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SYMMETRY REDUCTION AND INVARIANT SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFUSION EQUATION WITH A SOURCE TERM

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Abstract. We consider a problem on constructing invariant solutions to a nonlinear fractional differential equations of anomalous diffusion with a source. On the base of an earlier made group classification of the considered equation, for each case in the classification we construct the optimal systems of one–dimensional subalgebras of Lie algebras of infinitesimal operators of the point transformations group admitted by the equation. For each one-dimensional subalgebra of each optimal system we find the corresponding form of the invariant solution and made the symmetry reduction to an ordinary differential equation. We prove that there are three different types of the reduction equations (factor equations): a second order ordinary differential equation integrated by quadratures and two ordinary nonlinear fractional differential equations. For particular cases of the latter we find exact solutions.

Keywords: fractional diffusion equation, symmetry, optimal system of subalgebras, symmetry reduction, invariant solution.

Mathematics Subject Classification: 45K05, 70G65

1. Introduction

We consider a nonlinear equation of anomalous diffusion

$$D_t^\alpha u = (k(u)u_x)_x + f(u), \quad \alpha \in (0,2) \tag{1}$$

with the fractional Riemann-Liouville time derivative [1, 2]:

$$D_t^\alpha u = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{2}$$

Here $\Gamma(z)$ is the Gamma function and $n = [\alpha] + 1$.

As $\alpha \in (0,1)$, equation (1) is known as a subdiffusion equation, while for $\alpha \in (1,2)$ this is a diffusion-wave equation [3–6]. In the limiting case $\alpha = 1$ it becomes the classical diffusion equation, and as $\alpha = 2$, this is the wave equation.

At present, the most studied case is when equation (1) has no source term ($f(u) = 0$) and is linear ($k(u) = const$). The issues on solvability and uniqueness of the solutions to the corresponding initial boundary value problems for such linear equations were studied by many authors, who showed that usually, the solutions to these problems are expressed in terms of Wright and Fox special functions [6–8].

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Nonlinear equations of anomalous diffusion were studied less intensively. This is due to considerable complicacy of studying nonlinear integral-differential equations, whose particular case is equation (1). Nevertheless, there is a series of general mathematical approaches allowing one to construct exact solutions to nonlinear differential equations and one of such approaches is the modern group analysis [9, 10].

The study of symmetry properties of fractional order equations is a rather complicated problem even for linear equations. The complicacy of the problem is first of all due to the properties of the integral-differential operator of fractional differentiation and is related to its non-locality. In contrast to the classical operator of integer differentiation, the actions of the fractional differentiation operator on the product of two functions is presented not by a finite sum but an infinite series, which is a so-called generalized Leibnitz rule [1]. A much more complicated form is for the fractional derivatives of a composite function. Here in the general case one has to deal with a four-multiple series [13]. As a result, the solving of generating equations for finding the group of point transformation turns out to be quite effortful.

The general methods of the group transformations for fractional differential equations were presented in works [13–18]. There were constructed the continuation formulae for the groups of point transformations for fractional differential variables, the algorithm of finding linear-autonomous symmetries for such equations was proposed and there were developed the algorithms for constructing conservation laws by known symmetries on the base of fractional differential generalization of Noether theorem.

The symmetry were first used for fractional differential equations of anomalous diffusion in works [11,12]. There were constructed invariant solutions to the linear equations of subdiffusion and to diffusion-wave equations with Riemann-Liouville derivatives corresponding to the group of non-uniform dilatations. It was shown that the solutions are represented via Wright functions.

In work [14], the group classification of nonlinear equation (1) without source was made. It was shown that such equation always admits a two-parametric group consisting of the group of translations along x and the group of non-uniform dilatations. This main group is extended for four particular cases of the diffusion coefficient $k(u)$: $k = const$, $k = u^{\sigma}$, $k = u^{2\alpha/(1-\alpha)}$ and $k = u^{-4/3}$.

In work [19], the results on group classification of the classical ($\alpha = 1$) diffusion equation with source of form (1), were provided as well as the invariant solutions corresponding to optimal systems of one-dimensional subalgebras for each case of group classification. The group classification for fractional differential equation (1) was made in work [20]. Some invariant solutions to this equations were also constructed in this work. However, the problem on constructing invariant solutions to this equations corresponding to optimal systems of subalgebras was just mentioned in [20].

In the present work we provide the solution to this problem for all one-dimensional sublagebras of finite-dimensional Lie algebras of infinitesimal operators of the groups of point transformations admitted by equation (1). It was proved that under the symmetry reduction, the original equation of anomalous diffusion is reduced to one of three ordinary differential equations called reduced equation or factor equations (in terms in Academician L.V. Ovsyannikov): one second order equation integrated by quadratures and two ordinary fractional differential equations, for which the admitted groups of linear autonomous symmetries were found and the corresponding invariant solutions are constructed.

2. Optimal systems of one-dimensional subalgebras

Since (1) is a scalar equation with two independent variables, we can use only one- and two-dimensional subalgebras for constructing invariant solutions. At that, the solutions constructed by two-dimensional subalgebras are particular cases of solutions constructed on corresponding

embedded one-dimensional subalgebras. This is why one-dimensional subalgebras are of the main interest.

The group classification of equation (1) was first made in work [20], its results are provided in the first three columns of Table 1. At that, the function $g(t,x)$ determining the infinite-dimensional algebra L_∞ is an arbitrary solution of the equation $D_t^\alpha g = g_{xx}$ in Case I.1 and I.2 and of the equation $D_t^\alpha g = g_{xx} + \delta g$ in Case I.3.

In the fourth column (L) of Table 1 the type of the corresponding algebra of infinitesimal operators is provided. Analysing Table 1, we see that the finite-dimensional part of Lie algebras of the groups of the operators admitted by equation (1) is exhausted by six types of algebras, two two-dimensional algebras, two three-dimensional algebras and two four-dimensional algebras:

$$\begin{array}{ll}
1)\ L_2 = 2A_1; & \\
2)\ L_2 = A_2: & [e_1,e_2]=e_2; \\
3)\ L_3 = A_2 \oplus A_1: & [e_1,e_2]=e_2; \\
4)\ L_3 = A_{3,8}: & [e_1,e_2]=e_1,\ [e_2,e_3]=e_3,\ [e_3,e_1]=2e_2; \\
5)\ L_4 = 2A_2: & [e_1,e_2]=e_2,\ [e_3,e_4]=e_4; \\
6)\ L_4 = A_{3,8} \oplus A_1: & [e_1,e_2]=e_1,\ [e_2,e_3]=e_3,\ [e_3,e_1]=2e_2.
\end{array}$$

Here e_i is the ith basis vector of the algebras and as usually, only nonzero commutator relations are shown.

In the fourth column (L) of Table 1 we also show the relation between the bases $\{e_i\}$ and $\{X_i\}$. For instance, in Case IV.5, the notation $< \frac{2}{\omega}(4-1), \frac{1}{\omega}(1+3-4), \frac{1}{\omega}(1+3), 2 >$ means that we have the change of the basis:

$$e_1 = \frac{2}{\omega}(X_4 - X_1),\ e_2 = \frac{1}{\omega}(X_1 + X_3 - X_4),\ e_3 = \frac{1}{\omega}(X_1 + X_3),\ e_4 = X_2.$$

For the algebras of smaller dimensions, the optimal systems of subalgebras are well-known. For instance, for real Lie algebras of dimension three and four such optimal systems were constructed in work [21]. The corresponding optimal systems of one-dimensional sublagebras were used as a base in the present work. In the series of cases, to simplify the constructing of invariant solutions, these systems were additionally transformed by means of the groups of internal automorphisms of the algebras. We also took into consideration that equation (1) is invariant w.r.t. the reflection $\bar{x} = -x$. The final form of optimal systems of one-dimensional subalgebras for the considered equation is given in the last column (ΘL_1) of Table 1. All optimal systems are written in the basis $\{X_i\}$. As usually, only the indices of the corresponding operators are indicated, that is, the notation $2 + \beta 3$ means the subalgebra with the operator $X_2 + \beta X_3$, where β is an arbitrary real constant. By Table 1 we see that the dimension of the corresponding optimal systems of one-dimensional subalgebras for equation (1) does not exceed seven.

3. Symmetry reduction

For each case of each optimal system of one-dimensional subalgebras in Table 1 we found the form of the corresponding invariant solution and made the symmetry reduction of equation (1). The results are provided in Table 2. The reduction is made in such a way that the form of the fractional Riemann-Liouville derivative is preserved. We note that such approach is not the only possible one. In work [11], to construct the invariant solution on the group of non-uniform dilatations, the reduction to an equation with a frational differential operator of Erdélyi-Kober type was employed.

Table 1: Group classification of equation (1) and optimal systems of one-dimensional subalgebras
(Here $\varepsilon = \pm 1$, $\delta = \pm 1$, $\beta, \gamma \in \mathbb{R}$, $\rho, \lambda \geqslant 0$, $\omega = 2/\sqrt{3}$)

$k(u)$	$f(u)$	X	L	ΘL_1
I. $k = 1$	1. $f = 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2}{\alpha} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$, $X_3 = u \frac{\partial}{\partial u}$, $X_\infty = g(t,x) \frac{\partial}{\partial u}$	$L_3 \oplus L_\infty$, $L_3 = A_2 \oplus A_1$ $< -2, 1, 3 >$	1. $\{1\}$ 2. $\{1+3\}$ 3. $\{2+\beta 3\}$ 4. $\{3\}$
	2. $f = \delta$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2}{\alpha} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} +$ $+ \frac{2\delta}{\Gamma(1+\alpha)} t^\alpha \frac{\partial}{\partial u}$ $X_3 = \left(u - \frac{\delta}{\Gamma(\alpha+1)} t^\alpha \right) \frac{\partial}{\partial u}$, $X_\infty = g(t,x) \frac{\partial}{\partial u}$	$L_3 \oplus L_\infty$ $L_3 = A_2 \oplus A_1$ $< -2, 1, 3 >$	1. $\{1\}$ 2. $\{1+3\}$ 3. $\{2+\beta 3\}$ 4. $\{3\}$
	3. $f = \delta u + \chi$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = [u - \chi t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha)] \frac{\partial}{\partial u}$, $X_\infty = g(t,x) \frac{\partial}{\partial u}$	$L_2 \oplus L_\infty$, $L_2 = 2A_1$ $< 1, 2 >$	1. $\{1\}$ 2. $\{\beta 1 + 2\}$
	4. $f = \delta u^\gamma$ $(\gamma \neq 0,\ 1)$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2}{\alpha} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{2}{1-\gamma} u \frac{\partial}{\partial u}$	$L_2 = A_2$ $< -2, 1 >$	1. $\{1\}$ 2. $\{2\}$
II. $k = u^\sigma$ $(\sigma \neq 0,$ $-\frac{4}{3},\ \frac{2\alpha}{1-\alpha})$	1. $f = 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2}{\alpha} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$, $X_3 = \frac{\sigma}{\alpha} t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}$	$L_3 = A_2 \oplus A_1$ $< -2, 1, -3 >$	1. $\{1\}$ 2. $\{1+3\}$ 3. $\{2+\beta 3\}$ 4. $\{3\}$
	2. $f = \delta u^\gamma$ $(\gamma \neq \sigma + 1)$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2(1-\gamma)}{\alpha(\sigma+1-\gamma)} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} +$ $+ \frac{2}{\sigma+1-\gamma} u \frac{\partial}{\partial u}$	$L_2 = A_2$ $< -2, 1 >$	1. $\{1\}$ 2. $\{2\}$
	3. $f = \delta u^{\sigma+1}$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\sigma}{\alpha} t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}$	$L_2 = 2A_1$ $< 1, 2 >$	1. $\{1\}$ 2. $\{\beta 1 + 2\}$
III. $k = u^{\frac{2\alpha}{1-\alpha}}$	1. $f = 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = t \frac{\partial}{\partial t} + \frac{\alpha-1}{2} u \frac{\partial}{\partial u}$, $X_3 = x \frac{\partial}{\partial x} - \frac{\alpha-1}{\alpha} u \frac{\partial}{\partial u}$, $X_4 = t^2 \frac{\partial}{\partial t} + (\alpha - 1) t u \frac{\partial}{\partial u}$	$L_4 = 2A_2$ $< -3, 1, 2, 4 >$	1. $\{1\}$ 2. $\{1+2\}$ 3. $\{1+4\}$ 4. $\{2\}$ 5. $\{\beta 2 + 3\}$ 6. $\{3 + \varepsilon 4\}$ 7. $\{4\}$
	2. $f = \delta u^\gamma$ $(\gamma \neq \frac{1+\alpha}{1-\alpha})$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{2(1-\gamma)(1-\alpha))}{\alpha[1-\gamma+\alpha(1+\gamma)]} t \frac{\partial}{\partial t} +$ $+ x \frac{\partial}{\partial x} + \frac{2(1-\alpha)}{1-\gamma+\alpha(1+\gamma)} u \frac{\partial}{\partial u}$	$L_2 = A_2$ $< -2, 1 >$	1. $\{1\}$ 2. $\{2\}$

Continuation of Table 1

$k(u)$	$f(u)$	X	L	ΘL_1
	3. $f=\delta u^{\frac{1+\alpha}{1-\alpha}}$	$X_1=\frac{\partial}{\partial x}$, $X_2=t\frac{\partial}{\partial t}+\frac{(\alpha-1)}{2}u\frac{\partial}{\partial u}$, $X_3=t^2\frac{\partial}{\partial t}+(\alpha-1)tu\frac{\partial}{\partial u}$	$L_3=A_2\oplus A_1$ $<2,3,1>$	1. $\{1\}$ 2. $\{\beta 1+2\}$ 3. $\{1+3\}$ 4. $\{3\}$
IV. $k=u^{-\frac{4}{3}}$	1. $f=0$	$X_1=\frac{\partial}{\partial x}$, $X_2=\frac{2}{\alpha}t\frac{\partial}{\partial t}+\frac{3}{2}u\frac{\partial}{\partial u}$, $X_3=x\frac{\partial}{\partial x}-\frac{3}{2}u\frac{\partial}{\partial u}$, $X_4=x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$	$L_4=A_{3,8}\oplus A_1$ $<1,3,-4,2>$	1. $\{1\}$ 2. $\{1+2\}$ 3. $\{1+\beta 2-4\}$ 4. $\{2\}$ 5. $\{\rho 2+3\}$
	2. $f=\delta u$	$X_1=\frac{\partial}{\partial x}$ $X_2=x\frac{\partial}{\partial x}-\frac{3}{2}u\frac{\partial}{\partial u}$, $X_3=x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$	$L_3=A_{3,8}$ $<1,2,-3>$	1. $\{1\}$ 2. $\{2\}$ 3. $\{1-3\}$
	3. $f=\delta u^{\gamma}$ $(\gamma\neq-\frac{1}{3},1)$	$X_1=\frac{\partial}{\partial x}$, $X_2=\frac{6(1-\gamma)}{\alpha(1+3\gamma)}t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}+$ $+\frac{6}{1+3\gamma}u\frac{\partial}{\partial u}$	$L_2=A_2$ $<2,1>$	1. $\{1\}$ 2. $\{2\}$
	4. $f=u^{-\frac{1}{3}}$	$X_1=\frac{\partial}{\partial x}$, $X_2=4t\frac{\partial}{\partial t}+3\alpha u\frac{\partial}{\partial u}$, $X_3=e^{\omega x}\frac{\partial}{\partial x}-\sqrt{3}e^{\omega x}u\frac{\partial}{\partial u}$, $X_4=e^{-\omega x}\frac{\partial}{\partial x}+\sqrt{3}e^{-\omega x}u\frac{\partial}{\partial u}$	$L_4=A_{3,8}\oplus A_1$ $<\frac{1}{\omega}3,-\frac{1}{\omega}1,$ $-\frac{1}{\omega}4,2>$	1. $\{1\}$ 2. $\{\rho 1+2\}$ 3. $\{2+\varepsilon 3\}$ 4. $\{\beta 2+3-4\}$ 5. $\{3\}$
	5. $f=-u^{-\frac{1}{3}}$	$X_1=\frac{\partial}{\partial x}$, $X_2=4t\frac{\partial}{\partial t}+3\alpha u\frac{\partial}{\partial u}$, $X_3=\cos(\omega x)\frac{\partial}{\partial x}+$ $+\sqrt{3}\sin(\omega x)u\frac{\partial}{\partial u}$, $X_4=\sin(\omega x)\frac{\partial}{\partial x}-$ $-\sqrt{3}\cos(\omega x)u\frac{\partial}{\partial u}$	$L_4=A_{3,8}\oplus A_1$ $<\frac{2}{\omega}(4-1),$ $\frac{1}{\omega}(1+3-4),$ $\frac{1}{\omega}(1+3),2>$	1. $\{1\}$ 2. $\{\rho 1+2\}$ 3. $\{1+2+3\}$ 4. $\{1+4\}$ 5. $\{\lambda 2+3\}$
	6. $f=u^{-\frac{1}{3}}+$ $+\chi u$	$X_1=\frac{\partial}{\partial x}$, $X_2=e^{\omega x}\frac{\partial}{\partial x}-\sqrt{3}e^{\omega x}u\frac{\partial}{\partial u}$, $X_3=e^{-\omega x}\frac{\partial}{\partial x}+\sqrt{3}e^{-\omega x}u\frac{\partial}{\partial u}$,	$L_3=A_{3,8}$ $<\frac{1}{\omega}2,-\frac{1}{\omega}1,$ $-\frac{1}{\omega}3>$	1. $\{1\}$ 2. $\{2\}$ 3. $\{2-3\}$
	7. $f=$ $=-u^{-\frac{1}{3}}+\chi u$	$X_1=\frac{\partial}{\partial x}$, $X_2=\cos(\omega x)\frac{\partial}{\partial x}+$ $+\sqrt{3}\sin(\omega x)u\frac{\partial}{\partial u}$, $X_3=\sin(\omega x)\frac{\partial}{\partial x}-$ $-\sqrt{3}\cos(\omega x)u\frac{\partial}{\partial u}$	$L_3=A_{3,8}$ $<\frac{2}{\omega}(3-1),$ $\frac{1}{\omega}(1+2-3),$ $\frac{1}{\omega}(1+2)>$	1. $\{1\}$ 2. $\{1+3\}$ 3. $\{2\}$

Table 2: Invariant solutions and the corresponding reduced equations for equation (1)
(Here $\varepsilon = \pm 1$, $\delta = \pm 1$, β, $\gamma \in \mathbb{R}$, $\rho, \lambda \geqslant 0$, $\omega = 2/\sqrt{3}$)

N_k	N_f	N_Θ	Solution	Reduced equation
I	1	1	$u = \varphi(t)$	$D_t^\alpha \varphi = 0$
		2	$u = e^x \varphi(t)$	$D_t^\alpha \varphi = \varphi$
		3	$u = x^{-\beta}\varphi(\tau),\ \tau = tx^{-\frac{2}{\alpha}}$	$D_\tau^\alpha \varphi = \frac{4}{\alpha^2}\tau^2\varphi'' + \frac{2}{\alpha}(\frac{2}{\alpha} + 1 + 2\beta)\tau\varphi' + \beta(\beta+1)\varphi$
		4	—	—
	2	1	$u = \varphi(t)$	$D_t^\alpha \varphi = \delta$
		2	$u = e^x\varphi(t) + \frac{\delta t^\alpha}{\Gamma(1+\alpha)}$	$D_t^\alpha \varphi = \varphi$
		3	$u = x^{-\beta}\varphi(\tau) + \frac{\delta t^\alpha}{\Gamma(1+\alpha)}$, $\tau = tx^{-\frac{2}{\alpha}}$	$D_\tau^\alpha \varphi = \frac{4}{\alpha^2}\tau^2\varphi'' + \frac{2}{\alpha}(\frac{2}{\alpha} + 1 + 2\beta)\tau\varphi' + \beta(\beta+1)\varphi$
		4	$u = \frac{\delta t^\alpha}{\Gamma(1+\alpha)}$	—
	3	1	$u = \varphi(t)$	$D_t^\alpha \varphi = \delta\varphi + \chi$
		2	$u = e^{-\beta x}\varphi(t) +$ $+\chi t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha),\ (\beta \neq 0)$	$D_\tau^\alpha \varphi = (\beta^2 + \delta)\varphi$
			$u = \chi t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha)\ (\beta = 0)$	—
	4	1	$u = \varphi(t)$	$D_t^\alpha \varphi = \delta\varphi^\gamma\ (\gamma \neq 0,\ 1)$
		2	$u = x^{-\beta}\varphi(\tau),\ \tau = tx^{-\frac{2}{\alpha}}$, $\beta = \frac{2}{1-\gamma}\ (\gamma \neq 0,\ 1)$	$D_\tau^\alpha \varphi = \frac{4}{\alpha^2}\tau^2\varphi'' + \frac{2}{\alpha}(\frac{2}{\alpha} + 1 + 2\beta)\tau\varphi' +$ $+\beta(\beta+1)\varphi + \delta\varphi^\gamma$
II	1	1	$u = \varphi(t)$	$D_t^\alpha \varphi = 0$
		2	$u = e^{-x}\varphi(\tau),\ \tau = te^{-\frac{\sigma}{\alpha}x}$	$D_\tau^\alpha \varphi = \frac{\sigma^2}{\alpha^2}\tau^2\varphi^\sigma\varphi'' + \frac{\sigma^3}{\alpha^2}\tau^2\varphi^{\sigma-1}(\varphi')^2 +$ $+\frac{\sigma}{\alpha}\left[\frac{\sigma}{\alpha} + 2(1+\sigma)\right]\tau\varphi^\sigma\varphi' + (\sigma+1)\varphi^{\sigma+1}$
		3	$u = x^{-\beta}\varphi(\tau),\ \tau = tx^\nu$, $\nu = -\frac{\beta\sigma+2}{\alpha}$,	$D_\tau^\alpha \varphi = \nu^2\tau^2\varphi^\sigma\varphi'' + \sigma\nu^2\tau^2\varphi^{\sigma-1}(\varphi')^2 +$ $+\nu[\nu - 1 - 2\beta(\sigma+1)]\tau\varphi^\sigma\varphi' +$ $+\beta[1 + \beta(1+\sigma)]\varphi^{\sigma+1}$
		4	$u = t^{-\frac{\alpha}{\sigma}}\varphi(x)$	$\varphi\varphi'' + \sigma(\varphi')^2 - \frac{\Gamma(1-\alpha/\sigma)}{\Gamma(1-\alpha-\alpha/\sigma)}\varphi^{2-\sigma} = 0$
	2	1	$u = \varphi(t)$	$D_t^\alpha \varphi = \delta\varphi^\gamma$
		2	$u = x^{-\beta}\varphi(\tau),\ \tau = tx^\nu$, $\nu = -\frac{\beta\sigma+2}{\alpha},\ \beta = -\frac{2}{\sigma+1-\gamma}$	$D_\tau^\alpha \varphi = \nu^2\tau^2\varphi^\sigma\varphi'' + \sigma\nu^2\tau^2\varphi^{\sigma-1}(\varphi')^2 +$ $+\nu[\nu - 1 - 2\beta(\sigma+1)]\tau\varphi^\sigma\varphi' +$ $+\beta[1 + \beta(1+\sigma)]\varphi^{\sigma+1} + \delta\varphi^\gamma$
	3	1	$u = \varphi(t)$	$D_t^\alpha \varphi = \delta\varphi^{\sigma+1}$
		2	$u = e^{-\frac{x}{\beta}}\varphi(\tau),\ \tau = te^{\nu x}$, $\nu = -\frac{\sigma}{\alpha\beta}\ (\beta \neq 0)$	$D_\tau^\alpha \varphi = \nu^2\tau^2\varphi^\sigma\varphi'' + \sigma\nu^2\tau^2\varphi^{\sigma-1}(\varphi')^2 +$ $+\nu\left[\nu - \frac{2(\sigma+1)}{\beta}\right]\tau\varphi^\sigma\varphi' + \left(\frac{\sigma+1}{\beta^2} + \delta\right)\varphi^{\sigma+1}$
			$u = t^{-\frac{\alpha}{\sigma}}\varphi(x)\ (\beta = 0)$	$\varphi\varphi'' + \sigma(\varphi')^2 + \delta\varphi^2 - \frac{\Gamma(1-\alpha/\sigma)}{\Gamma(1-\alpha-\alpha/\sigma)}\varphi^{2-\sigma} = 0$

Continuation of Table 2

N_k	N_f	N_Θ	Solution	Reduced equation
III	1	1	$u=\varphi(t)$	$D_t^\alpha\varphi=0$
		2	$u=e^{\frac{\alpha-1}{2}x}\varphi(\tau)$, $\tau=te^{-x}$, $\sigma=\frac{2\alpha}{1-\alpha}$	$D_\tau^\alpha\varphi=\tau^2\varphi^\sigma\varphi''+\sigma\tau^2\varphi^{\sigma-1}(\varphi')^2+$ $+(\alpha+2)\tau\varphi^\sigma\varphi'+\frac{1-\alpha^2}{4}\varphi^{\sigma+1}$
		3	$u=(1+tx)^{\alpha-1}\varphi(\tau)$, $\tau=\frac{t}{1+tx}$, $\sigma=\frac{2\alpha}{1-\alpha}$	$D_\tau^\alpha\varphi=\tau^4\varphi^\sigma\varphi''+\sigma\tau^4\varphi^{\sigma-1}(\varphi')^2+$ $+2(\alpha+2)\tau^3\varphi^\sigma\varphi'+(1-\alpha)(2+\alpha)\tau^2\varphi^{\sigma+1}$
		4	$u=t^{\frac{\alpha-1}{2}}\varphi(x)$	$\varphi\varphi''+\frac{2\alpha}{1-\alpha}(\varphi')^2-\frac{\Gamma(1/2+\alpha/2)}{\Gamma(1/2-\alpha/2)}\varphi^{\frac{4\alpha-2}{\alpha-1}}=0$
		5	$u=x^\nu\varphi(\tau)$, $\tau=tx^{-\beta}$, $\nu=\frac{2-\alpha\beta}{\sigma}$, $\sigma=\frac{2\alpha}{1-\alpha}$	$D_\tau^\alpha\varphi=\beta^2\tau^2\varphi^\sigma\varphi''+\sigma\beta^2\tau^2\varphi^{\sigma-1}(\varphi')^2+$ $+\beta[\beta+1-2\gamma(\sigma+1)]\tau\varphi^\sigma\varphi'+$ $+\nu[\nu(\sigma+1)-1]\varphi^{\sigma+1}$
		6	$u=x^{\frac{2}{\sigma}}\varphi(\tau)\times$ $\times(1-\varepsilon\tau\ln(x))^{1-\alpha}$, $\tau=\frac{t}{1+\varepsilon t\ln(x)}$, $\sigma=\frac{2\alpha}{1-\alpha}$	$D_\tau^\alpha\varphi=\tau^4\varphi^\sigma\varphi''+\sigma\tau^4\varphi^{\sigma-1}(\varphi')^2+$ $+\frac{\alpha+2}{\alpha}(2\alpha\tau-\varepsilon)\tau^2\varphi^\sigma\varphi'+$ $+\frac{1-\alpha}{\alpha^2}[1-\alpha(\alpha+2)\varepsilon\tau+\alpha^2(\alpha+2)\tau^2]\varphi^{\sigma+1}$
		7	$u=t^{\alpha-1}\varphi(x)$	$\varphi\varphi''+\frac{2\alpha}{1-\alpha}(\varphi')^2=0$
	2	1	$u=\varphi(t)$	$D_t^\alpha=\delta\varphi^\gamma$
		2	$u=x^{-\beta}\varphi(\tau)$, $\tau=tx^\nu$, $\nu=-\frac{\beta\sigma+2}{\alpha}$, $\beta=-\frac{2}{\sigma+1-\gamma}$, $\sigma=\frac{2\alpha}{1-\alpha}$ $\left(\gamma\neq\frac{1+\alpha}{1-\alpha}\right)$	$D_\tau^\alpha\varphi=\nu^2\tau^2\varphi^\sigma\varphi''+\sigma\nu^2\tau^2\varphi^{\sigma-1}(\varphi')^2+$ $+\nu[\nu-1-2\beta(\sigma+1)]\tau\varphi^\sigma\varphi'+$ $+\beta[1+\beta(1+\sigma)]\varphi^{\sigma+1}+\delta\varphi^\gamma$
	3	1	$u=\varphi(t)$	$D_t^\alpha\varphi=\delta\varphi^{\frac{1+\alpha}{1-\alpha}}$
		2	$u=e^{\frac{1-\alpha}{2}\nu x}\varphi(\tau)$, $\tau=te^{\nu x}$, $\nu=-\frac{1}{\beta}$, $\sigma=\frac{2\alpha}{1-\alpha}$ $(\beta\neq0)$	$D_\tau^\alpha\varphi=\nu^2\tau^2\varphi^\sigma\varphi''+\sigma\nu^2\tau^2\varphi^{\sigma-1}(\varphi')^2+$ $+(\alpha+2)\nu^2\tau\varphi^\sigma\varphi'+\left(\delta+\frac{1-\alpha^2}{4}\nu^2\right)\varphi^{\sigma+1}$
			$u=t^{\frac{\alpha-1}{2}}\varphi(x)$ $(\beta=0)$	$\varphi\varphi''+\frac{2\alpha}{1-\alpha}(\varphi')^2+\delta\varphi^2-\frac{\Gamma(1/2+\alpha/2)}{\Gamma(1/2-\alpha/2)}\varphi^{\frac{2-4\alpha}{1-\alpha}}=0$
		3	$u=(1+tx)^{\alpha-1}\varphi(\tau)$, $\tau=\frac{t}{1+tx}$, $\sigma=\frac{2\alpha}{1-\alpha}$	$D_\tau^\alpha\varphi=\tau^4\varphi^\sigma\varphi''+\sigma\tau^3\varphi^{\sigma-1}(\varphi')^2+$ $+2(\alpha+2)\tau^3\varphi^\sigma\varphi'+$ $+[\delta-(\alpha-1)(\alpha+2)\tau^2]\varphi^{\sigma+1}$
		4	$u=t^{\alpha-1}\varphi(x)$	$\varphi\varphi''+\frac{2\alpha}{1-\alpha}(\varphi')^2+\delta\varphi^2=0$
IV	1	1	$u=\varphi(t)$	$D_t^\alpha\varphi=0$
		2	$u=e^{\frac{3}{2}x}\varphi(\tau)$, $\tau=te^{-\frac{2}{\alpha}x}$	$D_\tau^\alpha\varphi=\frac{4}{\alpha^2}\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{16}{3\alpha^2}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+\frac{2}{\alpha}\left(1+\frac{2}{\alpha}\right)\tau\varphi^{-\frac{4}{3}}\varphi'-\frac{3}{4}\varphi^{-\frac{1}{3}}$
		3	$u=\frac{(1+x)^{\frac{3}{4}\beta-\frac{3}{2}}}{(1-x)^{\frac{3}{4}\beta+\frac{3}{2}}}\varphi(\tau)$, $\tau=t\left(\frac{1-x}{1+x}\right)^{\frac{\beta}{\alpha}}$	$D_\tau^\alpha\varphi=\frac{4\beta^2}{\alpha^2}\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{16}{3}\frac{\beta^2}{\alpha^2}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+\frac{2\beta^2}{\alpha^2}(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'+\left(3-\frac{\beta^2}{4}\right)\varphi^{-\frac{1}{3}}$
		4	$u=t^{\frac{3}{4}\alpha}\varphi(x)$	$\varphi\varphi''-\frac{4}{3}(\varphi')^2-\frac{\Gamma(1+3\alpha/4)}{\Gamma(1-\alpha/4)}\varphi^{\frac{10}{3}}=0$
		5	$u=x^{\frac{3}{2}(\rho-1)}\varphi(\tau)$, $\tau=tx^{-2\frac{\rho}{\alpha}}$	$D_\tau^\alpha\varphi=\frac{4\rho^2}{\alpha^2}\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{16}{3}\frac{\rho^2}{\alpha^2}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+\frac{\rho}{\alpha}\left(\frac{4\rho}{\alpha}+2-\rho\right)\tau\varphi^{-\frac{4}{3}}\varphi'-\frac{3}{4}(1+\rho)\varphi^{-\frac{1}{3}}$

Continuation of Table 2

N_k	N_f	N_Θ	Solution	Reduced equation
	2	1	$u=\varphi(t)$	$D_t^\alpha\varphi=\delta\varphi$
		2	$u=x^{-\frac{3}{2}}\varphi(t)$	$D_t^\alpha\varphi=\frac{3}{4}\varphi^{-\frac{1}{3}}+\delta\varphi$
		3	$u=(1-x^2)^{-\frac{3}{2}}\varphi(t)$	$D_t^\alpha\varphi=3\varphi^{-\frac{1}{3}}+\delta\varphi$
	3	1	$u=\varphi(t)$	$D_t^\alpha\varphi=\delta\varphi^\gamma$
		2	$u=x^{-\beta}\varphi(\tau),\ \tau=tx^\nu,$ $\nu=\frac{6(1-\gamma)}{\alpha(1+3\gamma)},\ \beta=\frac{6}{1+3\gamma}$	$D_\tau^\alpha\varphi=\nu^2\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{4}{3}\nu^2\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+\nu\left(\frac{2}{3}b-1+\nu\right)\tau\varphi^{-\frac{4}{3}}\varphi'+\left(b-\frac{b^2}{3}\right)\varphi^{-\frac{1}{3}}+\delta\varphi^\gamma$
	4	1	$u=\varphi(t)$	$D_t^\alpha\varphi=\varphi^{-1/3}$
		2	$u=t^{\frac{3}{4}\alpha}\varphi(x)\ (\rho=0)$	$\varphi\varphi''-\frac{4}{3}(\varphi')^2+\varphi^2-\frac{\Gamma(1+3\alpha/4)}{\Gamma(1-\alpha/4)}\varphi^{\frac{10}{3}}=0$
			$u=e^{-\frac{3\alpha}{\rho}x}\varphi(\tau),\ \tau=te^{\frac{4}{\rho}x}$ $(\rho>0)$	$D_\tau^\alpha\varphi=\frac{16}{\rho^2}\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{64}{3\rho^2}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+\frac{8}{\rho^2}(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'+\left(1-\frac{3\alpha^2}{\rho^2}\right)\varphi^{-\frac{1}{3}}$
		3	$u=e^{-\frac{3}{2}\omega x-3\alpha\psi(x)}\varphi(\tau),$ $\tau=te^{4\psi(x)},\ \psi(x)=\frac{\varepsilon}{\omega}e^{-\omega x}$	$D_\tau^\alpha\varphi=\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{4}{3}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+(1+2\alpha)\tau\varphi^{-\frac{4}{3}}\varphi'-3\alpha^2\varphi^{-\frac{1}{3}}$
		4	$u=\psi^{-\frac{3}{4}\alpha}\sinh^{-\frac{3}{2}}(\omega x)\varphi(\tau),$ $\tau=t\psi(x),$ $\psi=\left[\tanh\left(\frac{\omega x}{2}\right)\right]^{-\frac{4\beta}{\omega}}$	$D_\tau^\alpha\varphi=16\beta^2\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{64}{3}\beta^2\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+8\beta^2(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'+(1-3\alpha^2\beta^2)\varphi^{-\frac{1}{3}}$
		5	$u=e^{-\frac{3}{2}\omega x}\varphi(t)$	$D_t^\alpha\varphi=0$
	5	1	$u=\varphi(t)$	$D_t^\alpha\varphi=-\varphi^{-\frac{1}{3}}$
		2	$u=t^{\frac{3}{4}\alpha}\varphi(x)\ (\rho=0)$	$\varphi\varphi''-\frac{4}{3}(\varphi')^2-\varphi^2-\frac{\Gamma(1+3\alpha/4)}{\Gamma(1-\alpha/4)}\varphi^{\frac{10}{3}}=0$
			$u=e^{\frac{3\alpha}{\rho}x}\varphi(\tau),$ $\tau=te^{-\frac{4}{\rho}x}\ (\rho>0)$	$D_\tau^\alpha\varphi=\frac{16}{\rho^2}\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{64}{3\rho^2}\varphi^{-\frac{7}{3}}\tau^2(\varphi')^2+$ $+\frac{8}{\rho^2}(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'-\left(1+\frac{3\alpha^2}{\rho^2}\right)\varphi^{-\frac{1}{3}}$
		3	$u=\psi^{-\frac{3}{4}\alpha}(x)\cos^{-\frac{3}{2}}(\omega x)\varphi(\tau),$ $\tau=t\psi(x),\ \psi=e^{-\frac{4}{\omega}\tan(\omega x/2)}$	$D_\tau^\alpha\varphi=4\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{16}{3}\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+2(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'-\frac{3}{4}\alpha^2\varphi^{-\frac{1}{3}}$
		4	$u=\sin^{-3}\left(\frac{\omega x}{2}+\frac{\pi}{4}\right)\varphi(t)$	$D_t^\alpha\varphi=0$
		5	$u=\psi^{-\frac{3}{4}\alpha}(x)\cos^{-\frac{3}{2}}(\omega x)\varphi(\tau),$ $\tau=t\psi(x),$ $\psi(x)=\tan^{-\frac{4\lambda}{\omega}}\left(\frac{\omega x}{2}+\frac{\pi}{4}\right),$	$D_\tau^\alpha\varphi=16\lambda^2\tau^2\varphi^{-\frac{4}{3}}\varphi''-\frac{64}{3}\lambda^2\tau^2\varphi^{-\frac{7}{3}}(\varphi')^2+$ $+8\lambda^2(\alpha+2)\tau\varphi^{-\frac{4}{3}}\varphi'+(1-3\alpha^2\lambda^2)\varphi^{-\frac{1}{3}}$
	6	1	$u=\varphi(t)$	$D_t^\alpha\varphi=\varphi^{-\frac{1}{3}}+\chi\varphi$
		2	$u=e^{-\frac{3}{2}\omega x}\varphi(t)$	$D_t^\alpha\varphi=\chi\varphi$
		3	$u=\sinh^{-\frac{3}{2}}(\omega x)\varphi(t)$	$D_t^\alpha\varphi=\varphi^{-\frac{1}{3}}+\chi\varphi$
	7	1	$u=\varphi(t)$	$D_t^\alpha\varphi=-\varphi^{-\frac{1}{3}}+\chi\varphi$
		2	$u=\sin^{-3}\left(\frac{\omega x}{2}+\frac{\pi}{4}\right)\varphi(t)$	$D_t^\alpha\varphi=\chi\varphi$
		3	$u=\cos^{-\frac{3}{2}}(\omega x)\varphi(t)$	$D_t^\alpha\varphi=\varphi^{-\frac{1}{3}}+\chi\varphi$

Nevertheless, the keeping of the fractional differentiation operator under the reduction seems to be reasonable since it gives an opportunity to consider various types of reduced equations from the same position. The type of fractional derivative (2) is kept, in particular, under the

following change of the variables:

$$u(x,t) = f(x)\varphi(\tau),\ \tau = tg(x),$$

where $f(x)$, $g(x)$ are arbitrary functions. Then

$$D_t^\alpha u = f(x)g^\alpha(x)D_\tau^\alpha\varphi(\tau).$$

The most part of the cases in Table 2 corresponds to such change of the variables.

Another type of the change of variables allowing to keep the type of the fractional differentiation operator under the symmetry reduction is

$$u(x,t) = f(x)\varphi(\tau),\ \tau = \frac{t}{1+tg(x)},$$

which corresponds to the projective group of point transformations. In this case

$$D_t^\alpha u = f(x)(1+\tau g(x))^{1-\alpha}D_\tau^\alpha\varphi(\tau).$$

Such change of variables corresponds to cases III.1.3, III.1.6 and III.3.3 of Table 2, in which the one-dimensional subalgebras include the generator of projective transformation

$$X = t^2\frac{\partial}{\partial t} + (\alpha-1)tu\frac{\partial}{\partial t}.$$

Table 2 implies

Proposition 1. *The symmetry reduction of nonlinear fractional differential equaiton of anomalous diffusion (1) corresponding to optimal systems of one-dimensional subalgebras of Lie algebras of infinitesimal operators of the group of point transformations of this equation provided in Table 1 leads to the equation of one of the following forms:*

$$\varphi\varphi'' + \sigma(\varphi')^2 + \delta\varphi^2 + \varepsilon\frac{\Gamma(1-\alpha/\sigma)}{\Gamma(1-\alpha-\alpha/\sigma)}\varphi^{2-\sigma} = 0, \quad \sigma \notin [0,\alpha]; \tag{3}$$

$$D_\tau^\alpha\varphi = A\tau^2(\varphi^\sigma\varphi')' + B\tau\varphi^\sigma\varphi' + C\varphi^{\sigma+1} + \delta\varphi^\gamma; \tag{4}$$

$$\begin{aligned} D_\tau^\alpha\varphi = {} & \tau^4(\varphi^{\frac{2\alpha}{1-\alpha}}\varphi')' + (2+\alpha)\left[2\tau - \frac{\varepsilon}{\alpha}\right]\tau^2\varphi^{\frac{2\alpha}{1-\alpha}}\varphi' + \\ & + (1-\alpha)(2+\alpha)\left[\tau^2 - \frac{\varepsilon}{\alpha}\tau + \frac{\varepsilon^2}{\alpha^2(2+\alpha)} + \frac{\delta}{(1-\alpha)(2+\alpha)}\right]\varphi^{\frac{1+\alpha}{1-\alpha}}. \end{aligned} \tag{5}$$

Here $\alpha \in (0,1)\cup(1,2)$; $\varphi = \varphi(\tau)$; $\delta = 0,\ \pm1$; $\varepsilon = 0,\ 1$.

4. Solutions of reduced equations

As the result of the made symmetry reduction, the problem on constructing invariant solutions to equations (1) is reduced to the problem on finding solutions to ordinary differential equations (3)–(5).

Equation (3) is integrated by quadratures:

$$\int \frac{d\varphi}{\sqrt{\psi(\varphi, C_1)}} = \tau \pm C_2, \tag{6}$$

where

$$\psi(\varphi, C_1) = \begin{cases} C_1\varphi^{-2\sigma} + \dfrac{a}{2+\sigma}\varphi^{2-\sigma} - \dfrac{\delta}{1+\sigma}\varphi^2, & (\sigma \neq -1)\cup(\delta \neq 0),\ \sigma \neq -2; \\ C_1\varphi^2 + a\varphi^3 - 2\delta\varphi^2\ln(\varphi), & \sigma = -1,\ \delta \neq 0; \\ C_1\varphi^4 + a\varphi^4\ln(\varphi) + \delta\varphi^2, & \sigma = -2. \end{cases}$$

Here $a = 2\Gamma(1-\alpha/\sigma)/\Gamma(1-\alpha-\alpha/\sigma)$, C_1 and C_2 are integration constants.

The general solutions to nonlinear ordinary fractional differential equations (4) and (5) are unknown at present. We can construct either the general solution to linear ($\sigma = 0$) equation of form (4) as well as some invariant solutions to nonlinear equations (4) and (5).

In accordance with Table 2, as $\sigma = 0$, one can choose the following linear particular cases of equation (4):

$$D_\tau^\alpha \varphi = 0; \tag{7}$$

$$D_\tau^\alpha \varphi = \lambda\varphi + \delta, \quad \lambda \in \mathbb{R},\ \delta = \pm 1; \tag{8}$$

$$\alpha^2 D_\tau^\alpha \varphi = 4\tau^2\varphi'' + 2\left(2\alpha\beta + \alpha + 2\right)\tau\varphi' + \alpha^2\beta(\beta+1)\varphi, \quad \beta \in \mathbb{R}. \tag{9}$$

The general solutions to equations (7) and (8) are well-known [1] and they are of the form

$$\varphi(\tau) = \sum_{k=1}^{n} C_k \tau^{\alpha-k}, \quad \varphi(\tau) = \sum_{k=1}^{n} C_k \tau^{\alpha-k} E_{\alpha,\alpha+1-k}(\lambda\tau^\alpha) + \delta\tau^\alpha E_{\alpha,\alpha+1}(\lambda\tau^\alpha),$$

respectively, where C_k are arbitrary constants, $n = [\alpha] + 1$ and $E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n+\mu)}$ $(z \in \mathbb{C})$ is a function of Mittag-Leffler type.

The general solution to equation (9) can be represented in terms of the Wright function $\phi(\rho,\mu;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\rho n+\mu)}$, $(z \in \mathbb{C})$:

$$\varphi(\tau) = \tau^{-\frac{\alpha\beta}{2}} \left[C_1\phi\left(-\alpha/2, 1-\alpha\beta/2; \tau^{-\frac{\alpha}{2}}\right) + C_2\phi\left(-\alpha/2, 1-\alpha\beta/2; -\tau^{-\frac{\alpha}{2}}\right)\right],$$

where C_1 and C_2 are arbitrary constants. In the particular case $\beta = 0$, this solution was constructed first in work [11].

Invariant solutions to nonlinear equations (4) and (5) can be constructed by the methods of group analysis. In accordance with Table 2, we select two main particular forms of nonlinear ($\sigma \neq 0$) equation (4): $A = B = C = 0$, $\delta \neq 0$ and $A \neq 0$, $\delta = 0$. In the first case (4) becomes

$$D_\tau^\alpha \varphi = \delta\varphi^\gamma, \quad \delta \neq 0, \quad \gamma \neq 0,\ 1. \tag{10}$$

This equation is a particular case of the equation $D_\tau^{\alpha+1}\varphi = f(\tau, \varphi, D_\tau^\alpha\varphi)$, whose group classification in the class of linear autonomous symmetries of the form

$$X = \xi(\tau)\frac{\partial}{\partial\tau} + \left[\eta_0(\tau) + \eta_1(\tau)\varphi\right]\frac{\partial}{\partial\varphi}$$

was given in work [16]. The results of this classification implies

Proposition 2. *Nonlinear equation (10) for $\alpha \in (0,1) \cup (1,2)$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0,\ 1$ admits a one-parametric group of dilatations with the operator*

$$X_1 = (1-\gamma)\tau\frac{\partial}{\partial\tau} + \alpha\varphi\frac{\partial}{\partial\varphi}. \tag{11}$$

In the particular case $\gamma = (1+\alpha)/(1-\alpha)$ the group is extended by the projective point transformation with the operator

$$X_2 = \tau^2\frac{\partial}{\partial\tau} + (\alpha-1)\tau\varphi\frac{\partial}{\partial\varphi}. \tag{12}$$

The operator X_1 in (11) is associated the invariant solution to equation (10):

$$\varphi(\tau) = a\tau^\nu, \quad a = \left[\delta\,\frac{\Gamma(1+\nu)}{\Gamma(1+\gamma\nu)}\right]^{-\frac{\nu}{\alpha}}, \quad \nu = \frac{\alpha}{1-\gamma}. \tag{13}$$

This solution exists as $\gamma \notin [1, 1+\alpha]$, $(\nu > -1)$, under additional conditions

$$\gamma \neq \frac{1}{1-\alpha}, \quad \alpha \in (1,2); \qquad \gamma \neq \frac{1+n}{1+n-\alpha}, \quad n = [\alpha], \quad \alpha \in (1,2) \cup (1,2),$$

corresponding to non-vanishing of the coefficient a at the infinity and zero, respectively.

As $\gamma = (1+\alpha)/(1-\alpha)$, by means of the operator X_2 (12), via solution (13) we get an one-parametric family of the solutions:

$$\varphi(\tau) = a[\tau(1-b\tau)]^{\frac{\alpha-1}{2}}, \quad a = \left[\frac{\delta}{\pi}\cos\frac{\pi\alpha}{2}\Gamma^2\left(\frac{1+\alpha}{2}\right)\right]^{\frac{1-\alpha}{2\alpha}},$$

where b is an arbitrary constant. As $\alpha \in (0,1)$, this solution is the general solution to the corresponding equation.

Now we consider equation (4) as $\delta = 0$ and $A \neq 0$. Its group classification is made in work [22]. The following proposition holds.

Proposition 3. *As $\delta = 0$ and $A \neq 0$, nonlinear ($\sigma \neq 0$) equation (4) admits the one-parametric group of point dilatation transformations with the operator*

$$X_1 = -\sigma\tau\frac{\partial}{\partial\tau} + \alpha\varphi\frac{\partial}{\partial\varphi}. \tag{14}$$

In the particular case $\sigma = -2$, $B = 2\alpha A$, $C = \alpha(1-\alpha)A$, by the projective point transformation with the operator X_2 in (12) the group is extended to a two-parametric one.

Operator (14) generates an invariant solution to equation (4):

$$\varphi(\tau) = a\tau^{-\frac{\alpha}{\sigma}}, \quad a = \left[\frac{\sigma^2}{\alpha(\alpha+\sigma+\alpha\sigma)A - \alpha\sigma B + \sigma^2 C}\,\frac{\Gamma(1-\alpha/\sigma)}{\Gamma(1-\alpha-\alpha/\sigma)}\right]^{\frac{1}{\sigma}}. \tag{15}$$

This solution exists for $\sigma \notin [0,\alpha]$, $\sigma \neq \alpha/(1-\alpha)$ as $\alpha \in (0,1)\cup(1,2)$ and for $\sigma \neq \alpha/(2-\alpha)$ as $\alpha \in (1,2)$ as well as under obvious additional conditions:

$$\sigma \neq \frac{\alpha}{2C}\left[B - (1+\alpha)A \pm \sqrt{[(1+\alpha)A - B]^2 - 4AC}\right], \quad C \neq 0;$$

$$\sigma \neq \frac{\alpha A}{B - (1+\alpha)A}, \quad C = 0, \quad B \neq (1+\alpha)A.$$

In the particular case $\sigma = -2$, $B = 2\alpha A$, $C = \alpha(1-\alpha)A$, the operator X_2 in (12) corresponds to the invariant solution of equation (4) $\varphi(\tau) = a\tau^{\alpha-1}$ with an arbitrary constant a. Also as $A > 0$, by means of this operator, via solution (15) we construct a one-parametric (with an arbitrary constant b) family of solutions to equation (4):

$$\varphi(\tau) = a\tau^{\frac{\alpha}{2}}(1-b\tau)^{\frac{\alpha}{2}-1}, \quad a = \sqrt{\frac{(2-\alpha)\pi A}{2\sin(\pi\alpha/2)\Gamma^2(\alpha/2)}}.$$

Finally, we consider equation (5). We have

Proposition 4. *For each ε and δ, equation (5) admits one-parametric group of linear autonomous symmetries with the operator X_2 in (12). As $\varepsilon = 0$ and $\delta = 0$, the group is extended to a two-parametric one by the dilatation transformation with the operator*

$$X_1 = 2\alpha\tau\frac{\partial}{\partial\tau} + (\alpha-1)(\alpha+2)\varphi\frac{\partial}{\partial\varphi}.$$

As a result, for $\varepsilon = \delta = 0$ and each $\alpha \in (0,1)\cup(1,2)$, and for $\varepsilon = 1$, $\delta = -1$ and $\alpha = (\sqrt{5}-1)/2$, equation (5) has the invariant solution $\varphi(\tau) = a\tau^{\alpha-1}$ with an arbitrary constant a corresponding to the operator X_2. In the case $\varepsilon = 0$ and $\delta = 0$, by the operators X_1 and X_2, we construct one more one-parametric family of invariant solutions:

$$\varphi(\tau) = a\tau^{\nu}(1-b\tau)^{\alpha-1-\nu}, \quad a = \left[\frac{\Gamma(\nu+2-\alpha)}{\Gamma(\nu)}\right]^{\frac{\nu}{\alpha+2}}, \quad \nu = \frac{(\alpha-1)(\alpha+2)}{2\alpha},$$

where b is an arbitrary constant.

The issue on finding general solutions to nonlinear ordinary differential equations (4) and (5) remains open.

The constructed solutions to the reduced equations allow one to recover the corresponding exact solutions of the original equation of anomalous diffusion (1).

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