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MODULO-LOXODROMIC MEROMORPHIC FUNCTIONS IN $\mathbb{C} \setminus \{0\}$

A.YA. KHRYSTIYANYN, $\boxed{\text{A.A. KONDRATYUK}}$

Abstract. We introduce modulo-loxodromic functions and study their representations, zeroes and poles distribution. We also show that each modulo-loxodromic meromorphic function in $\mathbb{C} \setminus \{0\}$ is Julia exceptional.

Keywords: loxodromic meromorphic function, modulo-loxodromic function, Julia exceptional function.

Mathematics Subject Classification: 30D35, 30D45

1. Introduction

In the work [1, p. 133], which A. Ostrowski [2] called "besonders schöne und überraschende", G. Julia gave an example of a meromorphic in the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ function such that

$$f(qz) = f(z) \tag{1}$$

for some non-zero q, $|q| \neq 1$, and all $z \in \mathbb{C}^*$. He noted that the family $\{f_n(z)\}$, $f_n(z) = f(q^n z)$ is normal [3] in $\mathbb{C}^*$ since $f_n(z) = f(z)$ for all $z \in \mathbb{C}^*$.

The functions satisfying (1) are called multiplicatively periodic. The theory of meromorphic in $\mathbb{C} \setminus \{0\}$ multiplicatively periodic functions was developed by O. Rausenberger [4]. G. Valiron [5] (see also [6]) called these functions *loxodromic* since for non-real q, the points, at which such function takes the same value, are located on a logarithmic spirals (the images of loxodromes under the stereographic projection). They give a simple construction [5], [6] of elliptic functions, which are well known due to the works of K. Jacobi, N. Abel, and K. Weierstrass.

We consider *modulo-loxodromic* functions.

2. Modulo-loxodromic meromorphic functions

Definition 1. *A meromorphic in $\mathbb{C}^*$ function f is said to be* modulo-loxodromic *with a multiplicator q if there exists q $(0 < |q| < 1)$ such that $|f(qz)| = |f(z)|$, $z \in \mathbb{C}^*$.*

We denote by $|\mathcal{L}|_q$ and $\mathcal{L}_q$ the sets of all modulo-loxodromic and loxodromic functions with a multiplicator q, respectively.

It is obvious that $\mathcal{L}_q \subset |\mathcal{L}|_q$. However, there are modulo-loxodromic functions in $\mathbb{C}^*$ which are not loxodromic.

Indeed, consider an entire function with the zero sequence $\{q^{-n}\}$, $n \in \mathbb{N}$, where $0 < |q| < 1$,

$$h(z) = \prod_{n=1}^{\infty}(1 - q^n z).$$

A.Ya. Khrystiyanyn, $\boxed{\text{A.A. Kondratyuk}}$, Modulo-loxodromic meromorphic functions in $\mathbb{C} \setminus \{0\}$.

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The function

$$P(z) = (1-z)h(z)h\left(\frac{1}{z}\right) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)\left(1-\frac{q^n}{z}\right)$$

is called the *Schottky-Klein prime function* [8].

This function is holomorphic in $\mathbb{C}^*$ with zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. It was introduced by Schottky [9] and Klein [10] for studying the conformal mappings of doubly-connected domains.

Now consider the function

$$f(z) = \frac{P(e^{-i\alpha}z)}{P(z)}, \quad \frac{\alpha}{\pi} \notin \mathbb{Q}. \tag{2}$$

Taking into consideration that

$$P(qz) = -\tfrac{1}{z}P(z),$$

we have $f(qz) = e^{i\alpha}f(z)$ and $|f(qz)| = |f(z)|$, $z \in \mathbb{C}^*$. Hence, $f \in |\mathcal{L}|_q$ and $f \notin \mathcal{L}_q$.

Furthermore, $f \notin \mathcal{L}_q$ for each q. Indeed, suppose that there exists a non-zero σ, $|\sigma| \neq 1$ such that $f(\sigma z) = f(z)$ foreach $z \in \mathbb{C}^*$. We observe that $f(e^{i\alpha}) = 0$. So, $f(\sigma e^{i\alpha}) = 0$ and since the only zeroes of f are $q^k e^{i\alpha}$, $k \in \mathbb{Z}$, we obtain that there exists $k_0 \in \mathbb{Z}$ such that $\sigma = q^{k_0}$. This implies that $f(\sigma z) = f(q^{k_0}z) = e^{ik_0\alpha}f(z)$ and the last value cannot be equal to $f(z)$ for any $k_0 \in \mathbb{Z}$ due to the choice of α in (2).

3. Representation of modulo-loxodromic functions

Let f be a meromorphic in $\mathbb{C}^*$ modulo-loxodromic function with a multiplicator q.

First, we suppose that f is holomorphic in $\mathbb{C}^*$. Then f is bounded in a neighbourhood of the origin since $|f|$ is determined by its values in $A_q = \{z \in \mathbb{C} : |q| < |z| \leqslant 1\}$. Hence, the origin is a removable singularity of f. We have that f is holomorphic and bounded in $\mathbb{C}$. Therefore, by the Liouville theorem $f = const$.

If f is not holomorphic, then there exists at least one pole $b \in \mathbb{C}^*$ of f. Taking into consideration the modulo-loxodromity of f, we conclude that $q^n b$, where $n \in \mathbb{Z}$ are also the poles of f. Applying similar arguments to $1/f$, we obtain that f is either constant or has at least one zero $a \in \mathbb{C}^*$. In the latter case $q^n a$, where $n \in \mathbb{Z}$ must also be the zeroes of f.

Thus, we have only two mutually excluding possibilities. Either function f is constant or it has an infinite number of zeroes and poles.

The function $\log|f|$ is loxodromic δ-subharmonic function [7]. Applying [7, Thm. 3.3], we conclude that the function f has the same number of zeroes and poles (taken counting multiplicities) in each annulus $\{z : |q|r < |z| \leqslant r\}$, $r > 0$.

Let $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m$ be the zeroes and poles of f in $\{z : |q| < |z| \leqslant 1\}$, respectively. Then all the zeroes of f have the form $a_k q^n$, where $n \in \mathbb{Z}$, $k = 1, 2, \dots, m$, while all the poles of f are given by $b_k q^n$, where $n \in \mathbb{Z}$, $k = 1, 2, \dots, m$, and there exists $p \in \mathbb{Z}$ such that

$$\frac{|a_1 \cdots a_m|}{|b_1 \cdots b_m|} = |q|^p. \tag{3}$$

A Nevanlinna type characteristic of a function meromorphic in $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leqslant +\infty$, was introduced in [11]. Namely,

$$T_0(r, f) = m_0(r, f) + N_0(r, f), \qquad 1 < r < R_0,$$

where

$$m_0(r, f) = m(r, f) + m\left(\frac{1}{r}, f\right) - 2m(1, f),$$

$$m(t, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(te^{i\theta})|\, d\theta, \qquad \frac{1}{R_0} < t < R_0,$$

and

$$N_0(r,f)=\int_1^r \frac{n_0(t,f)}{t}dt,$$

$n_0(t,f)$ is the number of poles of f in the annulus $1/t<|z|\leqslant t$ taken counting multiplicities.

The function $T_0(r,f)$ is nonnegative, nondecreasing, continuous and convex with respect to $\log r$ on $[1,R_0)$ ([11]). By the definition we have $T_0(r,0)=0$.

Taking into consideration the above observations on zeroes and poles of f and analysing the proof of Theorem 1 in [12], we obtain that the statement of this Theorem 1 is valid for our function f. That is,

$$T_0(r,f)=\frac{m}{\log\frac{1}{|q|}}\log^2 r+O(\log r), \qquad r>1.$$

So, the function f is of order 0 and applying the representation theorem for a meromorphic in $\mathbb{C}^*$ function of finite order ([13]), we get the following representation of f.

Theorem 1. *Each modulo-loxodromic function f has the representation*

$$f(z)=Cz^p\prod_{k=1}^{m}\frac{\prod\limits_{n=0}^{+\infty}\left(1-\dfrac{q^nz}{a_k}\right)\prod\limits_{n=1}^{+\infty}\left(1-\dfrac{q^na_k}{z}\right)}{\prod\limits_{n=0}^{+\infty}\left(1-\dfrac{q^nz}{b_k}\right)\prod\limits_{n=1}^{+\infty}\left(1-\dfrac{q^nb_k}{z}\right)}, \qquad z\in\mathbb{C}^* \tag{4}$$

where C is a constant and $p\in\mathbb{Z}$ satisfies condition (3)*.*

4. Zero and pole distribution

Let $\{a_j\}$, $\{b_j\}$, $j\in\mathbb{Z}$ be a pair of sequences in $\mathbb{C}^*$, $p\in\mathbb{Z}$. Denote

$$\mathfrak{M}_p(r)=\begin{cases} r^p\dfrac{\prod\limits_{1<|a_j|\leqslant r}\frac{r}{|a_j|}}{\prod\limits_{1<|b_j|\leqslant r}\frac{r}{|b_j|}}, & r>1;\\[4ex] r^p\dfrac{\prod\limits_{r<|a_j|\leqslant 1}\frac{|a_j|}{r}}{\prod\limits_{r<|b_j|\leqslant 1}\frac{|b_j|}{r}}, & 0<r\leqslant 1.\end{cases}$$

Theorem 2. *The sequence of zeroes $\{a_j\}$ and the sequence of poles $\{b_j\}$ of a modulo-loxodromic function satisfy the following conditions*

(*i*) *the number of a_j and b_j in each annulus of the form $\{z: r<|z|<2r\}$, $r>0$ is bounded by an absolute constant;*

(*ii*) *the difference between the numbers of a_j and b_k in each annulus $\{z: r_1<|z|<r_2\}$, $0<r_1<r_2<+\infty$ is bounded by an absolute constant;*

(*iii*) *there exists $C_1>0$ such that $\left|\frac{a_j}{b_k}-1\right|>C_1$ for all $j,k\in\mathbb{Z}$;*

(*iv*) *the function $\mathfrak{M}_p(r)$, where $p\in\mathbb{Z}$ satisfies condition* (3)*, is continuous and bounded for $r>0$.*

Proof. Let f be a modulo-loxodromic function. As we have established, either function f is constant or it has infinitely many zeroes and poles. The proof of the theorem in the former case is trivial, so we can focus only on the latter one. Next we use the representation (4).

(i) First we remark that there exists a unique $n_0 \in \mathbb{Z}_+$ such that $\dfrac{1}{|q|^{n_0}} \leqslant 2 < \dfrac{1}{|q|^{n_0+1}}$. This n_0 is equal to $\left[\dfrac{\log 2}{\log \frac{1}{|q|}}\right]$.

Since each annulus $\left\{z : \dfrac{r}{|q|^k} < |z| \leqslant \dfrac{r}{|q|^{k+1}}\right\}$, where $k \in \mathbb{Z}$, contains the same number of zeroes of f, say m, and

$$(r, 2r] = \left(\bigcup_{k=0}^{n_0-1}\left(\frac{r}{|q|^k}, \frac{r}{|q|^{k+1}}\right]\right) \cup \left(\frac{r}{|q|^{n_0}}, 2r\right]$$

it follows that the annulus $\{z : r < |z| \leqslant 2r\}$ contains at least $n_0 m$ and less than $(n_0+1)m$ zeroes of f. The same is true about the poles of f.

(ii) In the same way as in (i), we can find unique $n_1, n_2 \in \mathbb{Z}$ such that

$$|q|^{n_1+1} < r_1 \leqslant |q|^{n_1} < |q|^{n_2} < r_2 \leqslant |q|^{n_2-1}.$$

Hence,

$$(r_1, r_2) = (r_1, |q|^{n_1}] \cup \left(\bigcup_{k=n_1}^{n_2-1}(|q|^k, |q|^{k+1}]\right) \cup (|q|^{n_2}, r_2).$$

Each annulus of the form $\{z : |q|^{k+1} < |z| \leqslant |q|^k\}$, where $k \in \mathbb{Z}$, contains the same amount of zeroes and poles of f taken counting multiplicities; we have denoted this number by m. Therefore, the difference between the numbers of zeroes and poles of f in the annulus $\{z : r_1 < |z| < r_2\}$ is no greater than $2m$ because of the choice of n_1, n_2.

(iii) Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ be the zeroes and the poles of f in $\{z : |q| < |z| \leqslant 1\}$, respectively. Then all the zeroes of f have the form $\alpha_{\mu,k} = a_k q^\mu$, where $\mu \in \mathbb{Z}$, $k = 1, 2, \ldots, m$. The same is true about the poles of f, namely $\beta_{\nu,k} = b_k q^\nu$, where $\nu \in \mathbb{Z}$, $k = 1, 2, \ldots, m$. So, $\frac{\alpha_{\mu,j}}{\beta_{\nu,k}} = \frac{a_j}{b_k} q^l$, where $l \in \mathbb{Z}$.

We need to show that there exists $C > 0$ such that the inequality

$$\left|\frac{a_j}{b_k} q^l - 1\right| > C$$

holds for all $j, k \in \{1, 2, \ldots, m\}$, and $l \in \mathbb{Z}$.

Suppose that for each $\varepsilon > 0$ there exist $j, k \in \{1, 2, \ldots, m\}$, and $l \in \mathbb{Z}$ such that

$$\left|\frac{a_j}{b_k} q^l - 1\right| \leqslant \varepsilon. \tag{5}$$

Without loss of generality we can assume that $|l| \leqslant 2$. Indeed, taking into consideration the location of a_j, b_k, we have

$$\left|\frac{a_j}{b_k} q^l\right| \leqslant \frac{1}{|q|}|q|^l \leqslant |q|, \qquad l \geqslant 2.$$

In the same way we get

$$\left|\frac{a_j}{b_k} q^l\right| \geqslant |q||q|^l \geqslant \frac{1}{|q|}, \qquad l \leqslant -2.$$

Thus, for all $j, k \in \{1, 2, \ldots, m\}$, and $l \geqslant 2$

$$\left|\frac{a_j}{b_k} q^l - 1\right| \geqslant 1 - |q|,$$

and for $l \leqslant -2$

$$\left|\frac{a_j}{b_k}q^l - 1\right| \geqslant \frac{1}{|q|} - 1.$$

Let now $|l| < 2$. Choose

$$\varepsilon = \frac{1}{2}\min\{|a_j q^l - b_k| : j, k \in \{1, 2, \ldots, m\}, -1 \leqslant l \leqslant 1\}.$$

Then (5) implies

$$|a_j q^l - b_k| \leqslant \varepsilon |b_k| \leqslant \varepsilon\,.$$

That is,

$$|a_j q^l - b_k| \leqslant \frac{1}{2}\min\{|a_j q^l - b_k| : j, k \in \{1, 2, \ldots, m\}, -1 \leqslant l \leqslant 1\}$$

which gives a contradiction.

(iv) We recall that f satisfies representation (4). Clearly, we can assume that $C \neq 0$. Consider the integral means $I(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\,,\ r > 0$.

Let $z = re^{i\theta}$. We have for $r > 1$ [16, p. 8]

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - \frac{z}{a_j}\right| d\theta = \log^+ \frac{r}{|a_j|}\,,$$

and, if $|a_j| \leqslant 1$,

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|1 - \frac{a_j}{z}\right| d\theta = 0\,.$$

The same is true for b_j.

Thus,

$$I(r) = p\log r + \sum_{|a_j|>1} \log^+ \frac{r}{|a_j|} - \sum_{|b_j|>1} \log^+ \frac{r}{|b_j|} + \log|C|\,, \quad r > 1\,.$$

In the same way for $0 < r \leqslant 1$ we obtain

$$I(r) = p\log r + \sum_{|a_j|\leqslant 1} \log^+ \frac{|a_j|}{r} - \sum_{|b_j|\leqslant 1} \log^+ \frac{|b_j|}{r} + \log|C|\,.$$

Hence,

$$\mathfrak{M}_p(r) = \frac{1}{|C|}\exp I(r), \quad r > 0\,. \tag{6}$$

Since $I(r)$ is convex with respect to $\log r$ and therefore, is continuous, $I(r)$ is bounded on $[|q|, 1]$. It follows from the definition of a modulo-loxodromic function that $I(|q|^k r) = I(r)$ for each $k \in \mathbb{Z}$. Then, we conclude that $I(r)$ remains bounded for all $r > 0$ that completes the proof.

$\square$

5. Julia exceptionality

Definition 2. *A meromorphic in $\mathbb{C}^*$ function f is called Julia exceptional* [3] *if for some q, $0<|q|<1$, the family $\{f_n(z)\}$, $n\in\mathbb{Z}$, where $f_n(z)=f(q^n z)$, is normal* [3] *in $\mathbb{C}^*$.*

The following theorem is a generalization of one "remarkably complete" result of A. Ostrowski [2], [3] for meromorphic functions f with two essential singularities. This theorem was originally formulated without proof by A. Eremenko [14] and later was proved by L. Radchenko [15]. We propose here its following version.

Theorem A. I. *Two sequences $\{a_j\}$, $\{b_j\}$ in $\mathbb{C}^*$ are sequences of zeroes and poles of a Julia exceptional in $\mathbb{C}^*$ function f, respectively, if and only if they satisfy conditions (i) – (iii) of Theorem 2 and*

(*iv*) *there exist $p\in\mathbb{Z}$ and $C_2>0$, $C_3>0$ such that $\mathfrak{M}_p(|a_j|)\leqslant C_2$ and $\mathfrak{M}_p(|b_j|)\geqslant C_3$ for each $j\in\mathbb{Z}$.*

II. *If $\{a_j\}$, $\{b_j\}$, and p satisfy (i) - (iv), then the function*

$$\Pi(z)=z^p\frac{\prod\limits_{|a_j|\leqslant 1}\left(1-\dfrac{a_j}{z}\right)\prod\limits_{|a_j|>1}\left(1-\dfrac{z}{a_j}\right)}{\prod\limits_{|b_j|\leqslant 1}\left(1-\dfrac{b_j}{z}\right)\prod\limits_{|b_j|>1}\left(1-\dfrac{z}{b_j}\right)},$$

is Julia exceptional in $\mathbb{C}^*$, and vice versa, each non-rational Julia exceptional in $\mathbb{C}^*$ function f satisfies the representation

$$f(z)=C\cdot\Pi(z)$$

where $\{a_j\}$, $\{b_j\}$, p satisfy (i) - (iv), and C is a constant.

As an immediate consequence of Theorem 2 and Theorem A [2], [3] we obtain the following theorem.

Theorem 3. *Each modulo-loxodromic function is Julia exceptional in $\mathbb{C}^*$.*

Indeed, using the representation of modulo-loxodromic function given by Theorem 1, we observe that conditions (*i*)-(*iii*) in Theorem A coincide with those of Theorem 2 and condition (*iv*) of Theorem A is implied immediately by condition (*iv*) of Theorem 2.

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Andriy Yaroslavovych Khrystiyanyn,
Ivan Franko National University of Lviv,
1 Universytetska Str.,
79000 Lviv, Ukraine
E-mail: `khrystiyanyn@ukr.net`

Andriy Andriyovych Kondratyuk,
Ivan Franko National University of Lviv,
1 Universytetska Str.,
79000 Lviv, Ukraine
E-mail: `kond@franko.lviv.ua`