete Hill equation [2, 9], which will be useful later.

We consider the Hill equation

$$(Ly)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{4}$$
$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n \in \mathbb{Z}\,,$$

with a spectral parameter λ and a period $N > 0$. We denote by $\theta_n(\lambda)$, $n \in \mathbb{Z}$ and $\varphi_n(\lambda)$, $n \in \mathbb{Z}$, the solutions of equation (4) obeying the initial conditions

$$\theta_0(\lambda) = 1, \quad \theta_1(\lambda) = 0, \qquad \varphi_0(\lambda) = 0, \quad \varphi_1(\lambda) = 1.$$

By $\lambda_1, \lambda_2, \ldots, \lambda_{2N}$ we denote the roots of the equation

$$\Delta^2(\lambda) - 4 = 0,$$

while $\mu_1, \mu_2, \ldots, \mu_{N-1}$ stand for the roots of the equation

$$\theta_{N+1}(\lambda) = 0,$$

where $\Delta(\lambda) = \theta_N(\lambda) + \varphi_{N+1}(\lambda)$. As it is known (see [2]), all λ_i, $i = 1, 2, \dots, 2N$ and μ_j, $j = 1, 2, \dots, N-1$ are real, the roots μ_j are simple, while among λ_i there can be double roots. It is easy to see that

$$\Delta^2(\lambda) - 4 = \left(\prod_{j=1}^{N} a_j\right)^{-2} \prod_{j=1}^{2N} (\lambda - \lambda_j), \tag{5}$$

$$\theta_{N+1}(\lambda) = -a_0 \left(\prod_{j=1}^{N} a_j\right)^{-1} \prod_{j=1}^{N-1} (\lambda - \mu_j). \tag{6}$$

We introduce the notation

$$\sigma_j = sign[\theta_N(\mu_j) - \varphi_{N+1}(\mu_j)], \quad j = 1, 2, \dots, N-1.$$

Definition 1. *A set of numbers μ_j, $j = 1, 2, \dots, N-1$ and a sequence of signs σ_j, $j = 1, 2, \dots, N-1$ is called spectral parameters of Hill equation (4).*

Definition 2. *The set of the spectral parameters $\{\mu_j, \sigma_j\}_{j=1}^{N-1}$ and the numbers λ_i, $i = 1, 2, \dots, 2N$, is called spectral data of Hill equation (4).*

Finding spectral data and studying their properties is call the direct spectral problem for the discrete Hill equation. Recovering of the coefficients a_n, b_n for the Hill equation by its spectral data is called the inverse spectral problem for equation (4).

The following lemma holds true.

Lemma 1. *The identities hold:*

$$b_{k+1} = \frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2}\sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k}), \tag{7}$$

$$a_k^2 = \frac{\lambda_1^2 + \lambda_{2N}^2}{8} + \frac{1}{8}\sum_{j=1}^{N-1} \left(\lambda_{2j}^2 + \lambda_{2j+1}^2 - 2\mu_{j,k}^2\right)$$
$$- \frac{1}{4}\left[\frac{\lambda_1 + \lambda_{2N}}{2} + \frac{1}{2}\sum_{j=1}^{N-1} (\lambda_{2j} + \lambda_{2j+1} - 2\mu_{j,k})\right]^2 - \frac{1}{2}\sum_{j=1}^{N-1} \frac{\sigma_{j,k}\sqrt{\prod_{i=1}^{2N} (\mu_{j,k} - \lambda_i)}}{\prod_{\substack{i=1\\ i\neq j}}^{N-1} (\mu_{j,k} - \mu_{i,k})}, \tag{8}$$

where $\mu_{j,k}$, $j = 1, 2, \dots, N-1$, are the roots of the equation $\theta_{N+1,k}(\lambda) = 0$. Here $\theta_{n,k}(\lambda)$, $n \in \mathbb{Z}$, is a solution of the equation

$$a_{n+k-1}y_{n-1} + b_{n+k}y_n + a_{n+k}y_{n+1} = \lambda y_n, \quad n \in \mathbb{Z},$$

satisfying the initial conditions $\theta_{0,k}(\lambda) = 1$, $\theta_{1,k}(\lambda) = 0$.

The proof of identities (7) and (8) was given in work [11].

3. Spectral parameters evolution

In this section we prove the main result of the work.

Theorem. *If the functions $a_n(t)$, $b_n(t)$, $n \in \mathbb{Z}$, is the solution of problem (1)–(3), then the spectrum of Hill operator (3) is independent of t, while the spectral parameters $\mu_j(t)$, $j =$*

$1,\ 2,\ \ldots,\ N-1$, *satisfy the following system of equations*

$$\dot{\mu}_j(t) = 2\frac{\sigma_j(t)\cdot\sqrt{\prod\limits_{k=1}^{2N}(\mu_j(t)-\lambda_k)}}{\prod\limits_{\substack{k=1\\k\neq j}}^{N-1}(\mu_j(t)-\mu_k(t))}\cdot[\mu_j(t)+b_1(t)],$$

where

$$b_1(t) = \frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{k=1}^{N-1}(\lambda_{2k}+\lambda_{2k+1}-2\mu_k(t)).$$

Proof. We denote by $y^j(t) = (y_0^j(t),\ y_1^j(t),\ \ldots,\ y_N^j(t))^T$, $j = 1,\ 2,\ \ldots,\ N-1$, the normalized vector eigenfunctions associated with the eigenvalues $\lambda = \mu_j(t)$, $j = 1,\ 2,\ \ldots,\ N-1$, of the following boundary value problem

$$\begin{cases}(L(t)y)_n \equiv a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1} = \lambda y_n, & 1\leq n\leq N\\ y_1 = 0,\quad y_{N+1} = 0.\end{cases}$$

It was shown in work [11] that

$$\dot{\mu}_j(t) = \sum_{n=1}^{N}(2\dot{a}_n(t)y_n^jy_{n+1}^j+\dot{b}_n(t)(y_n^j)^2). \tag{9}$$

Thanks to (1), identity (9) can be rewritten as

$$\begin{aligned}\dot{\mu}_j(t) =& \sum_{n=1}^{N}2[a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2)]y_n^jy_{n+1}^j\\ &+\sum_{n=1}^{N}[2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})](y_n^j)^2.\end{aligned} \tag{10}$$

In identity (10) we introduce the following notation

$$\begin{aligned}F_n =& 2[a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2)]y_n^jy_{n+1}^j\\ &+[2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})](y_n^j)^2.\end{aligned} \tag{11}$$

Let us find a sequence u_n such that $u_{n+1}-u_n = F_n$. We seek u_n as

$$u_n = A_n(y_n^j)^2+2B_ny_n^jy_{n+1}^j+C_n(y_{n+1}^j)^2, \tag{12}$$

where $A_n = A_n(t,\ \mu_j)$, $B_n = B_n(t,\ \mu_j)$ and $C_n = C_n(t,\ \mu_j)$ are unknown coefficients. Taking into consideration the identity

$$y_{n+2}^j = \frac{1}{a_{n+1}}[(\mu_j-b_{n+1})y_{n+1}^j-a_ny_n^j],$$

we get

$$\begin{aligned}&(A_{n+1}-C_n)(y_{n+1}^j)^2-A_n(y_n^j)^2-2B_ny_n^jy_{n+1}^j+\frac{2B_{n+1}}{a_{n+1}}y_{n+1}^j[(\mu_j-b_{n+1})y_{n+1}^j-a_ny_n^j]\\ &+\frac{C_{n+1}}{a_{n+1}^2}(\mu_j-b_{n+1})^2(y_{n+1}^j)^2-\frac{2C_{n+1}}{a_{n+1}^2}a_n(\mu_j-b_{n+1})y_n^jy_{n+1}^j+\frac{C_{n+1}}{a_{n+1}^2}a_n^2(y_n^j)^2 = F_n.\end{aligned} \tag{13}$$

By (13) we obtain

$$-B_n - \frac{a_n}{a_{n+1}}B_{n+1} - \frac{a_n(\mu_j - b_{n+1})}{a_{n+1}^2}C_{n+1} = a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2), \tag{14}$$

$$\begin{aligned} &-C_{n-1} + \frac{2}{a_n}(\mu_j - b_n)B_n + \frac{1}{a_n^2}(\mu_j - b_n)^2 C_n + \frac{a_n^2}{a_{n+1}^2}C_{n+1} \\ &\quad = 2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1}). \end{aligned} \tag{15}$$

It is easy to see that

$$C_n = 2a_n^2(\mu_j + b_n), \quad B_n = a_n(a_{n-1}^2 - a_n^2 + b_n^2 - \mu_j^2)$$

solve systems (14) and (15). By (12) we have

$$\begin{aligned} \dot{\mu}_j(t) =& \sum_{n=1}^{N}\{2[a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2)]y_n^j y_{n+1}^j\} \\ &+ \sum_{n=1}^{N}\{[2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1})](y_n^j)^2\} \\ =& C_{N+1}(y_{N+2}^j)^2 - C_1(y_2^j)^2 = 2a_0^2(\mu_j + b_1)[(y_N^j)^2 - (y_0^j)^2]. \end{aligned} \tag{16}$$

Taking into consideration the identities

$$\left\|\theta^j\right\|^2 = \sum_{n=1}^{N}(\theta_n^j)^2 = a_N\theta_N^j(\theta_{N+1}^j)'\big|_{\lambda=\mu_j}, \quad (\theta^j)' = \frac{d\theta^j}{d\lambda}, \quad (y_0^j)^2 = \frac{(\theta_0^j)^2}{\|\theta^j\|^2}, \quad (y_N^j)^2 = \frac{(\theta_N^j)^2}{\|\theta^j\|^2},$$

equation (16) can be rewritten as

$$\dot{\mu}_j(t) = 2a_0\left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)}\right)\frac{1}{(\theta_{N+1}^j)'\big|_{\lambda=\mu_j(t)}} \cdot [\mu_j(t) + b_1]. \tag{17}$$

Employing the identity

$$\theta_N(\lambda, t)\varphi_{N+1}(\lambda, t) - \theta_{N+1}(\lambda, t)\varphi_N(\lambda, t) = 1$$

we have

$$\begin{aligned} \Delta^2(\mu_j(t)) - 4 =& [\theta_N^j(\mu_j(t), t) - \varphi_{N+1}^j(\mu_j(t), t)]^2 + 4\theta_N^j(\mu_j(t), t)\varphi_{N+1}^j(\mu_j(t), t) - 4 \\ =& [\theta_N^j(\mu_j(t), t) - \varphi_{N+1}^j(\mu_j(t), t)]^2 = \left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)}\right)^2. \end{aligned}$$

This implies

$$\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)} = \sigma_j(t)\sqrt{\Delta^2(\mu_j(t)) - 4}, \tag{18}$$

where

$$\sigma_j(t) = sign\left(\theta_N^j(\mu_j(t), t) - \frac{1}{\theta_N^j(\mu_j(t), t)}\right), \quad j = 1,\ 2,\ \ldots,\ N-1.$$

It follows from expansions (5) and (6) that

$$\Delta^2(\lambda) - 4 = \left(\prod_{k=1}^{N} a_k\right)^{-2} \prod_{k=1}^{2N} (\lambda - \lambda_k), \tag{19}$$

$$\theta_{N+1}(\lambda, t) = -a_0 \left(\prod_{j=1}^{N} a_j\right)^{-1} \prod_{k=1}^{N-1} (\lambda - \mu_k(t)). \tag{20}$$

Differentiating expansion (20) w.r.t. λ and letting $\lambda = \mu_j(t)$, we get

$$\theta'_{N+1}(\lambda)\big|_{\lambda=\mu_j(t)} = -a_0 \left(\prod_{k=1}^{N} a_k\right)^{-1} \prod_{\substack{k=1 \\ k\neq j}}^{N-1} (\mu_j(t) - \mu_k(t)). \tag{21}$$

Substituting (18), (19) and (21) into (17) and taking into consideration identity (7), we obtain identity (9).

Now let us show that $\lambda_k(t)$ is independent of t. Let $\left\{g_n^k(t)\right\}$ be a normalized eigenfunction of the operator $L(t)$ associated with the eigenvalue $\lambda_k(t)$, $k = 1,\ 2,\ \ldots,\ 2N$, that is,

$$a_{n-1}g_{n-1}^k + b_n g_n^k + a_n g_{n+1}^k = \lambda_k g_n^k.$$

Differentiating this identity w.r.t. t, multiplying by g_n^k and summing up w.r.t. n, we obtain

$$\frac{d\lambda_k}{dt} = \sum_{n=1}^{N} \left(2\dot{a}_n(t) g_n^k g_{n+1}^k + \dot{b}_n(t) \left(g_n^k\right)^2\right). \tag{22}$$

Employing equations (1), identity (22) can be rewritten as

$$\begin{aligned} \dot{\lambda}_k(t) =& \sum_{n=1}^{N} \{2[a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2)]g_n^k g_{n+1}^k\} \\ &+ \sum_{n=1}^{N} \{[2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1})](g_n^k)^2\}. \end{aligned} \tag{23}$$

Similar to (16), by (23) we get $\dot{\lambda}_k(t) = 0$. Thus, identity (21) and the independence of t for the eigenvalues $\lambda_k(t)$ complete the proof. □

Remark. *This theorem gives a way for solving problem (1)-(3). In order to do it, we first find the spectral data* λ_i, $\mu_j(0)$, $\sigma_j(0)$ *by given sequences* $\{a_n^0\}$ *and* $\{b_n^0\}$. *Then, employing the provided in work [12] algorithm for solving the inverse problem, we define* $\mu_{j,k}(0)$, $\sigma_{j,k}(0)$. *Applying the proven theorem, we calculate* $\mu_{j,k}(t)$, $\sigma_{j,k}(t)$. *In view of the independence of* k *and* t *for the eigenvalues* $\lambda_{i,k}$, *by employing identities (7), (8), we find* $a_k(t)$, $b_k(t)$.

Corollary. *If* $N = 2p$ *and a number* p *is the period for the initial sequences* $\{a_n^0\}$ *and* $\{b_n^0\}$, *then the roots of the equation* $\Delta(\lambda) + 2 = 0$ *are double. Since the Lyapunov function corresponding to the coefficients* $a_n(t)$ *and* $b_n(t)$ *coincides with* $\Delta(\lambda)$, *by an analogue of the inverse Borg theorem for the discrete Hill equation (see [18]), the number* p *is also a period for the solution* $a_n(t)$, $b_n(t)$ *in the variable* n.

Let us demonstrate the application of the main theorem for the solution to problem (1), (2) subject to the initial conditions

$$(a_n^0)^2 = \frac{5}{8} - (-1)^n \frac{3}{8}, \quad b_n^0 = \frac{1}{2}, \quad n \in \mathbb{Z}.$$

In this case

$$N = 2, \quad \lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 = 1, \quad \lambda_4 = 2, \quad \mu(0) = \frac{1}{2}, \quad \sigma(0) = 1.$$

Employing the above remark, we obtain

$$a_n^2(t) = \frac{1}{2}\left(1 + \mu(t) - \mu^2(t)\right) - (-1)^n \frac{1}{2}\sqrt{\mu(t)(\mu(t) - 2)(\mu^2(t) - 1)},$$

$$b_n(t) = \frac{1}{2} - \frac{(-1)^n}{2}(1 - 2\mu(t)), \quad n \in \mathbb{Z},$$

where $\mu(t)$ is determined by the equation

$$\frac{d\mu(t)}{dt} = \sqrt{\mu(t)(2 - \mu(t))(1 - \mu^2(t))}$$

subject to the initial condition $\mu(0) = \frac{1}{2}$. Solving this problem (see [19]), we find that

$$\mu(t) = \frac{sn^2\left(t - t_0,\, \frac{\sqrt{3}}{2}\right)}{1 + cn^2\left(t - t_0,\, \frac{\sqrt{3}}{2}\right)},$$

where $\mathrm{sn}\left(t_0,\, \frac{\sqrt{3}}{2}\right) = -\sqrt{\frac{2}{3}}$. Thus,

$$\begin{aligned} a_n^2(t) =& \frac{5}{8} - \frac{1}{2}\left(\frac{1 - 3\mathrm{cn}^2\left(t - t_0, \frac{\sqrt{3}}{2}\right)}{1 + \mathrm{cn}^2\left(t - t_0, \frac{\sqrt{3}}{2}\right)}\right)^2 \\ & - (-1)^n \frac{2\mathrm{sn}\left(t - t_0, \frac{\sqrt{3}}{2}\right)\mathrm{cn}\left(t - t_0, \frac{\sqrt{3}}{2}\right)\mathrm{dn}\left(t - t_0, \frac{\sqrt{3}}{2}\right)}{\left(1 + \mathrm{cn}^2\left(t - t_0, \frac{\sqrt{3}}{2}\right)\right)^2}, \\ b_n(t) =& \frac{1}{2} - \frac{(-1)^n}{2}\left(\frac{3\mathrm{cn}^2\left(t - t_0, \frac{\sqrt{3}}{2}\right) - 1}{1 + \mathrm{cn}^2\left(t - t_0, \frac{\sqrt{3}}{2}\right)}\right), \quad n \in \mathbb{Z}, \end{aligned}$$

where sn, cn and dn are elliptic Jacobi functions.

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