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ON UNCONDITIONAL EXPONENTIAL BASES IN WEAKLY WEIGHTED SPACES ON SEGMENT

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Abstract. We show that the existence of unconditional exponential bases is not determined by the growth characteristics of a weight function. In order to do this, we construct examples of convex weights with arbitrarily slow growth near the boundary such that unconditional exponential bases do not exist in the corresponding space.

Keywords: Hilbert spaces, entire functions, unconditional exponential bases, Riesz bases.

Mathematics Subject Classification: 30B50, 30D20

1. Introduction

In the present work we consider Hilbert space of the form

$$L_2(W)=\left\{f\in L_{\rm loc}(-1,1):\|f\|^2=\int_{-1}^{1}|f(t)|^2W^2(t)dt<\infty\right\},$$

where W is positive continuous integrable function on $(-1,1)$.

In the classical case $W(t)\equiv 1$, the Fourier system $\{e^{\pi nit}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis. It is obvious that in other cases there can be no orthonormal exponential bases in spaces $L_2(W)$. The notion of Riesz basis was introduced in [1] and it denotes the image of the orthonormalized basis under a bounded invertible operator.

A basis $\{e_k,\ k=1,2,\ldots\}$ in a Hilbert space H is called unconditional [2] if for some constants $c,C>0$ and each element

$$x=\sum_{k=1}^{\infty}x_ke_k$$

the relation

$$c\sum_{k=1}^{\infty}|x_k|^2\|e_k\|^2\leqslant\|x\|^2\leqslant C\sum_{k=1}^{\infty}|x_k|^2\|e_k\|^2$$

holds true. An unconditional basis $\{e_k,\ \ k=1,2,...\}$ becomes Riesz basis if and only if $0<\inf\|e_k\|\leqslant\sup\|e_k\|<\infty$.

The problem on Riesz basis property for a given exponential system $\{e^{\lambda_kt}\}$ in the classical space L_2 was studied in details. In work [3], there was obtained a criterion saying that the generating function for this system should satisfy the Muckenhoupt condition. In the weighted space with an unbounded weight function there can be no Riesz bases. This fact was proved in work [4].

Unconditional bases were considered also in Hilbert subspaces of the space $H(D)$ of functions analytic in a bounded convex domain $D\subset\mathbb{C}$. For the Smirnov space $E_2(D)$ on a convex polygon

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D there were constructed unconditional exponentials bases [5]. In work [6], there was considered the existence of exponential bases in $E_2(D)$ on a convex domain D with a smooth boundary. It was proved in [7] that in Smirnov spaces on convex domains containing a smooth arc on the boundary, there are no exponential bases. It was shown in [8] that in Bergman spaces $B_2(D)$ on convex domains with a point of non-zero curvature on the boundary, there are no exponential bases.

In work [9] there was proved an analogue of this result in weighted spaces $L_2(e^{-h(t)})$ with a convex function h: under certain regularity conditions for the growth of the weight function $h(t)$, if for each $k \in \mathbb{N}$

$$e^{h(t)}(1-|t|)^k \longrightarrow \infty, \ |t| \longrightarrow 1,$$

then there are no unconditional exponential bases in the space $L_2(e^{-h(t)})$.

All the aforementioned problems can be formulated for one model of weighted spaces of entire functions, if by the Fourier-Laplace transform we pass to an equivalent problem on unconditional bases of reproducing kernels in Hilbert spaces of entire functions.

Let X be some Hilbert space of functions, in which the system of all exponentials $e^{\lambda z}$, $\lambda \in \mathbb{C}$, is complete.The the Fourier-Laplace transform mapping each linear continuous functional $S \in X^*$ into the function

$$\widehat{S}(\lambda) = S(e^{\lambda z}), \quad \lambda \in \mathbb{C},$$

makes a one-to-one correspondence between the dual space X^* and some space of functions $\widehat{X}$. Under natural conditions for the original space X, the space $\widehat{X}$ turns out to be the Hilbert space of entire functions with a structure endowed by X^*, in which point functionals $F \longrightarrow F(z)$ turn out to be bounded for all $z \in \mathbb{C}$. Thus, by the self-adjointness of Hilbert spaces there arises a reproducing kernel (see [10]) $K(\lambda, z)$:

$$(F(\lambda), K(\lambda, z))_{\widehat{X}} = F(z), \ \forall F \in \widehat{X}.$$

It follows from simple functional analytic arguments that the exponential system $e^{\lambda_k z}$, $k \in \mathbb{Z}$, is an unconditional basis in X if and only if the system $K(\lambda, \lambda_k)$, $k \in \mathbb{Z}$, is an unconditional basis in $\widehat{X}$.

The problem on unconditional bases of reproducing kernels in weighted spaces of entire functions was studied in works [11]–[14], in which the weighted spaces of entire functions

$$H(\varphi) = \{F \in H(\mathbb{C}) : \ \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-2\varphi(z)} dm(z) < \infty\}$$

were considered. Here φ is a some subharmonic function on the plane, $dm(z)$ is the planar Lebesgue measure. In work [14], under assumption of some regularity for the growth of the function $\varphi(z) = \varphi(|z|)$, it was proved that if

$$\ln^2 t = o(\varphi(t)), \ t \longrightarrow \infty,$$

then the space $H(\varphi)$ contains no unconditional bases of reproducing kernel, while in the spaces with the weight $\varphi(t) = \ln^\alpha t$, $1 \leqslant \alpha \leqslant 2$, such bases exist.

In work [15], a general condition for the Bergman function of the weighted space of entire functions was proved. This condition ensured the absence of an unconditional bases of reproducing kernels.

The results of work [14] suggests some stability of the existence of unconditional bases in weighted spaces under a “perturbation” of the weight. The matter is that the spaces $H(\varphi)$ with $\varphi(\lambda) = O(\ln|\lambda|)$, $\lambda \longrightarrow \infty$, become finite dimensional and hence, the unconditional bases of reproducing kernels exist. In the present work we construct examples of convex functions h on the interval $(-1; 1)$ with an arbitrary slow growth at the end-points of the interval such that there are no exponential unconditional bases in the space $L_2(h)$.

2. Notations, preliminaries and formulation of the statements

The statement that two non-negative functions f, g satisfy the estimate

$$f(x) \leqslant Cg(x), \quad \forall x \in X,$$

for with some constant C is indicated by the symbol $\prec$:

$$f(x) \prec g(x), \quad x \in X.$$

The symbols $\succ$ have $\asymp$ the corresponding meaning.

It was proved in work [16] that the space $\widehat{L}_2(h)$ of Fourier-Laplace transform of continuous functionals on $L_2(e^{-h})$ considered as a normed space is isomorphic to the space of the entire functions of exponential type with the norm

$$\|F\|^2 := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|F(x+iy)|^2}{K(x)} dy d\widetilde{h}'(x),$$

where

$$\widetilde{h}(x) = \sup_{|t|<1}(xt - h(t))$$

is Young dual function for the function h and

$$K(x) = \|e^{(x+iy)t}\|^2_{\widehat{L}_2(h)} = \int_{-1}^{1} e^{2xt-2h(t)}dt.$$

If $\delta_z : F(\cdot) \longrightarrow F(z)$ is a point functional on $\widehat{L}_2(h)$, then by the definition of the Fourier-Laplace transform

$$\|\delta_z\|^2_{L_2^*(h)} = \|e^{zt}\|^2 = K(\operatorname{Re} z).$$

To simplify the notations, in what follows we write $K(z) := K(\operatorname{Re} z)$.

Given a continuous in $\overline{B}(z,r)$ function f we let

$$\|f\|_r = \max_{w\in\overline{B}(z,r)} |f(w)|.$$

Let $d(f,z,r)$ be the distance from the function f to the space of harmonic in $B(z,r)$ functions:

$$d(f,z,r) = \inf\{\|f-H\|_r,\ H \text{is harmonic in } \ B(z,r)\}.$$

For a continuous in $\mathbb{C}$ function u and a positive number p we let

$$\tau(u,z,p) = \sup\{r : \ d(u,z,r) \leqslant p\}.$$

If the function u depends on $\operatorname{Re} z$ only, that is, $u(z) = u(x)$, $z = x + iy$, $u(x)$ is a convex function, then by slightly different means, it is possible to define the characteristics of this convex functions comparable with $\tau(u,z,p)$.

For instance, by $\rho(u,x,p)$ we denote the maximal number $r > 0$ such that

$$\int_{-r}^{r} |u'(x+t) - u'(x)|dt \leqslant p.$$

Then it follows from Lemmata 2 and 5 in work [17] (see also [18]) that

$$\tau(u,x,p) \asymp \rho(u,x,p), \quad x \in \mathbb{R}.$$

The statement

$$K(x) \asymp \frac{1}{\rho(\widetilde{h},x,p)} e^{2\widetilde{h}(x)}. \tag{1}$$

was formulated in [17] and was proved in [18, Thm. 2].

In the present work we prove the following theorem.

Theorem 1. *For each continuous integrable positive function W on the interval $(-1;1)$ tending to zero as $|t| \longrightarrow 1$, there exists a convex function h such that $e^{h(t)} \leqslant \frac{1}{W(t)}$ as $|t|<1$ and there are no exponential bases in the space $L_2(e^{-h(t)})$.*

The proof is essentially based on Theorem 4 in work [9].

Theorem A. *Let $h(t)$ be a convex function on the interval $(-1;1)$,*

$$K(\lambda) = \int_{-1}^{1} e^{2\mathrm{Re}\,\lambda t - 2h(t)}\, dt.$$

Assume that for some $p>0$ there exists a sequence of segments $[a_m; b_m]$ and of positive numbers τ_m, $m=1,2,\dots,$ such that

1) for some positive number δ and for all $x \in [a_m; b_m]$

$$\delta\tau_m \leqslant \tau(\ln K, x, p) \leqslant \tau_m, \; m = 1,2,\dots,$$

2) the relation

$$\lim_{m\longrightarrow\infty} \frac{b_m - a_m}{\tau_m} = \infty$$

holds true.

Then there are no exponential bases in the space $L^2(e^{-h})$.

3. Construction of dual function $\widetilde{h}$

We take an arbitrary positive continuous monotonically increasing unbounded function $\alpha(t)$ on $[1;\infty)$ obeying the condition:

$$\alpha(2t) \leqslant A\alpha(t), \quad t>1, \tag{2}$$

for some constant $A \in (1;2)$.

This function also satisfies the condition:

$$\alpha(y) \leqslant A \cdot \left(\frac{y}{x}\right)^{\delta} \alpha(x), \quad x > 1, \tag{3}$$

for $y \geqslant x$ and $\delta = \frac{\ln A}{\ln 2}$. Indeed, let $n = \left[\log_2 \frac{y}{x}\right] + 1$, where the square brackets stand for the integer part. Then by (2) and the monotonicity of α we get the inequality

$$\alpha(y) \leqslant \alpha(2^n x) \leqslant A^n \alpha(x) \leqslant A \cdot A^{\log_2 \frac{y}{x}} \alpha(x) = A \cdot \left(\frac{y}{x}\right)^{\delta} \alpha(x).$$

Letting $x=1$ and taking consideration that $\delta<1$, we obtain the convergence of the improper integral

$$\int^{\infty} \frac{\alpha(t)dt}{t^2} < \infty.$$

Thus, the function

$$v(x) = \int_1^x \left(\int_t^{\infty} \frac{\alpha(s)ds}{s^2}\right) dt, \; x \geqslant 1,$$

is well-defined. It is concave on $[1;\infty)$. Indeed,

$$v''(x) = -\frac{\alpha(x)}{x^2} < 0.$$

We define a sequence of non-negative numbers T_n:

$$T_1 = 1, \; T_{n+1} = \max(\alpha^{(-1)}(n^2), 2T_n), \quad n \in \mathbb{N}, \tag{4}$$

where $\alpha^{(-1)}$ is the inverse function for α. The sequence T_n tends to infinity. For each $n \in \mathbb{N}$ we define the characteristic function of the segment $I_n = [T_n; 2T_n]$

$$\chi_n(t) = \begin{cases} 0, & t < T_n, \\ 1, & T_n \leqslant t \leqslant 2T_n, \\ 0, & 2T_n < t. \end{cases}$$

and

$$\beta_n(t) = \sqrt{\alpha(t)}\chi_n(t), \quad \beta(t) = \sum_{n=1}^{\infty} \beta_n(t), \ t > 1.$$

We let

$$u(x) = \int_1^x \left(\int_t^{\infty} \frac{\beta(s)ds}{s^2} \right) dt, \ x \geqslant 1.$$

Lemma 1. *The function $u(x)$ is concave, non-negative, linear outside the segments I_n and grows monotonically to infinity. For some constant $c > 0$, the estimate*

$$u(x) \leqslant c\alpha(x), \quad x \geqslant 1,$$

holds true. The derivative $u'(x)$ tends to zero.

Proof. The function α increases monotonically to infinity, hence, for each $M > 0$ from some index m the inequality $\beta(t) \geqslant M$ is satisfied on the segments $I_k, \ k \geqslant m$. Then for $t \in [T_k; \frac{3}{2}T_k]$ we have

$$u'(t) = \int_t^{\infty} \frac{\beta(s)ds}{s^2} \geqslant M \int_{\frac{3}{2}T_k}^{2T_k} \frac{ds}{s^2} = \frac{M}{6T_k}.$$

Therefore,

$$u\left(\frac{3}{2}T_m\right) = u(T_m) + \int_{T_m}^{\frac{3}{2}T_m} u'(s)ds \geqslant \frac{M}{12}.$$

Since the function u is increasing by the definition, it increases to infinity.

Let us estimate the derivative u' from above. Let

$$B_k = \int_{T_k}^{T_{k+1}} \frac{\beta(t)dt}{t^2} = \int_{T_k}^{2T_k} \frac{\sqrt{\alpha(t)}dt}{t^2}, \quad k \in \mathbb{N}.$$

Then by condition (2)

$$B_k \leqslant \frac{\sqrt{\alpha(2T_k)}}{2T_k} \leqslant \frac{\sqrt{A\alpha(T_k)}}{2T_k}. \tag{5}$$

Let $x \in [2T_n, T_{n+1}]$. Then

$$u'(x) = \int_x^{\infty} \frac{\beta(t)dt}{t^2} = \sum_{k=n+1}^{\infty} B_k \leqslant \sqrt{A} \sum_{k=n+1}^{\infty} \frac{\sqrt{\alpha(T_k)}}{2T_k}.$$

We employ relation (3) for $y = T_k$, $k \geqslant n+1$, and $x = T_{n+1}$:

$$\alpha(T_k) \leqslant A \cdot \left(\frac{T_k}{T_{n+1}}\right)^{\delta} \cdot \alpha(T_{n+1}).$$

We keep estimating u':

$$u'(x) \leqslant \frac{A\sqrt{\alpha(T_{n+1})}}{2T_{n+1}^{\frac{\delta}{2}}} \sum_{k=n+1}^{\infty} T_k^{\frac{\delta}{2}-1}.$$

By the definition of the sequence T_k, the estimate

$$T_k \geqslant 2^{k-(n+1)}T_{n+1}$$

holds true. Hence, for $\varepsilon = 1 - \frac{\delta}{2} > 0$ and $x \in [2T_n; T_{n+1}]$ we have

$$u'(x) \leqslant \frac{A\sqrt{\alpha(T_{n+1})}}{2T_{n+1}} \sum_{k=n+1}^{\infty} (2^{\varepsilon})^{n+1-k} = \frac{A\sqrt{\alpha(T_{n+1})}}{2T_{n+1}} \cdot \frac{2^{\varepsilon}}{2^{\varepsilon}-1} := A_1 \frac{\sqrt{\alpha(T_{n+1})}}{T_{n+1}}.$$

If $x \in [T_n; 2T_n]$, the latter inequality and (5) yield

$$u'(x) = \int_x^{2T_n} \frac{\beta(t)dt}{t^2} + u'(2T_n) \leqslant B_n + u'(2T_n) \leqslant \frac{\sqrt{A}}{2} \frac{\sqrt{\alpha(T_n)}}{T_n} + A_1 \frac{\sqrt{\alpha(T_{n+1})}}{T_{n+1}}.$$

By definition (4) of the sequence T_n

$$T_{n+1} = \alpha^{(-1)}(n^2)$$

or

$$T_{n+1} = 2T_n.$$

In each case

$$n \leqslant \sqrt{\alpha(T_{n+1})}. \tag{6}$$

In the first case $\sqrt{\alpha(T_{n+1})} = n$, this is why for $n \geqslant 2$

$$\sqrt{\alpha(T_{n+1})} \leqslant 2(n-1) \leqslant 2\sqrt{\alpha(T_n)}.$$

In the second case we employ property (2)

$$\sqrt{\alpha(T_{n+1})} = \sqrt{\alpha(2T_n)} \leqslant \sqrt{A}\sqrt{\alpha(T_n)}.$$

Therefore, for each $n \geqslant 2$

$$\sqrt{\alpha(T_{n+1})} \leqslant 2\sqrt{\alpha(T_n)}. \tag{7}$$

Thus, for $x \in [2T_n; 2T_{n+1}]$ and some constant A_0 the estimate

$$u'(x) \leqslant A_0 \frac{\sqrt{\alpha(T_{n+1})}}{T_{n+1}}, \quad n \in \mathbb{N},$$

holds true. We estimate $u(x)$ from above. Let $x \in [2T_n; 2T_{n+1}]$, then

$$\begin{aligned} u(x) = \int_1^x u'(t)dt &= u(2) + \sum_{k=1}^{n-1} \int_{2T_k}^{2T_{k+1}} u'(t)\,dt + \int_{2T_n}^x u'(t)dt \\ &\leqslant u(2) + 2A_0 \sum_{k=1}^{n-1} \frac{\sqrt{\alpha(T_{k+1})}}{T_{k+1}} (T_{k+1} - T_k) + A_0 \frac{\sqrt{\alpha(T_{n+1})}}{T_{n+1}} (x - 2T_n) \\ &\leqslant u(2) + 2A_0 \sum_{k=1}^{n-1} \sqrt{\alpha(T_{k+1})} + 2A_0\sqrt{\alpha(T_{n+1})} \\ &\leqslant u(2) + 2A_0(n-1)\sqrt{\alpha(T_n)} + 2A_0\sqrt{\alpha(T_{n+1})}, \quad n \in \mathbb{N}. \end{aligned}$$

By inequalities (6) and (7) it follows that

$$u(x) \leqslant c\alpha(x)$$

for some constants $c > 0$, $x \geqslant 1$. □

Normalizing the function α if needed, we assume that

$$u'(1) = \sum_{k=1}^{\infty} B_k < 1.$$

Then the function

$$\widetilde{h}(x) = |x| - u(|x|),\ |x| \geqslant 1, \quad \widetilde{h}(x) = 1,\ |x| \leqslant 1,$$

is a convex function on $\mathbb{R}$ decaying on $\mathbb{R}_-$ and increasing on the positive semi-axis. We let

$$h(t) = \sup_x (xt - \widetilde{h}(x)), \quad |t| < 1,$$

and let us prove that under appropriate choice of α the function h satisfies the assumptions of Theorem 1.

4. Estimate for characteristics τ

Lemma 2. *If the function α satisfies condition (2) and the functions u, $\widetilde{h}$ are defined by means of this function α then for $q < \frac{1}{4}$ and for each $p > 0$ the estimate*

$$\tau(\widetilde{h}, x, p) \asymp x(\alpha(x))^{-\frac{1}{4}} = o(x), \;\; x \in J_n, \quad n \in \mathbb{N},$$

holds in the intervals $J_n = [(1+q)T_n; (2-q)T_n]$.

Proof. We have shown earlier that $\tau(u, x, p) \asymp \rho(u, x, p)$, this is why in the proof we deal with the characteristics ρ. If $x \in J_n$ and $\rho < qT_n$, then

$$\int_{-\rho}^{\rho} |\widetilde{h}'(x+t) - \widetilde{h}'(x)| dt \geqslant \min_{T_n \leqslant y \leqslant 2T_n} |u''(y)|\rho^2.$$

Hence,

$$\rho(\widetilde{h}, x, p) \leqslant \sqrt{p(\min_{T_n \leqslant y \leqslant 2T_n} |u''(y)|)^{-1}}.$$

On the other hand,

$$\int_{-\rho}^{\rho} |\widetilde{h}'(x+t) - \widetilde{h}'(x)| dt \leqslant \max_{T_n \leqslant y \leqslant 2T_n} |u''(y)|\rho^2.$$

Therefore,

$$\rho(\widetilde{h}, x, p) \geqslant \sqrt{p(\max_{T_n \leqslant y \leqslant 2T_n} |u''(y)|)^{-1}}.$$

By property (2) of function α we arrive at the statement of the lemma. □

Lemma 3. *If the function α satisfies condition (2) and the functios u, $\widetilde{h}$ are defined by this function α, then for $q < \frac{1}{4}$ and for each $p > 0$ the estimate*

$$\tau(\ln K, x, p) \asymp x(\alpha(x))^{-\frac{1}{4}} = o(x), \;\; x \in J_n, \;\; n \in \mathbb{N},$$

holds true in the intervals $J_n = [(1+q)T_n; (2-q)T_n]$. Thus, by Theorem A, there are no bases of exponentials in the space $L_2(e^{-h})$.

Proof. Relation (1) can be written as

$$K(x) \asymp \frac{\sqrt[4]{\alpha(x)}}{x} e^{2\widetilde{h}(x)}.$$

We let

$$a(x) = \frac{\sqrt[4]{\alpha(x)}}{x}, \;\; x \geqslant 1.$$

Then by property (2), for $x \in J_n$ and some constant $C > 0$

$$|\ln a(x) - \ln T_n| \leqslant C.$$

We let $u_1(x) = \widetilde{h}(x)$, $u_2(x) = \ln K(x) - \ln a(T_n)$. Then

$$|u_1(x) - u_2(x)| = |\widetilde{h}(x) - \ln K(x) + \ln a(T_n)| \leqslant C.$$

in the interval J_n. By Lemma 4 in work [18] it follows that

$$\frac{p}{(p+C)} \rho(u_1, y, p) \leqslant \rho(u_2, y, p) \leqslant \frac{(p+C)}{p} \rho(u_1, y, p).$$

Hence, by Lemma 2,

$$\rho(\ln K, x, p) \asymp \rho(\widetilde{h}, x, p) \asymp x(\alpha(x))^{-\frac{1}{4}} = o(x), \ x \in J_n, \ n \in \mathbb{N}.$$

□

5. Proof of Theorem 1

Passing if needed to the function $W(t) := \min(W(t), W(-t))$, we can assume that the weight function W is positive, even and $W(t) \longrightarrow 0$, $|t| \longrightarrow 1$. Then passing if needed to the function

$$W(t) := \min_{|\tau| \leqslant t} W(\tau),$$

we can assume that this function is monotone on the intervals $(-1;0), (0;1)$. Finally, normalizing by a multiplicative constant, we assume that $W(t) \leqslant 1$. Thus, the function

$$a(t) = \ln \frac{1}{W(t)}, \quad |t| < 1,$$

is positive, even and monotone on $(0;1)$. We let

$$\widetilde{a}(x) = \sup_{|t|<1}(xt - a(t)), \quad x \in \mathbb{R}.$$

The function $\widetilde{a}(x)$ is convex on $\mathbb{R}$, even and possesses the easily checked properties:

$$0 < \ln a(0) \leqslant \widetilde{a}(x) < |x|, \quad \lim_{|x| \longrightarrow \infty} \frac{\widetilde{a}(x)}{|x|} = 1. \tag{8}$$

At that, the function $b(x) = x - \widetilde{a}(x)$ is concave and is unbounded in $\mathbb{R}_+$. Indeed, if t_x is the point at which the supremum is attained in the definition of $\widetilde{a}$, then

$$b(x) = a(t_x) + (1 - t_x)x,$$

and if $|t_x| \leqslant d < 1$ as $x \in \mathbb{R}_+$, then $b(x) \geqslant (1-d)x \longrightarrow \infty$, while if $\overline{\lim}_{x \longrightarrow \infty} t_x = 1$, then $b(x) \geqslant a(t_x) \longrightarrow \infty$. It follows from the concavity that $b'(x)$ is a decreasing function and by the unboudedness we get that $b'(x)$ is a non-negative function. Hence, the function $b(x)$ increases to infinity. We let

$$h_0(t) = \sup_x(xt - \widetilde{a}(x)), \ |t| < 1.$$

Then the function h is convex on $(-1;1)$ and

$$e^{h_0(t)} \leqslant a(t) = \frac{1}{W(t)}, \quad |t| < 1.$$

It remains to find a convex function $\widetilde{h}(x) \geqslant \widetilde{a}(x)$ on $\mathbb{R}$ having a structure described in Section 2. Then

$$h(t) = \sup_x(xt - \widetilde{h}(x)) \leqslant \sup_x(xt - \widetilde{a}(x)) = h_0(t) \leqslant a(t) \leqslant \frac{1}{W(t)},$$

by Lemma 3, there are no unconditional exponential bases in the space $L_2(e^{-h})$.

We define the function $\alpha(x)$ by recurrent relations on the segments $[2^n; 2^{n+1}]$. We choose a number $A \in (1;2)$. Let $l_0(x)$ be a linear function such that $l_0(1) = b(1)$, $l_0(2) = \min(b(2), \sqrt{A}b(1))$ and for $x \in [1;2]$ we let $\alpha(x) = l_0(x)$. By the concavity of $b(x)$ we have $\alpha(x) \geqslant b(x)$, $x \in [1;2]$. Once we have defined the function α on the segments $[2^k; 2^{k+1}]$ as $k \leqslant n-1$, by l_n we denote a linear function such that $l_n(2^n) = \alpha(2^n)$, $l_n(2^{n+1}) = \min(b(2^{n+1}), \sqrt{A}\alpha(2^n)$ and for $x \in [2^n; 2^{n+1}]$ we let $\alpha(x) = l_n(x)$. The function α

introduced in this way is continuous, increases to infinity and satisfy the inequality $\alpha(x) \leqslant b(x)$. Indeed, if for some sequence n_k we have $\alpha(2^{n_k}) = b(2^{n_k})$, then

$$\lim_{x\longrightarrow\infty} \alpha(x) = \lim_{k\longrightarrow\infty} b(2^{n_k}) = \infty,$$

and if after some index m $\alpha(2^n) = \sqrt{A}\alpha(2^{n-1})$, then $\alpha(2^n) = A^{\frac{n-m}{2}}\alpha(2^m) \longrightarrow \infty$ as $n \longrightarrow \infty$.

The function α satisfies condition (2). We take $x \in [2^n; 2^{n+1}]$, $n \in \mathbb{N}$. Then

$$\alpha(2x) \leqslant \alpha(2^{n+2}) \leqslant \sqrt{A}\alpha(2n+1) \leqslant A\alpha(2n) \leqslant A\alpha(x).$$

As in Section 3, by the function α we construct the concave increasing function u on $[1;\infty)$ and the convex function $\widetilde{h}$ on $\mathbb{R}$. By Lemma 1 we can normalized the function $u(x)$ by a multiplicative constant so that

$$u(x) \leqslant \alpha(x), \quad x \geqslant 1.$$

By construction, $\widetilde{h}(x) = x - u(x) \geqslant x - b(x) = \widetilde{a}(x)$. The proof is complete.

BIBLIOGRAPHY

1. N.K. Bari. *On bases in Hilbert space* // Dokl. Akad. Nauk SSSR. **54**, 383-386 (1946). (in Russian.)
2. S.V. Hruščev, N.K. Nikol'skii, B.S. Pavlov. *Unconditional bases of exponentials and of reproducing kernels* // Complex Analysis and Spectral Theory. Lecture Notes in Mathematics. **864**, 214-335 (1981).
3. B.S. Pavlov. *Basicity of an exponential system and Muckenhoupts condition* // Dokl. Acad. Nauk SSSR. **247**, 37-40 (1979). [Sov. Math. Dokl. **20**, 655-659 (1979).]
4. A.A. Putintseva. *Riesz bases in weighted spaces* // Ufimskij Matem. Zhurn. **3**:1, 47-52 (2011). [Ufa Math. J. **3**:1, 45-50 (2011).]
5. B.Ya. Levin, Yu.I. Lyubarskii. *Interpolation by means of special classes of entire functions and related expansions in series of exponentials* // Izv. AN SSSR. Ser. Matem. **39**:3, 657-702 (1975). [Math. USSR. Izv. **9**:3, 621-662 (1975).]
6. Yu.I. Lyubarskii. *Exponential series in Smirnov spaces and interpolation by entire functions of special classes* // Izv. AN SSSR. Ser. Matem. **52**:3, 559-580 (1988). [Math. USSR-Izv. **32**:3, 563-586 (1989).]
7. V.I. Lutsenko. *Unconditional bases of exponentials in Smirnov space.* PhD thesis. Institute of Mathematics, Ufa Sci. Center, RAS. 1992. (in Russian).
8. K.P. Isaev, R.S. Yulmukhametov. *Laplace transforms of functionals on Bergman spaces* // Izv. RAN. Ser. matem. **68**:1, 5-42 (2004). [Izv. Math. **68**:1, 3-41 (2004).]
9. R.A. Bashmakov, A.A. Makhota, K.V. Trunov. *On absence conditions of unconditional bases of exponents* // Ufimskij Matem. Zhurn. **7**:2, 19-34 (2015). [Ufa Math. J. **7**:2, 17-32 (2015).]
10. N. Aronszajn *Theory of reproducing kernels* // Trans. Amer. Math. Soc. **68**:3, 337-404 (1950).
11. K. Seip. *Density theorems for sampling and interpolation in the Bargmann-Fock space. I* // Reine Angew. Math. **1992**:429, 91-106 (1992).
12. K. Seip, R. Wallsten. *Density theorems for sampling and interpolation in the Bargmann-Fock space. II* // Reine Angew. Math. **1992**:429, 107-113 (1992).
13. A. Borichev, R. Dhuez, K. Kellay *Sampling and interpolation in large Bergman and Fock spaces* // J. Funct. Anal. **242**:2, 563-606 (2007).
14. A. Borichev, Yu. Lyubarskii. *Riesz bases of reproducing kernels in Fock type spaces* // J. Inst. Math. Jussieu. **9**:3, 449-461 (2010).
15. K.P. Isaev, R.S. Yulmukhametov. *Unconditional bases of reproducing kernels in Hilbert spaces of entire functions* // Ufimskij Matem. Zhurn. **5**:3, 67-77 (2013). [Ufa Math. J. **5**:3, 67-76 (2013).]
16. V.I. Lutsenko, R.S. Yulmukhametov. *Generalization of the Paley-Wiener theorem in weighted spaces* // Matem. zam. **48**:5, 80-87 (1990). [Math. Notes Acad. Sci. USSR. **48**:5, 1131-1136 (1990).]

17. V.V. Napalkov, R.A. Bashmakov, R.S. Yulmukhametov. *Asymptotic behavior of Laplace integrals and geometric characteristics of convex functions* // Doklady AN. **413**:1, 20-22 (2007). [Doklady Math. **75**:2, 190-192 (2007).]
18. R.A. Bashmakov, K.P. Isaev, R.S. Yulmukhametov. *On geometric characteristics of convex functions and Laplace integrals* // Ufimskij Matem. Zhurn. **2**:1, 3-16 (2010).

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