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DEGENERATE FRACTIONAL DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES WITH A σ-REGULAR PAIR OF OPERATORS

M. KOSTIĆ, V.E. FEDOROV

Abstract. We consider a degenerate fractional order differential equation $D_t^\alpha Lu(t) = Mu(t)$ in a Hausdorff sequentially complete locally convex space. Under the p-regularity of the operator pair (L, M), we find the phase space of the equation and the family of its resolving operators. We show that the identity image of the latter coincides with the phase space. We prove an unique solvability theorem and obtain the form of the solution to the Cauchy problem for the corresponding inhomogeneous equation. We give an example of application the obtained abstract results to studying the solvability of the initial boundary value problems for the partial differential equations involving entire functions on an unbounded operator in a Banach space, which is a specially constructed Frechét space. It allows us to consider, for instance, a periodic in a spatial variable x problem for the equation with a shift along x and with a fractional order derivative with respect to time t.

Keywords: fractional differential equation, degenerate evolution equation, locally convex space, σ-regular pair of operators, phase space, solution operator.

Mathematics Subject Classification: 34A08, 34G10, 47D99, 35R11

1. Introduction

We consider a linear differential equation

$$D_t^\alpha Lu(t) = Mu(t) + f(t), \tag{1}$$

where D_t^α is the fractional Caputo derivative of order $\alpha > 0$ [1], $\mathfrak{U}$ and $\mathfrak{V}$ are separated sequentially complete locally compact linear topological spaces, $L : \mathfrak{U} \to \mathfrak{V}$ is a linear continuous operator, $M : D_M \to \mathfrak{V}$ is a linear closed operator with a domain D_M dense in $\mathfrak{U}$. In what follows this equation is called degenerate since we assume that $\ker L \neq \{0\}$. In the work we consider the issues on unique solvability of the Cauchy problem

$$u^{(k)}(0) = u_k, k = 0, 1, \dots, m-1, \tag{2}$$

for equation (1). Here m is the smallest integer number greater than or equal to α.

Equations of such type in Banach spaces were considered in works [2, 3] in the non-degenerate case, when the operator L is continuously invertible and work [4] for a degenerate operator L and a strong (L, p)-sectorial operator M. There are also works [5]–[10], in which fractional differential equations were studied in locally convex spaces.

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The present work provides the generalizations of results of work [11], in which the solvability of Cauchy problem (2) for homogeneous equation (1) in a Banach space was studied by using the conditions of (L,σ)-boundedness of the operator [12] introduced in studying a degenerate equation of order $\alpha = 1$. In the present work we employ the notion of (L,σ)-regular operator used before in [13] for studying first order degenerate equation in locally convex spaces. By means of (L,σ)-regularity condition for the operator M, we obtain pairs of invariant subspaces of operators L and M and the original equation is reduced to a system of two equations on two subspaces. One of the obtained equations is resolved with respect the derivative and has a regular operator [14] at the sought function. The solvability condition of the Cauchy problem for such equation and its solution were found in work [14]. The other obtained equation has a nilpotent operator at the derivative. In the case of p-regular pair of operators (L,M) we show the unique solvability of such equation without initial data. In the homogeneous case the corresponding solution is trivial. This fact allows us to find the phase space and the family of resolving operator for the homogeneous equation. In the inhomogeneous case it leads to compatibility conditions for the initial data in the Cauchy problem and the right hand of the equation.

We also consider a generalized Showalter-Sidorov problem, in which the initial data is imposed only for the projection of the solution on the first subspace instead of the initial for the solution itself. This is why the difference between the solvability theorem for such problem and for the Cauchy problem is just the absence of the compatibility conditions that stress a natural character of such problem for degenerate evolution equations.

The family of the resolving operators found in the work is constructed explicitly by using Mittag-Leffler function and this family is used for the representation of the solution.

The obtain abstract results are used for studying the solvability of periodic in a spatial variable x Cauchy problem and Showalter-Sidorov problem for the equations of fractional order w.r.t. the time and of infinite order w.r.t. x with entire functions of the differentiation operator in x and with a shift in this variable. In order to do it, the shift in the spatial variable is represented as the action of an exponential function of the operator A of the differentiation in x. For the obtained operators we show the $(L,0)$-regularity of the operator M while considering the problem in the Fréchet space being the inductive limit of the scale of Fréchet space of the elements of A-exponential type in $D(A^\infty)$.

2. Inhomogeneous Cauchy problem for a non-degenerate equation

Let $\mathfrak{Z}$ be a separated sequentially complete locally convex space. By $\circledast_{\mathfrak{Z}}$ we denote a fundamental system of semi-norms in $\mathfrak{Z}$ defining a topology in this space.

Definition 1. *A linear continuous operator $A : \mathfrak{Z} \to \mathfrak{Z}$ is called regular (shortly $A \in \mathcal{R}(\mathfrak{Z})$) if there exists $C \in \mathbb{R}_+$ such that for each semi-norm $q \in \circledast_{\mathfrak{Z}}$ there exists a semi-norm $r \in \circledast_{\mathfrak{Z}}$ such that $q(A^n z) \leqslant C^n r(z)$ for all $z \in \mathfrak{Z}$, $n \in \mathbb{N}$.*

Remark 1. *The constant C in Definition 1 is called the regularity constant of the operator A. It is clear that the set of the regularity constants for a given operator is unbounded from above.*

Remark 2. *In the case of the Banach space $\mathfrak{Z}$, the regularity of the operator means exactly the belonging to class $\mathcal{L}(\mathfrak{Z})$ of linear continuous on the entire space operators.*

Remark 3. *In the case of a quasi-complete locally convex space, an operator A is regular in the sense of the above definition if and only if it is a regular element of the convex bornological algebra of continuous linear mappings from $\mathfrak{Z}$ into $\mathfrak{Z}$ with the bornology of equicontinuity* [14].

The regular spectrum $\sigma_r(A)$ of an operator A [14] is a set of $\mu \in \mathbb{C}$, for which there exists no regular operator $(\mu I - A)^{-1}$, and the regular resolvent set of operator A is the set $\rho_r(A) =$

$\mathbb{C}\setminus\sigma_r(A)$. The regular spectrum of a regular operator in a quasi-complete locally convex space is a non-empty compact set [14]. In this case of the spectral radius of a regular operator A we have $r_\sigma(A)=\inf\{C>0: q(A^n v)\leqslant C^n r(v)\}$.

We denote $g_\delta(t)=\Gamma(\delta)^{-1}t^{\delta-1}$ as $\delta>0$, $t>0$,

$$J_t^\delta h(t)=(g_\delta * h)(t)=\int\limits_0^t g_\delta(t-s)h(s)ds.$$

Let $\alpha>0$, m be the smallest integer greater than or equal to α, D_t^m is the usual derivative of order $m\in\mathbb{N}$, J_t^0 is the identical operator, D_t^α is the fractional Caputo derivative, that is, $D_t^\alpha f(t)=J_t^{m-\alpha}D_t^m f(t)$ in the case when the expression in the right hand side of this identity makes sense. In what follows we shall make use of the identity

$$D_t^\alpha f(t)=D_t^m J_t^{m-\alpha}\left(f(t)-\sum_{k=0}^{m-1} f^{(k)}(0)g_{k+1}(t)\right),$$

which is valid in the case when the expression in its right hand side is well-defined [1].

As α, $\beta>0$ we introduce the Mittag-Leffler function $E_{\alpha,\beta}(z)=\sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ and we consider the Cauchy problem

$$z^{(k)}(0)=z_k,\ k=0,1,\dots,m-1, \tag{3}$$

for the fractional differential equation

$$D_t^\alpha z(t)=Az(t)+f(t),\quad t\in[0,T), \tag{4}$$

where $T\in(0,+\infty]$, A is a regular operator in a sequentially complete locally convex space $\mathfrak{Z}$. A solution to problem (3), (4) is a function $z\in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{Z})$, for which $g_{m-\alpha}*\left(z-\sum\limits_{k=0}^{m-1} z^{(k)}(0)g_{k+1}\right)\in C^m(\mathbb{R}_+;\mathfrak{Z})$ and identities (3) and (4) are satisfied. Hereinafter $\mathbb{R}_+=\{x\in\mathbb{R}: x>0\}$, $\overline{\mathbb{R}}_+=\mathbb{R}_+\cup\{0\}$.

Theorem 1. *Let $A\in\mathcal{R}(\mathfrak{Z})$, $f\in C^m([0,T);\mathfrak{Z})$. Then for each $z_0\in\mathfrak{Z}$ there exists the unique solution to problem (3), (4). This solution is*

$$z(t)=\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)z_k+\int\limits_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds. \tag{5}$$

Proof. Differentiating series term by term, by the regularity of the operator A we obtain the identities

$$D_t^\alpha\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)z_k=A\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)z_k, \tag{6}$$

$$D_t^l\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)z_k|_{t=0}=z_l,\quad l=0,1,\dots,m-1, \tag{7}$$

$$D_t^\alpha \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds = J_t^{m-\alpha}D_t^m \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(As^\alpha)f(t-s)ds$$

$$= J_t^{m-\alpha}D_t^{m-1}\left(t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)\right)f(0) + J_t^{m-\alpha}D_t^{m-1}\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(As^\alpha)f'(t-s)ds$$

$$= J_t^{m-\alpha}\sum_{k=0}^{m-1} D_t^k\left(t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)\right)f^{(m-1-k)}(0) + J_t^{m-\alpha}\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(As^\alpha)f^{(m)}(t-s)ds$$

$$= J_t^{m-\alpha}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{A^n t^{\alpha(n+1)-1-k}}{\Gamma(\alpha n+\alpha-k)}f^{(m-1-k)}(0)$$

$$+ \int_0^t \frac{(t-\tau)^{m-1-\alpha}d\tau}{\Gamma(m-\alpha)}\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}(A(\tau-s)^\alpha)f^{(m)}(s)ds$$

$$= \sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{A^n t^{\alpha n+m-1-k}}{\Gamma(\alpha n+m-k)}f^{(m-1-k)}(0) \tag{8}$$

$$+ \int_0^t ds\int_s^t \frac{(t-\tau)^{m-1-\alpha}}{\Gamma(m-\alpha)}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}(A(\tau-s)^\alpha)f^{(m)}(s)d\tau$$

$$= \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)f^{(k)}(0) + \int_0^t ds\int_0^{t-s}\frac{(t-s-\sigma)^{m-1-\alpha}}{\Gamma(m-\alpha)}\sigma^{\alpha-1}E_{\alpha,\alpha}(A\sigma^\alpha)f^{(m)}(s)d\sigma$$

$$= \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)f^{(k)}(0) + \int_0^t \left[J_{t-s}^{m-\alpha}(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\right]f^{(m)}(s)ds$$

$$= \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)f^{(k)}(0) + \int_0^t (t-s)^{m-1}E_{\alpha,m}(A(t-s)^\alpha)f^{(m)}(s)ds$$

$$= f(t) - \int_0^t \left[\frac{d}{ds}E_{\alpha,1}(A(t-s)^\alpha)\right]f(s)ds = A\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds + f(t).$$

Here in the end we made m-multiple integration by parts. It follows from identities (6)-(8) that function (5) solves problem (3), (4).

If there exist solutions z_1 and z_2 to problem (3), (4), then their difference $z = z_1 - z_2$ solves Cauchy problem (3) with the initial data $z_k = 0$, $k = 0, 1, \dots, m-1$, for the homogeneous equation $D_t^\alpha z(t) = Az(t)$. We apply the operator J_t^α to the both sides of this identity to obtain $z(t) = \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}Az(s)ds$ (see [1, Formula (1.21)]). Then by [14, Thm. 4], the unique solution to this problem for a regular operator A in a sequentially complete locally convex space is $z \equiv 0$. □

3. σ-regular pair of operators

Let $\mathfrak{U}$, $\mathfrak{V}$ be sequentially complete locally convex spaces. By $\mathcal{L}(\mathfrak{U};\mathfrak{V})$ we denote the set of linear continuous operators acting from $\mathfrak{U}$ into $\mathfrak{V}$. The set of linear closed operators with domains dense in the space $\mathfrak{U}$ acting into $\mathfrak{V}$ is denoted by $\mathcal{C}l(\mathfrak{U};\mathfrak{V})$. If $\mathfrak{V}=\mathfrak{U}$, the corresponding notations are $\mathcal{L}(\mathfrak{U})$ and $\mathcal{C}l(\mathfrak{U})$, respectively.

Let $L, M \in \mathcal{C}l(\mathfrak{U};\mathfrak{V})$. By D_L, D_M we denote the domains of the operators L and M. We denote $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$.

A regular L-resolvent set of the operator M is the set $\rho^L_r(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{V};\mathfrak{U}),\ R^L_\mu(M) \in \mathfrak{R}(\mathfrak{U}),\ L^L_\mu(M) \in \mathfrak{R}(\mathfrak{V})\}$, a regular L-spectrum is its complement $\sigma^L_r(M) = \mathbb{C} \setminus \rho^L_r(M)$.

Below we shall need the following identities valid for all $\mu, \lambda \in \rho^L_r(M)$, $u \in D_L \cap D_M$:

$$\begin{aligned}
&(\mu L - M)^{-1}(\lambda L - M)u = u + (\lambda - \mu)(\mu L - M)^{-1}Lu,\\
&(\lambda L - M)(\mu L - M)^{-1} = I + (\lambda - \mu)L(\mu L - M)^{-1},\\
&(\mu L - M)^{-1} - (\lambda L - M)^{-1} = (\lambda - \mu)R^L_\mu(M)(\lambda L - M)^{-1},\\
&L(\mu L - M)^{-1}Mu = M(\mu L - M)^{-1}Lu.
\end{aligned} \tag{9}$$

Proposition 1. *Let $L, M \in \mathcal{C}l(\mathfrak{U};\mathfrak{V})$, the set $D_L \cap D_M$ is sequentially dense in $\mathfrak{U}$. Then*
(i) *$\rho^L_r(M)$ is an open set;*
(ii) *The operator function $(\mu L - M)^{-1}$, $R^L_\mu(M)$, $L^L_\mu(M)$ are strongly holomorphic on $\rho^L_r(M)$;*
(iii) *We can choose regularity constants for the operators $R^L_\mu(M)$ and $L^L_\mu(M)$ depending continuously on $\mu \in \rho^L_r(M)$.*

Proof. Let $\mu \in \rho^L_r(M)$, C_μ be the maximal of two regularity constants of the operators $R^L_\mu(M)$ and $L^L_\mu(M)$. Then

$$\{\lambda \in \mathbb{C} : |\lambda - \mu| < C_\mu^{-1}\} \subset \rho^L_r(M).$$

Indeed, in accordance with the second identity in (9), the continuous operator is well-defined:

$$[(\lambda L - M)(\mu L - M)^{-1}]^{-1} = \sum_{k=0}^{\infty}(\mu - \lambda)^k[L(\mu L - M)^{-1}]^k.$$

We multiply this identity by $L^L_\mu(M)$ from the left and for each $v \in \mathfrak{V}$, $q \in \circledast_{\mathfrak{V}}$ we obtain:

$$\begin{aligned}
q(L^L_\lambda(M)v) =& q\left(\sum_{k=0}^{\infty}(\lambda - \mu)^k (L^L_\mu(M))^{k+1}v\right) \leqslant \frac{C_\mu r(v)}{1 - |\lambda - \mu| C_\mu},\\
q((L^L_\lambda(M))^n v) \leqslant& \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}|\lambda - \mu|^{\sum_{i=1}^n k_i} q\left((L^L_\mu(M))^{n+\sum_{i=1}^n k_i} v\right)\\
\leqslant& \sum_{k_1=0}^{\infty}|\lambda - \mu|^{k_1}C_\mu^{k_1+1}\cdots\sum_{k_n=0}^{\infty}|\lambda - \mu|^{k_n}C_\mu^{k_n+1}r(v) \leqslant \frac{C_\mu^n r(v)}{(1 - |\lambda - \mu|C_\mu)^n}
\end{aligned}$$

for some $r \in \circledast_{\mathfrak{V}}$. Thus, operator $R^L_\lambda(M)$ is regular with the regularity constant $C_\mu(1 - |\lambda - \mu|C_\mu)^{-1}$.

The proof for the operator functions $(\mu L - M)^{-1}$, $R^L_\mu(M)$ is similar by employing the first identity in (9). This is why each point μ lies in $\rho^L_r(M)$ with some neighbourhood. Statements (ii) and (iii) are obviously implied by the proved facts. □

Definition 2. *Let $L \in \mathcal{L}(\mathfrak{U};\mathfrak{V})$, $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{V})$. The operator M is called (L,σ)-regular if*

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L_r(M)).$$

Sometimes, instead of (L,σ)-regular operator M, it is more convenient to speak about σ-regular pair of operators (L, M), which in our presentation is the same.

We take a (L,σ)-regular operator M and construct a closed contour

$$\gamma = \{\mu \in \mathbb{C} : |\mu| = R > a\}.$$

Then the integrals

$$Pu = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)ud\mu, \quad Qv = \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)vd\mu$$

are well-defined as the integrals of holomorphic functions over the closed contour.

It is easy to show that the operators P and Q are projectors. We let $\mathfrak{U}^0 = \ker P$, $\mathfrak{V}^0 = \ker Q$, $\mathfrak{U}^1 = \mathrm{im}P$, $\mathfrak{V}^1 = \mathrm{im}Q$, then $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{V} = \mathfrak{V}^0 \oplus \mathfrak{V}^1$. By L_k (M_k) we denote the restriction of the operator L (M) on $\mathfrak{U}^k$ $(D_{M_k} = D_M \cap \mathfrak{U}^k)$, $k = 0,1$. Moreover, by $\sigma^L_{r,k}(M)$ we denote the regular L_k-spectrum of the operator M_k, while $\rho^L_{r,k}(M)$ stands for its regular L_k-resolvent set.

Lemma 1. *Let the operator M be (L,σ)-regular. Then for each $u \in \mathfrak{U}$ $Pu \in D_M$.*

Proof. Indeed, by Statement (ii) in Proposition 1, the integral

$$\int\limits_{\gamma} MR^L_\mu(M)ud\mu = \int\limits_{\gamma} \mu LR^L_\mu(M)ud\mu$$

converges. Since the operator M is closed, we obtain the needed fact. □

Theorem 2. *Let the operator M be (L,σ)-regular. Then*
(i) $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{V}^k)$, $k = 0,1$;
(ii) $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{V}^0)$, $M_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{V}^1)$;
(iii) *There exists the operator* $L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1;\mathfrak{U}^1)$;
(iv) $\rho^L_{r,0}(M) = \mathbb{C}$ *and in particular, there exists the operator* $M_0^{-1} \in \mathcal{L}(\mathfrak{V}^0;\mathfrak{U}^0)$;
(v) *The operators* $S_1 = L_1^{-1}M_1$, $T_1 = M_1L_1^{-1}$ *are regular.*

The proof of the theorem can be found in [13].

We denote $H = M_0^{-1}L_0$. If for some $p \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ we have $H^p \neq \mathbb{O}$, $H^{p+1} = \mathbb{O}$, then the (L,σ)-regular operator M is called (L,p)-regular. At that, the pair of the operators (L,M) is called (L,p)-regular.

It is easy to show that the nilpotentness degree of the operator H or the absence of the nilpotentness determine the character of the singularity at the infinity for the operator function $(\mu L - M)^{-1}$ (see [12]).

An ordered set $\{\varphi_0, \varphi_1, \varphi_2, \ldots\} \subset \mathfrak{U}$ is called a chain of M-adjont vectors of the operator L if $\varphi_0 \in \ker L \setminus \{0\}$,

$$L\varphi_{k+1} = M\varphi_k, \quad k = 0,1,\ldots, \quad \varphi_l \notin \ker L, \quad l = 1,2,\ldots$$

The chain is finite if there exists an M-adjoint vector φ_p such that either $\varphi_p \notin D_M$ or $M\varphi_p \notin \mathrm{im}L$. The index of a M-adjoint vector in the chain is called the height of this vector. The linear span of all M-adjoint vectors of the operator L is called M-root lineal of the operator L.

Arguing as in the monograph [12], we can obtain the following result.

Theorem 3. *Let the operator M be (L,σ)-regular. Then a M-root lineal of the operator L is contained in $\mathfrak{U}^0$. At that, the following statements hold.*

(i) *The operator M is (L,p)-regular for $p \in \mathbb{N}$ if and only if M-root lineal consists of M-adjoint vectors of the operator L of height at most p and at that there exist a M-adjoint vector of the height p. In this case, the M-root space of the operator L coincides with $\mathfrak{U}^0$.*

(ii) *The operator M is $(L,0)$-regular if and only if* $\ker L = \mathfrak{U}^0$. *At that,* $\mathrm{im}L = \mathfrak{V}^1$ *and for each* $\varphi_0 \in \ker L \setminus \{0\}$ *we have* $\varphi_0 \notin D_M$ *or* $M\varphi_0 \notin \mathrm{im}L$.

Lemma 2. *Let the operator M be (L,σ)-regular $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > a\}$, $\alpha, \beta > 0$,*

$$U_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{\gamma} R_{\mu}^{L}(M)E_{\alpha,\beta}(\mu t^{\alpha})d\mu, \quad t \geqslant 0.$$

$$V_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{\gamma} L_{\mu}^{L}(M)E_{\alpha,\beta}(\mu t^{\alpha})d\mu, \ t \geqslant 0.$$

Then for each $t \geqslant 0$ we have $U_{\alpha,\beta}(t)P = PU_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)$, $V_{\alpha,\beta}(t)Q = QV_{\alpha,\beta}(t) = V_{\alpha,\beta}(t)$.

Proof. Let $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > a\}$, $\gamma_1 = \{\lambda \in \mathbb{C} : |\lambda| = R+1\}$, then

$$U_{\alpha,\beta}(t)P = PU_{\alpha,\beta}(t) = \frac{1}{(2\pi i)^2}\int_{\gamma}\int_{\gamma_1} R_{\lambda}^{L}(M)R_{\mu}^{L}(M)E_{\alpha,\beta}(\mu t^{\alpha})d\mu d\lambda$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma} R_{\mu}^{L}(M)E_{\alpha,\beta}(\mu t^{\alpha})d\mu \int_{\gamma_1}\frac{d\lambda}{\lambda-\mu} - \frac{1}{(2\pi i)^2}\int_{\gamma_1} R_{\lambda}^{L}(M)d\lambda\int_{\gamma}\frac{E_{\alpha,\beta}(\mu t^{\alpha})d\mu}{\lambda-\mu} = U_{\alpha,\beta}(t).$$

The statements of the lemma on the operators $V_{\alpha,\beta}(t)$ and Q can be proved in the same way. □

The above lemma implies the obvious corollary.

Corollary 1. *Let the operator M be (L,σ)-regular, $\alpha, \beta > 0$. Then for each $t \geqslant 0$ we have $\mathfrak{U}^0 \subset \ker U_{\alpha,\beta}(t)$, $\operatorname{im} U_{\alpha,\beta}(t) \subset \mathfrak{U}^1$, $\mathfrak{V}^0 \subset \ker V_{\alpha,\beta}(t)$, $\operatorname{im} V_{\alpha,\beta}(t) \subset \mathfrak{V}^1$.*

Lemma 3. *Let the operator M be (L,σ)-regular, $\alpha, \beta > 0$. Then for each $t \geqslant 0$ we have $U_{\alpha,\beta}(t) = E_{\alpha,\beta}(L_1^{-1}M_1 t^{\alpha})P$, $V_{\alpha,\beta}(t) = E_{\alpha,\beta}(M_1 L_1^{-1} t^{\alpha})Q$.*

Proof. By Theorem 2, $S \equiv L_1^{-1}M_1 \in \mathcal{R}(\mathfrak{U}^1)$. By Lemma 2, for $t \geqslant 0$,

$$U_{\alpha,\beta}(t) = U_{\alpha,\beta}(t)P = \frac{1}{2\pi i}\int_{\gamma} R_{\mu}^{L_1}(M_1)PE_{\alpha,\beta}(\mu t^{\alpha})d\mu$$

$$= \frac{1}{2\pi i}\int_{\gamma}(\mu I - S)^{-1}PE_{\alpha,\beta}(\mu t^{\alpha})d\mu = \frac{1}{2\pi i}\int_{\gamma}\sum_{k=0}^{\infty}\mu^{-k-1}S^kP\sum_{n=0}^{\infty}\frac{\mu^n t^{\alpha n}}{\Gamma(\alpha n+\beta)}d\mu$$

$$= \sum_{k=0}^{\infty}\frac{S^k t^{\alpha k}}{\Gamma(\alpha k+\beta)}P = E_{\alpha,\beta}(St^{\alpha})P.$$

The second identity can be proved in the same way. □

Remark 4. *In the same way we can show that if the operator L is continuously invertible, then $U_{\alpha,\beta}(t) = E_{\alpha,\beta}(L^{-1}Mt^{\alpha})$ for regular operator $L^{-1}M$ and $V_{\alpha,\beta}(t) = E_{\alpha,\beta}(ML^{-1}t^{\alpha})$ if the operator ML^{-1} is regular.*

4. Homogeneous degenerate equation

A solution to the equation

$$D_t^{\alpha}Lu(t) = Mu(t), \quad t \in \overline{\mathbb{R}}_+, \tag{10}$$

is a function $u \in C(\overline{\mathbb{R}}_+; D_M)$ obeying

$$Lu \in C^{m-1}(\overline{\mathbb{R}}_+; \mathfrak{V}), \quad g_{m-\alpha} * \left(Lu - \sum_{k=0}^{m-1}(Lu)^{(k)}(0)g_{k+1}\right) \in C^m(\overline{\mathbb{R}}_+; \mathfrak{V}),$$

at that, for all $t \in \overline{\mathbb{R}}_+$ identity (10) holds true.

We shall consider equation (10) together with the equivalent equation on the space $\mathfrak{V}$

$$D_t^\alpha L_\beta^L(M)v(t) = M(\beta L - M)^{-1}v(t), \quad t \in \mathbb{R}_+, \tag{11}$$

where $(\beta L - M)^{-1} \in \mathcal{L}(\mathfrak{V};\mathfrak{U})$. The relation between the solutions to equations (10) and (11) is given by the identity $u(t) = (\beta L - M)^{-1}v(t)$.

Lemma 4. *Let the operator M be (L,σ)-regular. Then for each $u_0 \in \mathfrak{U}$ ($v_0 \in \mathfrak{V}$), the vector function $u(t) = U_{\alpha,1}(t)u_0$ ($v(t) = V_{\alpha,1}(t)v_0$) solves equations (10) (11).*

Proof. Let $u_0 \in \mathfrak{U}$. Then

$$MU(t)u_0 = \frac{1}{2\pi i}\int_\gamma \mu L R_\mu^L(M)u_0 E_\alpha(\mu t^\alpha)d\mu - \frac{1}{2\pi i}\int_\gamma L u_0 E_\alpha(\mu t^\alpha)d\mu.$$

The obtained function is continuous in t. It is also obvious that $LU_{\alpha,1}(\cdot)u_0 \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{V})$. In view of the strong holomorphy in μ of the integrands, as $t \geqslant 0$, we obtain the identities

$$\begin{aligned}
D_t^\alpha LU(t)u_0 =& \frac{1}{2\pi i}\int_\gamma LR_\mu^L(M)u_0 J_t^{m-\alpha}\sum_{n=1}^\infty \frac{\mu^n t^{\alpha n - m}}{\Gamma(\alpha n - m + 1)}d\mu \\
=& \frac{1}{2\pi i}\int_\gamma LR_\mu^L(M)u_0 \sum_{n=1}^\infty \frac{\mu^n t^{\alpha(n-1)}B(m-\alpha, \alpha n - m + 1)}{\Gamma(m-\alpha)\Gamma(\alpha n - m + 1)}d\mu \\
=& \frac{1}{2\pi i}\int_\gamma LR_\mu^L(M)u_0 \sum_{n=1}^\infty \frac{\mu^n t^{\alpha(n-1)}}{\Gamma(\alpha(n-1)+1)}d\mu = \frac{1}{2\pi i}\int_\gamma \mu L R_\mu^L(M)u_0 E(\mu t^\alpha)d\mu \\
=& \frac{1}{2\pi i}\int_\gamma Lu_0 E(\mu t^\alpha)d\mu + \frac{1}{2\pi i}\int_\gamma MR_\mu^L(M)u_0 E(\mu t^\alpha)d\mu = 0 + MU(t)u_0
\end{aligned}$$

by the Cauchy theorem and the closedness of the operator M. □

A solution to the Cauchy problem

$$u(0) = u_0, \; u^{(k)}(0) = 0, \; k = 1, 2, \ldots, m-1, \tag{12}$$

for equation (10) is a solution $u \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{U})$ to this equation satisfying conditions (12).

The feature of equations like (10) with a degenerate operator at the derivative is that their solutions cover pointwise just some subspace of the original space, in which the equation is posed. Of course, this subspace plays an important role in studying the equation. In order to describe it formally, we follow [12] and introduce a definition.

The set $\mathfrak{P} \subset \mathfrak{U}$ is called the *phase space* of equation (10) if

(i) for each solution $u = u(t)$ to equation (10) we have $u(t) \in \mathfrak{P}$ for all $t \geqslant 0$;

(ii) for each $u_0 \in \mathfrak{P}$ there exists the unique solution to problem (10), (12).

It is obvious that if the phase space of the equation exists, it is unique.

Remark 5. *If the operator $L^{-1} \in \mathcal{L}(\mathfrak{V};\mathfrak{U})$ exists and at that $L^{-1}M$ is regular, then the phase space of equation (9) is the entire space $\mathfrak{U}$. Indeed, by* [14, Thm. 4]*, for each $u_0 \in \mathfrak{U}$ there exists the unique solution $u(t) = E_{\alpha,1}(L^{-1}Mt^\alpha)u_0$ to problem (10), (12).*

Lemma 5. *Let an operator $G \in \mathcal{L}(\mathfrak{Z})$ be nilpotent of degree $p \in \mathbb{N}_0$, there exist fractional derivatives $(D_t^\alpha G)^k g \in C([0,T);\mathfrak{Z})$ as $k = 0, 1, \ldots, p$, $T \in (0, +\infty]$. Then there exists the unique solution to the equation*

$$D_t^\alpha G z(t) = z(t) + g(t), \quad t \in [0, T), \tag{13}$$

and it is of the form

$$z(t) = -\sum_{k=0}^{p}(D_t^{\alpha}G)^k g(t), \quad t \in [0, T). \tag{14}$$

Proof. Let $z = z(t)$ be a solution to equation (13). We apply the operator G to the both sides of (13) to obtain the identity $GD_t^{\alpha}Gz(t) = Gz(t) + Gg(t)$. Then there exists the fractional derivative of order α of its right hand side, and hence, of its left hand side. Applying the operator D_t^{α} to the both sides of this identity, we obtain

$$(D_t^{\alpha}G)^2 z = D_t^{\alpha}Gz + D_t^{\alpha}Gg = z + g + D_t^{\alpha}Gg.$$

Repeating this procedure, at the p-th step we obtain the identity

$$(D_t^{\alpha}G)^{p+1} z = z + \sum_{k=0}^{p}(D_t^{\alpha}G)^k g.$$

By the continuity and the nilpotentness of the operator G we have

$$(D_t^{\alpha}G)^{p+1} z = (D_t^{\alpha})^{p+1}G^{p+1} z \equiv 0,$$

which implies identity (14). It yields the existence of solution to equation (13) that can be checked by substituting this function into the equation and it also yields the uniqueness. □

Theorem 4. *Let the operator M is (L, p)-regular. Then the phase space of equation (10) (of equation (11)) coincides with the space $\mathfrak{U}^1$ ($\mathfrak{V}^1$).*

Proof. Let $u = u(t)$ be a solution to equation (10). We let $u^0(t) \equiv (I - P)u(t)$, $u^1(t) \equiv Pu(t)$. Then by Theorem 2 we have

$$D_t^{\alpha}Hu^0(t) = u^0(t), \quad H \equiv M_0^{-1}L_0\,,$$

$$D_t^{\alpha}u^1(t) = Su^1(t), \quad S \equiv L_1^{-1}M_1\,. \tag{15}$$

According to Lemma 5, $u^0 \equiv 0$ and $u(t) = u^1(t) \in \mathfrak{U}^1$ for each $t \geqslant 0$.

By Theorem 2, the operator S is regular in the space $\mathfrak{U}^1$. Then for each $Pu_0 = u_0^1 \in \mathfrak{U}^1$ there exists the unique solution to the Cauchy problem $u^1(0) = u_0^1$, $u^{1(k)}(0) = 0$, $k = 1, 2, \ldots, m-1$, for equation (15) (see Remark 5). Hence, the solution exists also for equation (10) and it is of the form $u(t) = E_{\alpha,1}(St^{\alpha})u_0^1 = U_{\alpha,1}(t)u_0$.

The statement of the theorem on the phase space of equation (11) can be proved in the same way. □

Remark 6. *It follows from the proof of Theorem 4 that the Cauchy problem*

$$u^{(k)}(0) = u_k, \quad k = 0, 1, 2, \ldots, m-1, \tag{16}$$

for equation (10) under the condition of (L, p)-regularity for the operator M is reduced to the Cauchy problem for equation (15). It implies the equivalence of Cauchy problem (12) and general Cauchy problem (16) for equation (10) and the solvability of the latter problem for each given $u_k \in \mathfrak{U}^1$, $k = 0, 1, 2, \ldots, m-1$. According to [1] *and Remark 5, the solution to both problem (10), (16) and problem (10), (12) is of the form:*

$$u(t) = \sum_{k=0}^{m-1} J_t^k U_{\alpha,1}(t)u_k = \sum_{k=0}^{m-1} J_t^k E_{\alpha,1}(St^{\alpha})u_k = \sum_{k=0}^{m-1} t^k U_{\alpha,k+1}(t)u_k.$$

The family of operators $\{W(t) \in \mathcal{L}(\mathfrak{U}) : t \in \overline{\mathbb{R}}_+\}$ is called the family of resolving operators for equation (10) if there exists the phase space $\mathfrak{P}$ of this equation and for each $u_0 \in \mathfrak{P}$ the unique solution to problem (10), (12) is of the form $u(t) = W(t)u_0$ as $t \geqslant 0$.

It follows from the definition that for each $t \geqslant 0$, the family of the resolving operators satisies $\mathrm{im}W(t) \subset \mathfrak{P}$, $\mathrm{im}W(0) = \mathfrak{P}$

Theorem 5. *Let the operator M is (L,p)-regular. Then the family of the operators $\{U_{\alpha,1}(t) \in \mathcal{L}(\mathfrak{U}) : t \in \overline{\mathbb{R}}_+\}$ ($\{V_{\alpha,1}(t) \in \mathcal{L}(\mathfrak{U}) : t \in \overline{\mathbb{R}}_+\}$) is the family of resolving operators for equation (10), (12).*

Proof. If $u_0 \in \mathfrak{U}^1$, by the definition of the phase space there exists the unique solution to problem (12) for equation (10). Therefore it should coincide with the known solution $u(t) = U_{\alpha,1}(t)u_0$ to this problem. □

5. Cauchy problem and Showalter-Sidorov problem for inhomogeneous degenerate equation

Theorem 6. *Let the operator M be (L,p)-regular, $Qf \in C^m([0,T);\mathfrak{V})$, $T \in (0,+\infty]$, there exist fractional derivatives $(D_t^\alpha H)^n M_0^{-1}(I-Q)f \in C([0,T);\mathfrak{U})$ as $n = 0,1,\dots,p$, $u_k \in \mathfrak{U}$, $k = 0,1,\dots,m-1$, the identities*

$$-D_t^k \sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1}(I-Q)f(t)|_{t=0} = (I-P)u_k, \quad k = 0,1,\dots,m-1, \tag{17}$$

hold true. Then there exists the unique solution to problem (16) for the equation

$$D_t^\alpha Lu(t) = Mu(t) + f(t), \quad t \in [0,T), \tag{18}$$

and it is of the form as $t \in [0,T)$

$$u(t) = \sum_{k=0}^{m-1} t^k U_{\alpha,k+1}(t)u_k + \int_0^t (t-s)^{\alpha-1} U_{\alpha,\alpha}(t-s) L_1^{-1} Qf(s)ds - \sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1}(I-Q)f(t). \tag{19}$$

Proof. Arguing as in the proof of Theorem 4, we obtain

$$D_t^\alpha H u^0(t) = u^0(t) + M_0^{-1}(I-Q)f(t), \ H \equiv M_0^{-1}L_0, \tag{20}$$

$$D_t^\alpha u^1(t) = Su^1(t) + h(t), \ S \equiv L_1^{-1}M_1, \ h(t) = L_1^{-1}Qf(t). \tag{21}$$

By Lemma 5, there exists the unique solution to equation (20), it is of the form

$$u^0 = -\sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1}(I-Q)f.$$

It follows that to satisfy Cauchy conditions (16), we need to satisfy compatibility conditions (17).

By Theorem 2, we have $S \in \mathcal{R}(\mathfrak{U}^1)$. This is why by Theorem 1, there exists the unique solution to Cauchy problem $u^{1(k)}(0) = Pu_k$, $k = 0,1,\dots,m-1$ for equation (21) and it is of the form

$$u^1(t) = \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(S^\alpha t^\alpha)Pu_k + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(S^\alpha (t-s)^\alpha) h(s)ds.$$

By using Lemma 3 we complete the proof. □

Since in the definition of the solution the function Lu belongs to the class $C^{m-1}([0,T);\mathfrak{V})$ but not u, it could seem logical to consider the initial conditions $(Lu)^{(k)}(0) = v_k$, $k = 0,1,\dots m-1$, instead of conditions (16). However, as we see by Theorem 6, in this case we need the matching conditions of the form

$$\lim_{t\to 0+} D_t^k L \sum_{n=0}^{p} (D_t^\alpha H)^n M_0^{-1}(I-Q)f(t) = -L(I-P)u_k, \quad k = 0,1,\dots,m-1.$$

It seems to be more natural to consider the generalized Showalter-Sidorov problem

$$(Pu)^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, m-1, \tag{22}$$

quite often arising in applications.

A solution of problem (22) for equation (18) is the solution to this equation satisfying conditions (22). We observe that the existence of the derivatives $(Lu)^{(k)}(0)$ implies the existence of the derivatives $(Pu)^{(k)}(0)$. Indeed, $L_1^{-1}Q(Lu)^{(k)} = (L_1^{-1}QLu)^{(k)} = (L_1^{-1}LPu)^{(k)} = (Pu)^{(k)}$. In the case $p = 0$, by Theorem 3, the identity $\ker P = \ker L$ holds true and this is conditions (22) are equivalent to conditions for $(Lu)^{(k)}(0)$.

Theorem 7. *Let the operator M be (L,p)-regular, $Qf \in C^m([0,T);\mathfrak{V})$, there exist fractional derivatives $(D_t^\alpha H)^n M_0^{-1}(I-Q)f \in C([0,T);\mathfrak{U})$ as $n = 0, 1, \ldots, p$. Then there exists the unique solution to problem (18), (22) and it given by (19).*

The proof is similar the previous one. At that, the feature of the initial Showalter-Sidorov condition is such that it does not imply the initial conditions for the projection u^0 of the solution to equation (18) and to its derivatives. This is why there is no need in satisfying compatibility conditions (17).

6. Example

Let $\mathfrak{X}$ be a Banach space, $A \in \mathcal{C}l(\mathfrak{X})$. We equip the lineal $D(A^\infty) = \bigcap_{k=1}^\infty D(A^k)$ by the system of semi-norms $q_k(u) = \sum_{l=0}^k \|A^l u\|_{\mathfrak{X}}$, $k \in \mathbb{N}_0$, and we obtain the Frechét space, which we denote by $\mathfrak{D}_A$. For some $\tau > 0$ we denote by $\mathring{\mathfrak{E}}_{\mathfrak{A}}(\tau) = \{u \in D(A^\infty) : \overline{\lim}_{k\to\infty} \|A^k u\|_{\mathfrak{X}}^{1/k} \leqslant \tau\}$ the set of the elements of A-exponential type not exceeding τ. The maximal closed in the topology $\mathfrak{D}_A$ subspace of the space $\mathring{\mathfrak{E}}_{\mathfrak{A}}(\tau)$ is denoted by $\mathfrak{E}_\tau$. This set with the topology induced by the semi-norms q_k, $k \in \mathbb{N}_0$, is also a Frechét space. We denote $A_\tau = A|_{\mathfrak{E}_\tau}$. It was shown in [14, Ch. 7, Sect. 3] that $\sigma(A_\tau) \subset \sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda| \leqslant \tau\}$, at that, the operator A_τ is regular in $\mathfrak{E}_\tau$.

Let $L = G(A) = \sum\limits_{k=0}^\infty a_k A^k$, $M = J(A) = \sum\limits_{k=0}^\infty b_k A^k$, where $G(\lambda)$, $J(\lambda)$ are entire functions. By the said above, $L, M \in \mathcal{L}(\mathfrak{E}_\tau)$.

We consider the inductive scale of locally convex linear topological spaces $\{\mathfrak{E}_\tau : \tau \in \mathbb{N}\}$ and its inductive limit $\mathfrak{E}_\infty$. It is interior and even proper [15, Ch. 1, Append.]. The space $\mathfrak{E}_\infty = \bigcup_{\tau\in\mathbb{N}} \mathfrak{E}_\tau \subset \mathfrak{D}_A$ equipped by the semi-norms $q_{k,\tau}(u) = \sum_{l=0}^k \|A_\tau^l u\|_{\mathfrak{X}}$, $k \in \mathbb{N}_0$, $\tau \in \mathbb{N}$, is a separable locally convex space. By [15, Ch. 1, Append., Prop. 4.1], $G(A), J(A) \in \mathcal{L}(\mathfrak{E}_\infty)$.

We denote $\mathcal{G}_0 = \{\lambda \in \mathbb{R} : G(\lambda) = 0\}$.

Theorem 8. *Let $\mathfrak{X}$ be a Hilbert space, A be a self-adjoint operator in $\mathfrak{X}$, $\mathfrak{U} = \mathfrak{V} = \mathfrak{E}_\infty$, entire functions G, J have no common zeroes in the set $\sigma(A)$,*

$$\exists a > 0 \quad \forall \lambda \in \sigma(A) \setminus \mathcal{G}_0 \quad |J(\lambda)/G(\lambda)| \leqslant a.$$

Then the operator $J(A)$ is $(G(A),0)$-regular.

Proof. We denote by $\{\mathcal{E}_\lambda : \lambda \in \mathbb{R}\}$ the spectral family of the self-adjoint operator A. By the assumption of the theorem, in the case $|\mu| > a+1$ the inequality $|\mu - J(\lambda)/G(\lambda)| \geqslant |\mu| - a > 1$ holds true and therefore,

$$\int\limits_{\sigma(A)\setminus\mathcal{G}_0} \frac{d\mathcal{E}_\lambda u}{\mu - J(\lambda)/G(\lambda)} = R_\mu^L(M)u = L_\mu^L(M)u,$$

$$q_{\beta,\tau}([R_\mu^L(M)]^k u) = \sum_{l=0}^\beta \left\| \int\limits_{\sigma(A_\tau)\setminus\mathcal{G}_0} \frac{\lambda^l d\mathcal{E}_\lambda u}{(\mu - J(\lambda)/G(\lambda))^k} \right\|_{\mathfrak{X}} \leqslant \sum_{l=0}^\beta \left\| \int\limits_{\sigma(A_\tau)\setminus\mathcal{G}_0} \lambda^l d\mathcal{E}_\lambda u \right\|_{\mathfrak{X}} = q_{\beta,\tau}(u).$$

Moreover, $\mathfrak{U}^0 = \mathfrak{V}^0 = \operatorname{im} \int\limits_{\mathcal{G}_0} d\mathcal{E}_\lambda$, $L_0 = \mathbb{O}$, and this is why the operator M is $(L, 0)$-regular. $\square$

Let $\mathfrak{X} = L_2(0, 1)$,

$$A = i\frac{d}{dx}, \quad D(A) = \{v \in L_2(0, 1) : v' \in L_2(0, 1),\, v(0) = v(1)\}. \tag{23}$$

Then $\sigma(A) = \{\lambda_k = 2\pi k : k \in \mathbb{Z}\}$. The eigenfunctions associated with the eigenvalues λ_k are $\varphi_k(x) = e^{-2\pi kix}$. By the operator A we construct the space $\mathfrak{E}_\infty$ as it was done above.

We consider the initial boundary value problem

$$D_t^\alpha G(A)u(x, t) = J_1(A)u(x + h, t) + f(x), \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}}_+, \tag{24}$$

$$u(x, t) = u(x + 1, t), \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}}_+, \tag{25}$$

$$\frac{\partial^n u}{\partial x^n}(x, 0) = u_n(x), \quad n = 0, 1, \dots, m - 1,\; x \in (0, 1). \tag{26}$$

Theorem 9. *Let G and J_1 be entire functions satisfying $|G(2\pi k)| + |J(2\pi k)| \neq 0$ as $k \in \mathbb{Z}$, the set $\{J_1(2\pi k)/G(2\pi k) : k \in \mathbb{Z},\, G(2\pi k) \neq 0\}$ is bounded, $f(x, t) = f(x + 1, t)$ for all $(x, t) \in \mathbb{R} \times \overline{\mathbb{R}}_+$, $f \in C^m(\overline{\mathbb{R}}_+; \mathfrak{E}_\infty)$, $u_n \in \mathfrak{E}_\infty$,*

$$\int\limits_{\mathcal{G}_0} d\mathcal{E}_\lambda (J_1(\lambda)e^{-ih\lambda}u_n + f^{(n)}(\cdot, 0)) = 0, \quad n = 0, 1, \dots, m - 1.$$

Then problem (24)-(26) has the unique solution with the values in the space $\mathfrak{E}_\infty$ and it is of the form

$$\begin{aligned} u(x, t) = &\sum_{n=0}^{m-1} t^n \int\limits_{\sigma(A)\setminus\mathcal{G}_0} E_{\alpha,n+1}\left(t^\alpha \frac{J_1(\lambda)e^{-ih\lambda}}{G(\lambda)}\right) d\mathcal{E}_\lambda u_n \\ &+ \int\limits_0^t (t - s)^{\alpha-1} \int\limits_{\sigma(A)\setminus\mathcal{G}_0} E_{\alpha,\alpha}\left(t^\alpha \frac{J_1(\lambda)e^{-ih\lambda}}{G(\lambda)}\right) \frac{d\mathcal{E}_\lambda f(\cdot, s)}{G(\lambda)} ds - \int\limits_{\mathcal{G}_0} \frac{e^{ih\lambda} d\mathcal{E}_\lambda f(\cdot, s)}{J_1(\lambda)}. \end{aligned} \tag{27}$$

Proof. We let $\mathfrak{X} = L_2(0, 1)$, A is self-adjoint operator (23), $J(\lambda) = e^{-ih\lambda}J_1(\lambda)$. Then

$$J(A)v(x) = J_1(A) \sum_{k=0}^\infty \frac{h^k v^{(k)}(x)}{k!} = J_1 v(x + h)$$

for $v \in D(A^\infty)$. By Theorem 8 we obtain $(G(A), 0)$-regularity of the operator $J(A)$ in the space $\mathfrak{E}_\infty$. Theorem 6 implies the statement of the theorem. $\square$

In this case the Showalter-Sidorov condition can be imposed as

$$G(A)\left(\frac{\partial^n u}{\partial x^n}(x, 0) - u_n(x)\right) = 0, \quad n = 0, 1, \dots, m - 1,\; x \in (0, 1). \tag{28}$$

In the same way we obtain the following statement.

Theorem 10. *Let G and J_1 be entire functions satisfying $|G(2\pi k)| + |J(2\pi k)| \neq 0$ as $k \in \mathbb{Z}$, the set $\{J_1(2\pi k)/G(2\pi k) : k \in \mathbb{Z},\, G(2\pi k) \neq 0\}$ is bounded, $f(x, t) = f(x + 1, t)$ for all $(x, t) \in \mathbb{R} \times \overline{\mathbb{R}}_+$, $f \in C^m(\overline{\mathbb{R}}_+; \mathfrak{E}_\infty)$, $u_n \in \mathfrak{E}_\infty$ as $n = 0, 1, \dots, m - 1$. Then problem (24), (25), (28) has the unique solution with the values in the space $\mathfrak{E}_\infty$ and it has the form (27).*

Remark 7. *The assumptions of Theorem 9 are satisfied, for instance, by the polynomials G and J_1 having no common zeroes among the numbers $\{2\pi k : k \in \mathbb{Z}\}$ and satisfying $\deg G \geqslant \deg J_1$. Then equation (24) has the form*

$$D_t^\alpha \sum_{k=0}^{r} a_k u^{(k)}(x,t) = \sum_{l=0}^{s} b_l u^{(l)}(x+h,t) + f(x), \quad (x,t) \in \mathbb{R} \times \overline{\mathbb{R}}_+,$$

where $a_r \neq 0$, $b_s \neq 0$, $r \geqslant s$, $-ia_k$ are the coefficients of the polynomial G, $-ib_l$ are the coefficients of the polynomial J_1.

BIBLIOGRAPHY

1. E.G. Bajlekova. *Fractional evolution equations in Banach spaces.* PhD thesis, Eindhoven University of Technology (2001).
2. K. Balachandran, S. Kiruthika. *Existence of solutions of abstract fractional integrodifferential equations of Sobolev type* // Comput. Math. Appl. **64**:10, 3406-3413 (2012).
3. F. Li, J. Liang, H.-K. Xu. *Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions* // J. Math. Anal. Appl. **391**:2, 510-525 (2012).
4. V.E. Fedorov, A. Debbouche. *A class of degenerate fractional evolution systems in Banach spaces* // Differ. Uravn. **49**:12, 1616-1622 (2013). [Differ. Equ. **49**:12, 1569-1576 (2013).]
5. M. Kostić. *Abstract Volterra equations in locally convex spaces* // Sci. China Math. **55**:9, 1797-1825 (2012).
6. M. Kostić, C.-G. Li, M. Li. *On a class of abstract time-fractional equations on locally convex spaces* // Abstr. Appl. Anal. **2012**, id 131652 (2012).
7. M. Kostić. *Abstract time-fractional equations: existence and growth of solutions* // Fract. Calculus Appl. Anal. **14**:2, 301-316 (2014).
8. M. Kostić. *On the existence and uniqueness of solutions of certain classes of abstract multi-term fractional differential equations* // Funct. Anal. Appr. Comp. 2014. V. 6. P. 13–33.
9. M. Kostić, C.-G. Li, M. Li. *Abstract multi-term fractional differential equations* // Krag. J. Math. 2014. V. 38. P. 51–71.
10. M. Kostić. *Hypercyclic and topologically mixing properties of degenerate multi-term fractional differential equations* // Diff. Eqns. Dyn. Sys. **24**:4, 475-498 (2015).
11. V.E. Fedorov, D.M. Gordievskikh. *Resolving operators of degenerate evolution equations with fractional derivative with respect to time* // Izv. VUZ. Matem. 1, 71-83 (2015). [Russ. Math. (Izv. VUZ. Matem.). **59**:1, 60-70 (2015).]
12. G. A. Sviridyuk, V. E. Fedorov. *Linear Sobolev type equations and degenerate semigroups of operators.* VSP: Utrecht, Boston (2003).
13. V.E. Fedorov. *Strongly holomorphic groups of linear equations of Sobolev type in locally convex spaces* // Differ. Uravn. **40**:5, 702-712 (2004). [Differ. Equ. **40**:5, 753-765 (2004).]
14. Ya.V. Radyno. *Linear equations and bornology.* Izd-vo BGU, Minsk (1982). (in Russian).
15. L.R. Volevich, S.G. Gindikin. *Generalized functions and convolutions equations.* Fizmatlin, Moscow (1994). (in Russian).

Marko Kostić,
University of Novi Sad,
D. Obradović str. 6,
21125, Novi Sad, Serbia
E-mail: marco.s@verat.net

Vladimir Evegn'evich Fedorov,
Chelyabinsk State University,
Laboratory of quantum topology,
Br. Kashirinykh str. 129,
454001, Chelyabinks, Russia
E-mail: kar@csu.ru