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QUANTUM ASPECTS OF THE INTEGRABILITY OF THE THIRD PAINLEVÉ EQUATION AND A NON-STATIONARY SCHRÖDINGER EQUATION WITH THE MORSE POTENTIAL

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Abstract. In terms of solutions to isomonodromic deformations equation for the third Painlevé equation, we write out the simultaneous solution of three linear partial differential equations. The first of them is a quantum analogue of the linearization of the third Painlevé equation written in one of the forms. The second is an analogue of the non-stationary Schrödinger equation determined by the Hamiltonian structure of this ordinary differential equation. The third is a first order equation with the coefficients depending explicitly on the solutions to the third Painlevé equation. For the autonomous reduction of the third Painlevé equation this simultaneous solution defines solutions to a non-stationary quantum mechanical Schrödinger equation, which is equivalent to a non-stationary Schrödinger equation with a known Morse potential. These solutions satisfy also linear differential equations with the coefficients depending explicitly on the solutions of the corresponding autonomous Hamiltonian system. It is shown that the condition of global boundedness in the spatial variable of the constructed solution to the Schrödinger equation is related to determining these solutions to the classical Hamiltonian system by Bohr-Sommerfeld rule of the old quantum mechanics.

Keywords: quantization, linearization, Hamiltonian, nonstationary Schrödinger equation, Painlevé equations, isomonodromic deformations, Morse potential.

Mathematics Subject Classification: 34M55

1. Introduction

1.1. All six Painlevé equations can be obtained from the Hamiltonian systems of ordinary differential equations (ODE)

$$\lambda'_\tau = H'_\mu(\tau,\lambda,\mu), \quad \mu'_\tau = -H'_\lambda(\tau,\lambda,\mu), \tag{1}$$

with Hamiltonians quadratic in momentum μ

$$H(\tau,\lambda,\mu) = \alpha(\tau,\lambda)\mu^2 + \beta(\tau,\lambda)\mu + \gamma(\tau,\lambda) \tag{2}$$

by excluding μ [1]–[4]. The components λ and μ of the solutions to corresponding systems (1) are involved in the coefficients of the linear equations of the isomonodromic deformations methods (IDM)

$$V''_{\zeta\zeta} = P(\zeta,\tau,\lambda,\lambda'_\tau)V, \quad V'_\tau = B(\zeta,\tau,\lambda,\lambda'_\tau)V'_\zeta - \frac{B_\zeta(\zeta,\tau,\lambda,\lambda'_\tau)}{2}V, \tag{3}$$

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compatible on the solutions $\lambda(\tau)$ of the corresponding Painlevé equation. There exist various version L, A of pairs of form (3). But the results of papers [5]–[10] propose that, by explicit changes including, generally speaking, transforms of Fourier-Laplace type, all these pairs can be reduced to pairs written out in [11]).

1.2. This does not exhaust the connection of the Hamiltonians for Painlevé equation with their L, A pairs in IDM; for all solutions $\lambda(\tau)$ to each of the Painlevé equations there exist [12] Hamiltonians $H = H_j(\tau, \lambda, \mu) \quad (j = 1, \dots, 6)$ of form (2) such that $\lambda(\tau)$ is a component of solution to system (1) and such that the equations

$$\Psi'_\tau = H(\tau, \zeta, \frac{\partial}{\partial\zeta})\Psi \tag{4}$$

have solutions, which by the explicit formulae $\Psi = S(\tau, \zeta, \lambda(\tau), \lambda'_\tau(\tau))V$ are determined via simultaneous solutions to pairs (3) written out in [11]. Following the terminology of paper [9], we call these evolution equations similar to quantum-mechanical non-stationary Schrödinger equations ($\hbar = 2\pi h$, h is the Planck constant)

$$i\hbar\Psi'_t = H(t, \zeta, -i\hbar\frac{\partial}{\partial\zeta})\Psi, \tag{5}$$

as "quantization" of the Painlevé equations. In this way, L, A pairs (3) for the Painlevé equations determine [9] a kind of separation of variables allowing one to construct particular solutions to corresponding equations (4) in terms of linear ODEs with coefficients depending explicitly on ζ and on the solutions $\lambda(\tau)$ to nonlinear Painlevé equations.

Formulae for the third and fifth Painlevé equations in [12] involves some inaccuracies, which can be easily corrected. The main part of the paper begin with Section 2, where by means of formulae and arguments, the mentioned inaccuracies in [12] are corrected for the studied object in our paper, the third Painlevé equation (a, b, c, d are arbitrary complex constants)

$$\lambda''_{\tau\tau} = \frac{(\lambda'_\tau)^2}{\lambda} - \frac{\lambda'_\tau}{\tau} + \frac{2\lambda^2}{\tau^2}(d + 2a\lambda) + \frac{2b}{\tau} - \frac{4c}{\lambda}. \tag{6}$$

1.3. After [12], the connections between solutions to linear equations of IDM with evolution equations in the quantum theory was discussed in the series of works [9], [13]–[30]; starting from [19], the possibilities of constructing solution to known equation of quantum theory of a field were also studied. In particular, A.V. Zabrodin and A.V. Zotov [14] assume the connection described in Subsection 1.2 as a basis of their classification of isomonodromic Hamiltonian system of ODEs (1), (2); by isomonodromic we mean systems to which IDM can be applied. As the result of this classification, it was established that under some additional restrictions, the representation of these Hamiltonian systems as the compatibility of the equation

$$\Psi'_\tau = \frac{1}{2}\Psi''_{\zeta\zeta} + U(\zeta, \tau)\Psi \tag{7}$$

with the linear first order partial differential equation

$$\Psi'_\tau = B(\zeta, \tau, \lambda, \lambda'_\tau)\Psi'_\zeta + C(\zeta, \tau, \lambda, \lambda'_\tau)\Psi \tag{8}$$

is possible only in the cases when the coordinate part λ of these systems satisfies one of the Painlevé equations. The classification made in [14] show that the using of "quantizations" (4) can be assumed as a basis for a new definition of isomonodromic Hamiltonian systems of form (1), (2).

At the same time, in the opinion of the author of the present work, the classification by A.V. Zabrodin and A.V. Zotov has a disadvantage related with the aforementioned additional restrictions for the coefficients of equations (7), (8). Say, instead of additional conditions in [14], one can assume that the coefficient $B(\zeta, \tau, \lambda, \lambda'_\tau)$ in equation (8) is independent of λ'_τ

and the coefficient and the potential U in equation (7) are locally analytic functions of their arguments and can consider $\lambda(\tau)$ and $\lambda'_\tau(\tau)$ as independent variables. Under this condition, equations (7) and (8) turn out to be compatible only on a finite set of Hamiltonian systems (1), (2) equivalent to the known list of Painlevé equations. The arguments proving this statement are not so simple and short. But since their result just reproduces the classification by A.V. Zabrodin and A.V. Zotov, in the author's opinion, their detailed description does not seem to be necessary. However, for instance, the apriori assumption for B to be independent of λ'_τ does not look natural. In this sense, additional restrictions in [14] look much better but part of them miss complete naturalness. As it was said, for instance, in Remark 3 in work [31], certain disadvantages appear in other definition of Painlevé equations. This is why the classification of isomonodromic Hamiltonian systems of ODEs of form (1), (2) based no natural postulate seems to be still rather topical problem.

In order to make such classification, in [16], together with the evolution equation

$$Q'_\tau = \frac{1}{2}Q''_{zz} + [U(z,\tau) + \Gamma(\tau,\varphi,\varphi'_\tau)]Q, \tag{9}$$

there was proposed to employ the equation of the form

$$\begin{aligned} Q''_{\tau\tau} =& A(\tau,z,\varphi)Q''_{zz} + D(\tau,z,\varphi)(Q'_z)'_\tau + [E_1(\tau,z,\varphi)\varphi'_\tau + E_0(\tau,z,\varphi)]Q'_\tau \\ &+ [F_1(\tau,z,\varphi)\varphi'_\tau + F_0(\tau,z,\varphi)]Q'_z + [J_2(\tau,z,\varphi)(\varphi'_\tau)^2 + J_1(\tau,z,\varphi)\varphi'_\tau + J_0(\tau,z,\varphi)]Q. \end{aligned} \tag{10}$$

It should be compatible with (9) on the solutions of ODE

$$\varphi''_{\tau\tau} = f(\tau,\varphi).$$

The change

$$\varphi = \int_{\lambda_*}^{\lambda} \frac{d\nu}{\sqrt{\alpha(\tau,\nu)}}, \qquad \lambda_* - const,$$

transforms the second order ODE

$$\lambda''_{\tau\tau} = \frac{\alpha'_\lambda(\tau,\lambda)}{2\alpha(\tau,\lambda)}(\lambda'_\tau)^2 + \frac{\alpha'_\tau(\tau,\lambda)}{\alpha(\tau,\lambda)}\lambda'_\tau + M(\tau,\lambda) \tag{11}$$

to such form; the latter equation are to be satisfied by the components $\lambda(\tau)$ of Hamiltonian systems (1), (2). By means of explicit transformations, "quantizations" (4) of these Hamiltonian systems are easily reduced to the equations of form (9). The coefficients U, A, D, E_k, F_k, J_k of equations (9), (10) are assumed to be locally analytic functions of their arguments.

As it was mentioned in [16], for all Painlevé equations we have the statement on the compatibility of evolution equation (9) with equation (10) on the solutions of the corresponding equations

$$\varphi''_{\tau\tau} = f(\tau,\varphi);$$

at that, the coefficients A, D, F_k should be identically zero and equation (10) becomes a linear second order ODE with the independent variable τ. The naturalness of the assumption on the form of equation (10) was illustrated in [16] by considering a series of Painlevé equations since for these Painlevé equations, the compatible with (9) equations have exactly such form and by explicit changes they are reduced to "quantum" linearization of these ODEs; we also mention that these linearization appear as a result of the procedure introduced in the same work [16].

In Section 3 we show that the solutions to "quantizations" of ODEs (6) constructed in Section 2 are exact solutions to equations equivalent to "quantum" linearization of one form of the third Painlevé equation. And this confirms once again that the assumption on compatibility of equations (9) and (10) for isomonodormic Hamiltonian systems (1), (2) is natural.

1.4. In the general situations the solutions to Painlevé equations are transcendental; they are represented neither in closed form by quadratures no in terms of known classical special functions. But as it is known, for certain values of parameters in some of the Painlevé equations, such representations are possible. At that, equations (3) of IDM can be solved explicitly and therefore, there appears a possibility to construct explicitly particular solutions of corresponding evolution equations (4). There is a series of autonomous reductions of Painlevé equations such that it is possible to construct explicit solutions to the corresponding non-stationary quantum-mechanical Schrödginer equations (5) by means of their L, A pairs.

In this way, in Section 4 of the present paper we construct solutions to the non-stationary Schrödinger equations

$$i\hbar\Psi'_t = -\hbar^2\frac{\zeta^2\Psi''_{\zeta\zeta}}{2} - \alpha(2\zeta-\zeta^2)\Psi \quad (\alpha>0-const). \tag{12}$$

By the changes

$$\zeta=\exp(-x), \quad \Psi=\zeta^{\frac{1}{2}}\exp\left(-i\frac{\hbar t}{8}\right)G(x,t) \tag{13}$$

this equation is reduced to the Schrödinger equation

$$i\hbar G'_t = -\hbar^2\frac{G''_{xx}}{2} + \alpha(\exp(-2x)-2\exp(-x))G \tag{14}$$

with the well-known Morse potential.

Schrödinger equation (14) is determined by the Hamiltonian

$$H(q,p)=H_M(q,p)=\frac{p^2}{2}+\alpha(\exp(-2q)-2\exp(-q)) \tag{15}$$

of the autonomous Hamiltonian system

$$q'_t = H'_p(q,p), \qquad p'_t=-H'_q(q,p). \tag{16}$$

Excluding here the momentum p gives the ODE

$$q''_{tt}=2\alpha(\exp(-2q)-\exp(-q)), \tag{17}$$

which is point equivalent to a particular case of the third Painlevé equation (6)

$$\lambda''_{\tau\tau}=\frac{(\lambda'_\tau)^2}{\lambda}-\frac{\lambda'_\tau}{\tau}+\frac{4a\lambda^2}{\tau^2}(\lambda-1). \tag{18}$$

The solutions to Schrödinger equation (12), which we construct below, satisfy the first order equation

$$\Psi'_t = K(\hbar,t,\zeta,q(t),p(t))\Psi'_\zeta + M(\hbar,t,\zeta,q(t),p(t))\Psi, \tag{19}$$

where the coefficients depend on the solutions to systems (15), (16): these coefficient are expressed explicitly in terms of the coefficients P and B of the pair of equations (3) of IDM for the third Painlevé equation in [11].

Among these solutions, we select a discrete series Ψ_n $(n=1,2,\dots)$, whose elements vanishes as $\zeta=0$ and $\zeta\to\infty$. As it is shown in end of Section 4, among periodic nontrivial solutions to Hamiltonian systems (15), (16), this condition selects exactly the same solutions as the *old* version of Bohr-Sommerfeld formula [32, Ch. I, Sect. 15, Formula (17)]:

$$\oint p(q)dq = 2n\pi\hbar \qquad (n=1,2,\dots,\infty), \tag{20}$$

where the integration is made along the periodic trajectory with an energy $H=const.$

Remark 1. *For a wide class of energy Hamiltonians*

$$H(q,p) = \frac{p^2}{2} + U(q) \tag{21}$$

involving Hamiltonian (15), the modern version of the Bohr-Sommerfeld rule is given by the formula

$$\oint p(q)dq = 2(n+1/2)\pi\hbar \qquad (n = 0, 1, \dots, \infty).$$

This formula defines the leading term in the asymptotics for the discrete spectrum $H = H_n$ *as* $\hbar \to 0$ *for the linear differential operator* $H(\zeta, -i\hbar\frac{\partial}{\partial\zeta})$.

2. "Quantizations" of equation $PIII$

2.1. The pair of equations (3) of IDM for third Painlevé equation (6) written in [11] is given by the coefficients

$$B = \frac{\lambda\zeta}{\tau(\zeta-\lambda)}, \quad P = \frac{c\tau^2}{\zeta^4} - \frac{b\tau}{\zeta^3} + \frac{s(\tau)}{\zeta^2} + \frac{d}{\zeta} + a + \frac{3}{4(\zeta-\lambda)^2} - \frac{\lambda + \tau\lambda'_\tau}{2\lambda\zeta(\zeta-\lambda)}, \tag{22}$$

where

$$s(\tau) = \frac{\tau^2(\lambda'_\tau)^2}{4\lambda^2} + \frac{b\tau}{\lambda} - a\lambda^2 - \frac{c\tau^2}{\lambda^2} - d\lambda - 1/4.$$

The formulae

$$s(\tau) = \tau H_{(3)}(\tau, \lambda, \mu) - 1/4$$

and

$$s(\tau) = \tau H_{(III)}(\tau, \lambda, \mu)$$

express the functions s in terms of the Hamiltonians

$$H = H_{(3)} = \frac{\lambda^2\mu^2}{\tau} + \frac{b}{\lambda} - \frac{a\lambda^2}{\tau} - \frac{c\tau}{\lambda^2} - \frac{d\lambda}{\tau} \tag{23}$$

and, respectively,

$$H = H_{(III)} = \frac{\lambda^2\mu^2 + \lambda\mu}{\tau} + \frac{b}{\lambda} - \frac{a\lambda^2}{\tau} - \frac{c\tau}{\lambda^2} - \frac{d\lambda}{\tau} \tag{24}$$

of two Hamiltonian systems (1); excluding here the momentum μ leads us to ODE (6).

The change

$$V = \frac{\zeta}{\sqrt{(\zeta-\lambda)}}\Phi$$

transforms the simultaneous solutions to equations (3) with coefficients (22) into the simultaneous solutions of the pair of equations

$$\zeta^2\Phi''_{\zeta\zeta} + \zeta\Phi'_\zeta = \frac{\lambda}{\zeta-\lambda}\left(\zeta\Phi'_\zeta - \frac{\tau\lambda'_\tau - \lambda}{2\lambda}\Phi\right) + \left(\frac{c\tau^2}{\zeta^2} - \frac{b\tau}{\zeta} + d\zeta + a\zeta^2 + s(\tau) - \frac{\tau\lambda'_\tau}{2\lambda} + \frac{1}{2}\right)\Phi, \tag{25}$$

$$\tau\Phi'_\tau = \frac{\lambda}{\zeta-\lambda}\left(\zeta\Phi'_\zeta - \frac{\tau\lambda'_\tau - \lambda}{2\lambda}\Phi\right), \tag{26}$$

which, therefore, satisfy the evolution equation

$$\tau\Phi'_\tau = \zeta^2\Phi''_{\zeta\zeta} + \zeta\Phi'_\zeta - \left(\frac{c\tau^2}{\zeta^2} - \frac{b\tau}{\zeta} + d\zeta + a\zeta^2 + s(\tau) - \frac{\tau\lambda'_\tau}{2\lambda} + \frac{1}{2}\right)\Phi. \tag{27}$$

In its turn, the solutions to the latter equation are related to the solutions of the evolution equations

$$\Psi'_\tau = \frac{\zeta^2\Psi''_{\zeta\zeta}}{\tau} - \left(\frac{c\tau}{\zeta^2} - \frac{b}{\zeta} + \frac{d\zeta + a\zeta^2}{\tau}\right)\Psi \tag{28}$$

and

$$G'_\tau = \frac{\zeta^2 G''_{\zeta\zeta} + \zeta G'_\zeta}{\tau} - \left(\frac{c\tau}{\zeta^2} - \frac{b}{\zeta} + \frac{d\zeta + a\zeta^2}{\tau}\right)G. \tag{29}$$

by the formulae

$$\Phi = \left(\frac{\lambda}{\zeta}\right)^{1/2}\tau^{-1/4}\exp\left(-\int\limits_{\tau_*}^{\tau}\left(\frac{s(\nu)}{\nu}\right)d\nu\right)\Psi \tag{30}$$

and

$$\Phi = \left(\frac{\lambda}{\tau}\right)^{1/2}\exp\left(-\int\limits_{\tau_*}^{\tau}\left(\frac{s(\nu)}{\nu}\right)d\nu\right)G, \tag{31}$$

where τ_* is a constant.

By the operator relation

$$\frac{\partial}{\partial\zeta}\zeta - \zeta\frac{\partial}{\partial\zeta} = 1,$$

each of the above evolution equations can be symbolically written as "quantizations" of form (4) for the third Painlevé equation (6) according to each of two Hamiltonians (23), (24) of Hamiltonians systems (1). As we see, while writing out such type of "quantizations", there is some freedom related to the validity of this operator relation.

2.2. We note that from the point of view of consistency of "quantizing" substitutions

$$\lambda(\tau) \to \zeta, \qquad \mu(\tau) \to \frac{\partial}{\partial\zeta} \tag{32}$$

formed by two solutions of different "quantizations" of form (4) according to Hamiltonian system (1), (23) described in Subsection 2.1, the solution Ψ to equation (28) is more preferable than the solution G to equation (29).

Indeed, it follows from equation (26) and changes (30), (31) that Ψ and G are solutions to the first order equations

$$\left(\tau\Psi'_\tau + \left[-\frac{(\tau\lambda'_\tau)^2}{4\lambda^2} + \frac{\tau\lambda'_\tau}{2\lambda} + a\lambda^2 + d\lambda - \frac{b\tau}{\lambda} + \frac{c\tau^2}{\lambda^2}\right]\Psi\right)(\zeta - \lambda) = \lambda\left(\zeta\Psi'_\zeta - \frac{\tau\lambda'_\tau}{2\lambda}\Psi\right) \tag{33}$$

and, respectively,

$$\left(\tau G'_\tau + \left[-\frac{(\tau\lambda'_\tau)^2}{4\lambda^2} + \frac{\tau\lambda'_\tau}{2\lambda} + a\lambda^2 + d\lambda - \frac{b\tau}{\lambda} + \frac{c\tau^2}{\lambda^2} - \frac{1}{4}\right]G\right)(\zeta - \lambda) = \lambda\left[\zeta G'_\zeta + \left(\frac{1}{2} - \frac{\tau\lambda'_\tau}{2\lambda}\right)G\right]. \tag{34}$$

On each of the curves $\zeta = \lambda(\tau)$, the solution Ψ to equation (33) satisfies the identity

$$\Psi'_\zeta \equiv \frac{\lambda'_\tau(\tau)}{2\lambda^2(\tau)}\Psi, \tag{35}$$

where the function $\frac{\lambda'_\tau}{2\lambda^2}$ coincides with the momentum μ of Hamiltonian system (1), (23). On such curves, the solution G of equation (34) satisfies the identity

$$G'_\zeta \equiv \left(\frac{\lambda'_\tau(\tau)}{2\lambda^2(\tau)} - \frac{1}{2\lambda(\tau)}\right)G \tag{36}$$

and not the identity

$$G'_{\zeta} \equiv \frac{\lambda'_{\tau}(\tau)}{2\lambda^2(\tau)} G.$$

On the other hand, the function $(\frac{\lambda'_{\tau}}{2\lambda^2} - \frac{1}{2\lambda})$ coincides with the momentum μ of Hamiltonian system (1), (24). This is why, same identities (35) and (36) show that from the point of view of consistency of "quantizing" substitutions (32) formed by the constructed solutions of two "quantizations" (4) according to this Hamiltonian system, the solution G is more preferable than Ψ.

3. "Quantum" linearization of Painlevé equation III

3.1. To each ODE of form (11), we can associate Hamiltonian system (1), (2) with the coordinate $\lambda = \lambda(\tau)$. All such ODEs admit a quantum analogue of the linearization procedure introduced in work [16]. This procedure depending on two fixed numbers ε and k assumes the existence of two successive steps applied to the result of the linearization

$$\Lambda''_{\tau\tau} = [2K(\tau,\lambda)\lambda' + L(\tau,\lambda)]\,\Lambda'_{\tau} + [K'_{\lambda}(\tau,\lambda)(\lambda')^2 + L'_{\lambda}(\tau,\lambda)\lambda'_{\tau} + M'_{\lambda}(\tau,\lambda)]\Lambda \tag{37}$$

of ODE (11). Here

$$K(\tau,\lambda) = \frac{\alpha'_{\lambda}(\tau,\lambda)}{2\alpha(\tau,\lambda)}, \qquad L(\tau,\lambda) = \frac{\alpha'_{\tau}(\tau,\lambda)}{\alpha(\tau,\lambda)},$$

α and M locally analytic functions of their arguments. These steps are

a) at some points in the right hand side of linear ODE (37), the classical coordinates $\lambda(\tau)$ and momenta $\mu(\tau)$ of Hamiltonian system (1), (2) are replaced by their quantum analogues, x, and, respectively, the differential operator $k\frac{\partial}{\partial x}$ (k is constant);

b) in (37), the derivatives $\frac{\partial^m \Lambda}{\partial \tau^m}$ are replaced by $\varepsilon^m \frac{\partial^m \Phi}{\partial \tau^m}$ $(m = 0, 1, 2)$.

Of course, step a) is defined not strictly. However, in any case, the "quantum" linearization of each equation (11) obtained by these two step is reduced to an equation of form (10) by an explicit change. This change can be chosen so that "quantization" (4) in accordance with Hamiltonian (2) reduces it to an evolution equation of form (9).

For a series of Painlevé equations, in [16], the following fact was mentioned: each solution to each of these equations can be represented as the coordinate λ of Hamiltonian system (1),(2), for which there exists "quantization" (4) and such "quantum" linearization of this Painlevé equation, that corresponding equations (7) and (10) have a simultaneous solution explicitly expressed in terms of the simultaneous solution of the pair of equations (3) of IDM for this Painlevé equation.

The mentioned series of Painlevé equation involves:

– ODE (a, b, c, d are arbitrary constants)

$$\lambda''_{\tau\tau} = a_4(2\lambda^3 + \tau\lambda) + a_3(6\lambda^2 + \tau) + a_2\lambda + a_1,$$

which, as particular cases, involves the first and second Painlevé equations;

– the fourth Painlevé equation;

– ODE of Painlevé type 34. By non-point changes this ODE is reduced to the second Painlevé equations; for quantum aspect of the integrability of ODE of Painlevé type 34 see also paper [9].

In this section we show that this series also involves the ODE

$$y''_{\theta\theta} = \frac{(y'_{\theta})^2}{y} - 2d - \frac{4a}{y} - 2by^2\exp(\theta) + 4cy^3\exp(2\theta) \tag{38}$$

being one of the forms of the third canonical Painlevé equation. This equation appears from (6) under the point change

$$\lambda = \frac{1}{y}, \qquad \tau = \exp(\theta). \tag{39}$$

A solution to this equation defines the coordinate y and the momentum $\omega = \frac{y'_\theta}{2y^2}$ of the Hamiltonian system

$$y'_\theta = H'_\omega(\theta, y, \omega), \quad \omega'_\theta = -H'_y(\theta, y, \omega), \tag{40}$$

with the Hamiltonian

$$H(\theta, y, \omega) = H_{III}(\theta, y, \omega) = y^2\omega^2 - cy^2\exp(2\theta) + by\exp(\theta) - \frac{d}{y} - \frac{a}{y^2}. \tag{41}$$

3.2. The linearization of ODE (38) is of the form

$$Y''_{\theta\theta} = \frac{2y'_\theta}{y}Y'_\theta + \left(-\frac{(y'_\theta)^2}{y^2} + \frac{4a}{y^2} - 4by\exp(\theta) + 12cy^2\exp(2\theta)\right)Y. \tag{42}$$

We then note that the simultaneous solution $\Phi(\zeta, \lambda, \mu)$ of equations (26), (27) satisfies also the following linear second order ODE in variable τ:

$$\Phi''_{\tau\tau} = \left(\frac{\lambda'_\tau}{\lambda} - \frac{1}{\tau}\right)\Phi'_\tau + \left(\frac{a\lambda^2}{\tau^2} - \frac{b}{\zeta\tau} + \frac{c(\lambda + 2\zeta)}{\zeta^2\lambda}\right)\Phi. \tag{43}$$

Indeed, taking into consideration (27), by differentiating equation (26) w.r.t. the variable ζ we obtain the identity

$$\tau\Phi''_{\tau\zeta} = -\frac{(\lambda + \tau\lambda'_\tau)}{2\lambda\zeta}\tau\Phi'_\tau + \left(\frac{d\lambda}{\zeta} + \frac{a\lambda(\zeta + \lambda)}{\zeta} + \frac{b\tau}{\zeta^2} - \frac{c\tau^2(\lambda + \zeta)}{\zeta^3\lambda}\right)\Phi. \tag{44}$$

By this identity and by differentiating equation (26) with respect to variable τ we obtain immediately ODE (43).

And vice versa, the simultaneous solutions to equations (26) and (43) satisfy also ODE (25) and equation (27).

By the changes

$$\Phi = \left(\frac{\lambda}{\zeta}\right)^{1/2} W, \quad \zeta = \frac{1}{\rho} \tag{45}$$

and change (39), the simultaneous solutions to equations (25)–(27), (43), (44) defines the simultaneouss solution to the following four linear equations

$$\left(W'_\theta - \frac{y'_\theta}{2y}W\right)(\rho - y) = \rho\left(\rho W'_\rho - \frac{y'_\theta}{2y}W\right), \tag{46}$$

$$\begin{aligned} W'_\theta =& \rho^2 W''_{\rho\rho} + 2\rho W'_\rho - \Bigg(c\rho^2\exp(2\theta) - b\rho\exp(\theta) + \frac{a}{\rho^2} + \frac{d}{\rho} \\ &+ \frac{(y'_\theta)^2}{4y^2} - cy^2\exp(2\theta) + by\exp(\theta) - \frac{a}{y^2} - \frac{d}{y}\Bigg)W \\ =& \rho^2 W''_{\rho\rho} + 2\rho W'_\rho - \left(c\rho^2\exp(2\theta) - b\rho\exp(\theta) + \frac{a}{\rho^2} + \frac{d}{\rho} + H_{III}(\theta, y, \omega)\right)W, \end{aligned} \tag{47}$$

$$\rho^2 W''_{\rho\theta} = \frac{y'_\theta}{2y}\rho^2 W'_\rho - \frac{y'_\theta}{2y}\rho W'_\theta + \left(\frac{(y'_\theta)^2\rho}{4y^2} - \frac{d\rho}{y} - \frac{a(\rho + y)}{y^2} - b\exp(\theta)\rho^2 + c\exp(2\theta)(\rho^2 y + \rho^3)\right)W,$$

$$W''_{\theta\theta} = \left(\frac{(y'_\theta)^2}{4y^2} - \frac{d}{y} - \frac{a}{y^2} - b\exp(\theta)(\rho + y) + c\exp(2\theta)(\rho^2 + 2\rho y + 2y^2)\right)W,$$

where the functions H_{III} and $\omega = y'_\theta/(2y^2)$ define the Hamiltonian and the momentum for Hamiltonian system (40), (41). Two latter equations imply the equation

$$\varepsilon^2 W''_{\theta\theta} = 2\varepsilon\rho W''_{\rho\theta} + \varepsilon\frac{y'_\theta}{y}W'_\theta - 2\frac{y'_\theta\rho}{y}W'_\rho + \left(\frac{4a}{\rho y} - 4b\exp(\theta)y + 4c\exp(2\theta)(\rho y + 2y^2)\right)W. \quad (48)$$

for $\varepsilon = 2$. This equation leads to the linearization of the third Painlevé equation (42) as a result of formal substitutions

$$\rho \to y, \quad \frac{\partial}{\partial\rho} \to \omega = \frac{y'_\theta}{2y^2}, \quad \varepsilon^m\frac{\partial^m W}{\partial\theta^m} \to \frac{\partial^m Y}{\partial^m\theta} \quad (m = 0, 1, 2).$$

This is partial differential equation (48) is one of possible “quantum” linearizations of form (38) of the third Painlevé equation.

The form of changes (39), (45) implies that the solution Ψ of “quantization” (28) of form (6) of the third Painlevé equation in accordance with Hamiltonian (23) satisfies also the evolution equation

$$\Psi'_\theta = \rho^2\Psi''_{\rho\rho} + 2\rho\Psi'_\rho - \left(c\rho^2\exp(2\theta) - b\rho\exp(\theta) + \frac{a}{\rho^2} + \frac{d}{\rho}\right)\Psi.$$

This evolution equation can be symbolically written as “quantization”

$$\Psi'_\theta = H_{III}(\theta, \rho, \frac{\partial}{\partial\rho})\Psi$$

of form (38) of the third Painlevé equation determined by Hamiltonian system (40), (41). Thus, we indeed have the compatibility of evolution equation (47) obtained from this “quantization” by a simple explicit change and “quantum” linearization (48) of form (38) of the third Painlevé equation.

3.3. It follows from first order equation (46) for W that on the curves $\rho = y(\theta)$ determined by the coordinate y of Hamiltonian system (40), (41), the identity $W'_\rho \equiv \frac{y'_\theta}{2y^2}W = \omega(\theta)W$ holds true, where the function ω is the momentum of this Hamiltonian system.

The solution to each of other canonical Painlevé equations (as well as of Painlevé equation of type 34) can be represented as the coordinate for Hamiltonian system (1), (2), for which we have “quantization” (4) according to Hamiltonian (2) with a solution $\Psi(t,\zeta)$ such that

– an explicit change of form $\Psi = S(\tau, \zeta, \lambda(\tau), \lambda'_\tau(\tau))V$ expresses it in terms of simultaneous solutions V of pairs of equations (3) of IDM for the Painlevé equation for the coordinate λ;

– by means of explicit changes (including the change of independent spatial variable) it is reduced to the function Q being a simultaneous solution of equations of form (9) and (10);

– on the curves $\zeta = \lambda(\tau)$, the solution Ψ satisfies the identity $\Psi'_\zeta \equiv \mu(\tau)\Psi$.

Due to this reason, as it was mentioned in [16], the assumption on compatibility of linear equations (9), (10) can be completed by the assumption on the validity of the identity

$$(Q'_z - [\varphi'_\tau\nu(\tau,\varphi) + \xi(\tau,\varphi)]Q)\,|_{z=\varphi(\tau)} \equiv 0$$

for the simultaneous solution to these equations, where ν and ξ are locally analytic functions of their arguments. It was shown above that some quantum aspects of the integrability of the third Painlevé equation confirm not only the validity but also the naturalness of both these assumptions.

4. Non-stationary Schrödinger equation with Morse potential

4.1. Under the point change

$$\lambda = \exp(-q), \qquad \tau = \exp(\theta), \quad (49)$$

third Painlevé equation (6) becomes

$$q''_{\theta\theta} = -2\exp(-q)(d + 2a\exp(-q)) - 2b\exp(\theta + q) + 4c\exp(2(\theta + q)).$$

The reduction of this form of the Painlevé equation

$$b = c = 0, \qquad d = -2a \tag{50}$$

gives the autonomous ODE

$$q''_{\theta\theta} = -4a(\exp(-2q) - \exp(-q)). \tag{51}$$

For real positive a, by the changes

$$t = -\frac{2i}{\hbar}\theta, \quad a = \frac{2\alpha}{\hbar^2} \tag{52}$$

this ODE is reduced to ODE (17) (for physical applications of solutions to ODE (17) see [33].)

Employing the just described reduction and the results of the previous sections, below we construct simultaneous solutions $G(\hbar, t, x)$ to Schrödinger equation (14) and linear first order partial differential equation

$$(\exp(-x) - \exp(-q))\left(2G'_t - \frac{i\hbar G}{4} + \frac{2iH_M G}{\hbar} - q'_t G\right) = \exp(-q)\left[i\hbar\left(\frac{G}{2} - G'_x\right) + q'_t G\right], \tag{53}$$

with the coefficients written explicitly in terms of the solutions ($q(t)$,$p(t)$) to system (15), (16). Up to the end of this section we consider only periodic solutions to this system not coinciding with the trivial solution $p(t) = 0$, $q(t) = 0$. These periodic solutions satisfy the inequalities [34, Ch. III, Sect. 23, Fig. 3]

$$0 < -H_M < \alpha, \quad -\alpha \leqslant \alpha(\exp(-2q) - 2\exp(-q)) < H_M(p, q). \tag{54}$$

By changes (13), these solutions to the Schrödinger equation are related to the solutions of Schrödinger equations (12), which are determined by the conservative Hamiltonian system

$$\lambda'_t = H'_\mu(\lambda, \mu), \qquad \mu'_t = -H'_\lambda(\lambda, \mu), \tag{55}$$

with the Hamiltonian

$$H(\lambda, \mu) = H_{MM}(\lambda, \mu) = \frac{\lambda^2\mu^2}{2} - \alpha(2\lambda - \lambda^2). \tag{56}$$

4.2. In case(50) ODE (43) casts into the form

$$\Phi''_{\tau\tau} = (\frac{\lambda'_\tau}{\lambda} - \frac{1}{\tau})\Phi'_\tau + \frac{a\lambda^2}{\tau^2}\Phi, \tag{57}$$

whose coefficients depend on the solutions $\lambda(\tau)$ to autonomous ODE (18) begin the reduction of third Painlevé equation (6). At that, equations (27), (28) and (29) are reduced to simpler equations

$$\tau\Phi'_\tau = \zeta^2\Phi''_{\zeta\zeta} + \zeta\Phi'_\zeta + \left[2a\zeta - a\zeta^2 - \frac{(\tau\lambda'_\tau)^2}{4\lambda^2} - \frac{\tau\lambda'_\tau}{2\lambda} + 1/4 - 2a\lambda + a\lambda^2\right]\Phi, \tag{58}$$

$$\tau\Psi'_\tau = \zeta^2\Psi''_{\zeta\zeta} + [2a\zeta - a\zeta^2]\Psi, \tag{59}$$

$$\tau G'_\tau = \zeta^2 G''_{\zeta\zeta} + \zeta G'_\zeta + [2a\zeta - a\zeta^2]G. \tag{60}$$

Under the change of τ to the variable θ by formula (49), ODE (18) is transformed to the autonomous ODE

$$\lambda''_{\theta\theta} = \frac{(\lambda'_\theta)^2}{\lambda} + 4a\lambda^2(\lambda - 1),$$

which is obtained from ODE (51) by the change $\lambda = \exp(-q)$. First order equation (26), ODE (57) and evolution equations (58) – (60) are transformed, respectively, to the equations

$$\Phi'_\theta = \frac{\lambda}{\zeta - \lambda}\left(\zeta\Phi'_\zeta - \frac{\lambda'_\theta - \lambda}{2\lambda}\Phi\right), \tag{61}$$

$$\Phi''_{\theta\theta} = \frac{\lambda'_\theta}{\lambda}\Phi'_\theta + a\lambda^2\Phi, \tag{62}$$

$$\Phi'_\theta = \zeta^2\Phi''_{\zeta\zeta} + \zeta\Phi'_\zeta + [2a\zeta - a\zeta^2 - \frac{(\lambda'_\theta)^2}{4\lambda^2} - \frac{\lambda'_\theta}{2\lambda} + 1/4 - 2a\lambda + a\lambda^2]\Phi,$$

$$\Psi'_\theta = \zeta^2\Psi''_{\zeta\zeta} + [2a\zeta - a\zeta^2]\Psi,$$

$$G'_\theta = \zeta^2 G''_{\zeta\zeta} + \zeta G'_\zeta + [2a\zeta - a\zeta^2]G. \tag{63}$$

Under reduction (50), Hamiltonian (23) becomes

$$H = H_r(\tau, \lambda, \mu) = \frac{\lambda^2\mu^2 - a\lambda^2 + 2a\lambda}{\tau}. \tag{64}$$

Under the change of τ by the independent variable θ in accordance with formula (49), corresponding Hamiltonian system (1) is transformed in the conservative Hamiltonian system

$$\lambda_\theta = K'_\mu, \quad \mu'_\theta = -K'_\lambda \tag{65}$$

with the Hamiltonian independent of time θ:

$$K_{III} = \lambda^2\mu^2 - a\lambda^2 + 2a\lambda = \tau H_r(\tau, \lambda, \mu). \tag{66}$$

It follows from the first equation in this Hamiltonian system that

$$\mu\lambda = \frac{\lambda'_\theta}{2\lambda}. \tag{67}$$

According to (66) and (67), formula (31) can be rewritten as

$$G(\theta, \zeta) = Const\lambda^{-1/2}\exp\left((K_{III} + 1/4)\theta\right)\Phi(\theta, \zeta), \tag{68}$$

4.3. Schrödinger equation (14) and evolution equation (63) are related by the changes

$$t = -\frac{2i}{\hbar}\theta, \quad x = -\ln\zeta, \quad a = \frac{2\alpha}{\hbar^2}. \tag{69}$$

It follows from formulae (67) and (69) that under the changes

$$\lambda(\theta) = \exp(-q(t)), \qquad \hbar\lambda\theta\mu(\theta) = p(t)$$

system (65), (66) is transformed to Hamiltonian system (16) with Hamiltonian (15)

$$H_M = -\frac{\hbar^2}{2}K_{III}.$$

ODE (62) can be solved easily: the change

$$\Phi = \lambda^{1/2}\Upsilon \tag{70}$$

reduces ODE (62) to the linear ODE

$$\Upsilon''_{\theta\theta} = \Delta^2\Upsilon \tag{71}$$

with the coefficient

$$\Delta^2 = \frac{1}{4}\left(\frac{\lambda'_\theta}{\lambda}\right)^2 - a\lambda^2 + 2a\lambda = K_{III} = -\frac{2H_M}{\hbar^2}, \tag{72}$$

which is positive and constant thanks to the first inequality in (54). The general solution to the latter equation is

$$\Upsilon(\theta, \zeta) = R_+(\zeta)\exp(\Delta\theta) + R_-(\zeta)\exp(-\Delta\theta). \tag{73}$$

In what follows we consider separately two cases differing by the sign of nonlinear ODEs

$$\frac{\lambda'_\theta}{2\lambda} = \pm\sqrt{\Delta^2 + a\lambda^2 - 2a\lambda} \tag{74}$$

for $\lambda(\theta)$. Their real solutions are defined by the formulae

$$\delta_\pm \exp\left(\mp 2\Delta\theta\right) = \frac{\Delta^2 - a\lambda + \Delta\sqrt{\Delta^2 + a\lambda^2 - 2a\lambda}}{\lambda} \tag{75}$$

depending on constants $\delta\pm$.

Remark 2. *We assume the inequalities*

$$a > K_{III} > 0, \quad -a \leqslant a(\lambda^2 - 2\lambda) + K_{III} < 0, \tag{76}$$

which are equivalent to inequalities (54). Relations (69), (72), (76) mean that for real values t, both sides of ODE (74) are pure imaginary. Separating real and imaginary parts in identities (75), it is easy to write the dependence of periodic real trajectories $q(t)$ on the real independent variable t. But in what follows we do not need this dependence.

4.4. Together with ODE (71), the functions Υ satisfy also the equation

$$(\zeta - \lambda)\Upsilon'_\theta = \lambda\zeta\Upsilon'_\zeta + (\frac{\lambda}{2} - \zeta\frac{\lambda'_\theta}{2\lambda})\Upsilon, \tag{77}$$

which can be obtained from linear equation (61) by the change (70).

It will be shown in Subsection 5.1, that one of the functions $R\pm$ in representation (73) can be identically zero only in the case of the trivial solution $p = 0, q = 0$ to Hamiltonian system (15), (16). This is why hereafter in this section we assume that $R\pm(\zeta) \not\equiv 0$.

We substitute the right hand side of ODE (74) into equation (77) instead of $\lambda'_\theta/(2\lambda)$, while $\Upsilon(\theta, \zeta)$ is replaced by the right hand side of identity (73). Equating then the coefficients at the powers of $\sqrt{\Delta^2 + a\lambda^2 - 2a\lambda}$ and λ, we obtain that the functions R_+ and R_- satisfy the following systems of linear equations:

1) the system of equations

$$\begin{gathered} \Delta\zeta R'_-(\zeta) + (a\zeta + \frac{\Delta}{2} - \Delta^2)R_- = \delta_+\zeta R_+, \\ \delta_+(\Delta\zeta R'_+(\zeta) + (-a\zeta + \frac{\Delta}{2} + \Delta^2)R_+) = a(\Delta^2 - a)\zeta R_- \end{gathered} \tag{78}$$

in the case of ODE (74) with '+' sign;

2) the system of equations

$$\begin{gathered} \Delta\zeta R'_+(\zeta) + (-a\zeta + \frac{\Delta}{2} + \Delta^2)R_+ = -\delta_-\zeta R_-, \\ \delta_-(\Delta\zeta R'_-(\zeta) + (a\zeta + \frac{\Delta}{2} - \Delta^2)R_-) = -a(\Delta^2 - a)\zeta R_+ \end{gathered} \tag{79}$$

in the case of ODE (74) with '$-$' sign.

Both system (78) and system (79) imply that the functions $R_\pm$ satisfy the following linear second order equations:

$$\zeta^2 R''_\pm + \zeta R'_\pm + (-a\zeta^2 + 2a\zeta - (\Delta \pm \frac{1}{2})^2)R_\pm = 0. \tag{80}$$

By the changes

$$\xi = 2\sqrt{a}\zeta, \quad R_\pm = exp(-\xi/2)\xi^{\Delta\pm 1/2}F_\pm(\xi), \tag{81}$$

equations (80) are reduced to the ODE

$$\xi F''_\pm + (2(\Delta \pm 1/2) + 1 - \xi)F'_\pm + (\sqrt{a} - \frac{1}{2} - \Delta \mp \frac{1}{2})F_\pm = 0, \tag{82}$$

whose solutions are determined by the confluent hypergeometric functions.

4.5. The final solutions $G(\hbar,t,x)$ to Schrödginer equation (14) are written as

$$\begin{aligned} G(\hbar,t,x) &= \exp\left((\Delta^2+1/4)\theta\right)\Upsilon(\theta,\zeta) \\ &= \exp((\Delta+1/2)^2\theta)R_+(\xi)+\exp((\Delta-1/2)^2\theta)R_-(\xi), \end{aligned} \tag{83}$$

where the functions $R_\pm$ satisfy relations (78), (79).

These functions are sums of the pairs of the known solutions of the form $\exp(-iEh/t)R(x)$ with constants E. A non-evident shade is that functions (83) satisfy also linear partial differential equation (53), which can be obtained from equation (61) by changes (49), (68), (69) and (70). The coefficients of equation (53) are determined by the classical trajectories of Hamiltonian system (15), (16). That is, constructed solutions (83) to Schrödinger equation (12) describes a quantum-classical correspondence for this system, which was mentioned for no quantum-mechanical Schrödinger equation before work [16] (or, a little bit earlier, in short announcement [15]).

A similar quantum-classical connection for Hamiltonian system (55), (56) describes solutions Ψ to Schrödinger equations (12); these solutions can be obtained from solutions (83) of equations (14) by change (13) and these solutions Ψ satisfy also the first order partial differential equation

$$(\zeta-\lambda)\left(2\Psi'_t+\frac{\lambda'_t}{\lambda}\Psi+\frac{2iH_{MM}\Psi}{\hbar}\right)=\lambda\left(i\hbar\zeta\Psi'_\zeta-\frac{\lambda'_t}{\lambda}\Psi\right). \tag{84}$$

4.6. Hamiltonian systems (55), (56) and (15), (16) are equivalent: they are related by the transformation

$$\lambda=\exp(-q), \qquad \mu=-p\exp(q), \qquad H_M=H_{MM}. \tag{85}$$

This transformation in equation (84) allows us to write the latter in form (19). But the analytic properties of the corresponding solutions to Schrödinger equations (12) and (14) are different.

By change (13) we see that as $\zeta\to 0$ (that is, as $x\to\infty$), the solutions Ψ to equations (12) are smoother than the related by this change solutions G to equations (14). As we shall show in Subsections 4.7 and 4.8, this fact turns out to be rather essential while considering a natural issue on selecting a subset of the constructed solutions G and Ψ to the Schrödginer equations, which are globally bounded with respect to the spatial variables. Thus, due to the constructions in the present work used in the quantization of Hamiltonian system (16) with energy Hamiltonian (15), the most natural action associating Schrödginer equation (14) is not optimal and it is more preferable to transform first Hamiltonian system (15), (16) to the equivalent system (55), (56) by transformation (85). And then to consider the corresponding to this system solutions of Schrödinger equation (12).

Remark 3. *After the passing to the spatial variable $x=-\ln\zeta$, the latter equation becomes*

$$i\hbar\Psi'_t=-\hbar^2\frac{\Psi''_{xx}+\Psi'_x}{2}+\alpha(\exp(-2x)-2\exp(-x))\Psi.$$

It seems that it is quite a complicated issue on symbolic writing this equation as quantum-mechanical Schrödginer equation (5) determined directly by Hamiltonian system (15), (16).

Remark 4. *It follows from equation (84) that solutions (83), (13) to Schrödginer equation (12) satisfy the identity $i\hbar\Psi'_\zeta\equiv\mu(t)\Psi$ on the curve $\zeta=\lambda(t)$ determined by the coordinate component $\lambda(t)$ of Hamiltonian system (55), (56) for a particular value of Hamiltonian H_{MM}; in this identity, $\mu(t)$ is the momentum component of system (55), (56) at this value H_{MM}. That is, on such curves, the action of the quantum-mechanical momentum operator on the constructed solutions $\Psi(t,x,\hbar)$ differ by the sign from the multiplication of these solutions by the corresponding classical momenta. The solutions $G(t,x,\hbar)$ to Schrödinger equation (14) related to these solutions $\Psi(t,x,\hbar)$ of (12) by change (13) do not possess such property; it* *follows from equation (53) that on the curve $z = q(t)$ determined by the coordinate component $q(t)$ of Hamiltonian system (15), (16), the identity $i\hbar G'_t \equiv \left(\frac{i\hbar}{2} + q'_t\right) G$ holds true and not the identity $i\hbar G'_t \equiv q'_t G$.*

4.7 The condition of vanishing as $x \to \pm\infty$ for solutions (83) to equation (14) in the coefficients of equation (53) selects a discrete series of values $H_{M,n}$ of the corresponding Hamiltonians (15). This series of values is determined by the old version of the Bohr-Sommerfeld rule:

by inequalities (54), it follows from formula (20) that the corresponding values $H_{M,n}$ of Hamiltonians (15) are determined by the solutions to the sequence of equations ($e_n = (1 + H_{M,n}/\alpha)^{1/2}$)

$$\begin{aligned}
\oint p(q)dq =& 2\sqrt{2} \int\limits_{-\ln[1+e_n]}^{-\ln[1-e_n]} \sqrt{H_{M,n} - \alpha(\exp(-2q) - 2\exp(-q))}dq \\
=& 2\sqrt{2\alpha} \int\limits_{1-e_n}^{1+e_n} \frac{(e_n^2 - 1 - r^2 + 2r)^{1/2}}{r} dr \\
=& 2\sqrt{2\alpha} \left(\int\limits_{1-e_n}^{1+e_n} \frac{dr}{(e_n^2 - 1 - r^2 + 2r)^{1/2}} + (e_n^2 - 1) \int\limits_{1-e_n}^{1+e_n} \frac{dr}{r(e_n^2 - 1 - r^2 + 2r)^{1/2}} \right) \\
=& -2\sqrt{2\alpha} \left(\arcsin \frac{1-r}{e_n} - (1 - e_n^2)^{1/2} \arcsin \frac{e_n^2 - 1 + r}{re_n} \right) \Bigg|_{1-e_n}^{1+e_n} \\
=& 2\sqrt{2\alpha}\pi \left(1 + \sqrt{-H_{M,n}/\alpha} \right) = 2\pi n\hbar, \qquad (n = 1, 2, \ldots, \infty).
\end{aligned}$$

That is,

$$H_{M,n} = -\left(\sqrt{\alpha} - n\hbar/\sqrt{2}\right)^2, \tag{86}$$

where $n \in \mathbb{N}$ ranges from 1 to the value, for which we still have $\sqrt{2\alpha} > n\hbar$.

The only difference is that for simultaneous solutions of equations (14) and (53) vanishing as $x \to \pm\infty$, the natural n in formula (86) ranges from 1 to value, for which we still have $\sqrt{2\alpha} > (n + 1/2)\hbar$.

4.8. Indeed, let us assume for the coefficients of linear ODEs (80) that there exist solutions to these ODEs tending to zero as both as $\zeta \to 0$ and as $\zeta \to \infty$. As it has been mentioned, changes (81) reduce (80) to ODE (82) for the confluent hypergeometric function. According to [34, Ch. III, Sect. 23, Problem 4, Formula (2)], the criterion for the existence of bounded as $\xi \to 0$ solutions to equations (82), which grow at most a power of ξ as $\xi \to \infty$ is the relations

$$\Delta \pm \frac{1}{2} = \sqrt{a} - n - \frac{1}{2}, \quad \sqrt{a} > n + \frac{1}{2}$$

being satisfied for integer nonnegative n (at, the functions $F_\pm(\xi, n)$ are polynomials).

These relations imply that in order to have the global boundedness in x for solutions (83) to non-stationary Schrödginer equation (14) defined by functions (81) satisfying the relations (78), (79), it is necessary and sufficient to satisfy the relations

$$\Delta = \sqrt{a} - n, \quad n = 1, 2, \ldots, \quad \sqrt{a} > n + \frac{1}{2}.$$

The former of these restrictions is equivalent to formula (86), while the latter means that the nonnegative integer n ranges in the set described in the last paragraph of Subsection 4.7.

If we impose the condition of vanishing as $\zeta \to 0$ and as $\zeta \to \infty$ for the simultaneous solutions Ψ to Schrödinger equation (12) and first order equation (84) which are obtained from solutions (83) under changes (13), among the periodic solutions to Hamiltonian system (15), (16) (differing from $\lambda(t) = 1$, $\mu(t) = 0$) in the coefficients to equation (84), this condition selects the set of classical trajectories coinciding with the set described in the second paragraph in Section 4.7.

5. Concldging remarks

5.1. Consider the solutions to ODE (71) of the form

$$\Upsilon = R_\pm(\zeta)\exp(\pm\Delta\theta). \tag{87}$$

If these solutions satisfy also first order linear equation (77) yields the relations

$$\pm\,\Delta(\zeta-\lambda) = \lambda\frac{\zeta R'_\pm(\zeta)}{R_\pm(\zeta)} + \frac{\lambda}{2} - \zeta\frac{\lambda'_\theta}{2\lambda}. \tag{88}$$

Differentiating them w.r.t. the variable ζ allows us to conclude that

$$\frac{\lambda'_\theta}{2\lambda^2} \pm \frac{\Delta}{\lambda} = \left(\frac{\zeta R'_\pm(\zeta)}{R_\pm(\zeta)}\right)'_\zeta = \vartheta_\pm,$$

where $\vartheta_\pm$ are some constants. Hence, the functions $R_\pm$ should solve the linear ODEs ($\upsilon_\pm$ are constants)

$$\frac{\zeta R'_\pm(\zeta)}{R_\pm(\zeta)} = \vartheta_\pm\zeta + \epsilon_\pm. \tag{89}$$

Substituting this expression into equation (88), in view of the ODE

$$\frac{\lambda'_\theta}{2\lambda} \pm \Delta = \vartheta_\pm\lambda, \tag{90}$$

we get the formulae

$$\epsilon_\pm = -\frac{1}{2} \mp \Delta. \tag{91}$$

The first parts of changes (49) and (52), transform ODE (90) into the ODE

$$\frac{iq'_t}{\hbar} = \vartheta_\pm\exp(-q) \mp \Delta. \tag{92}$$

These ODEs and the formula

$$\Delta^2 = -\frac{2}{\hbar^2}H_M = -\frac{2}{\hbar^2}\left[\frac{(q'_t)^2}{2} + \alpha(\exp(-2q) - 2\exp(-q))\right]$$

for the Hamiltonian H_M of Hamiltonian system (15), (16) imply the double identity

$$-\frac{(q'_t)^2}{\hbar^2} = \Delta^2 + \frac{2\alpha}{\hbar^2}(\exp(-2q) - 2\exp(-q)) = \Delta^2 \mp 2\vartheta_\pm\Delta\exp(-q) + \vartheta_\pm^2\exp(-2q).$$

Assume that $q_t \not\equiv 0$. Then the above double identity yields the identities

$$\Delta = \frac{\sqrt{2\alpha}}{\hbar}, \qquad \vartheta_\pm = \pm\Delta, \tag{93}$$

guaranteeing, in particular, that the right hand side of ODE (92) is real, while the left hand side is pure imaginary for real $q(t)$.

Thus, for such function q, functions (87) can satisfy both ODE (71) and linear equation (77) is possible only for the trivial solutions

$$p(t) = q'_t(t) \equiv 0, \qquad q(t) \equiv 0$$

of Hamiltonian system (15), (16). In this case, together with identity (91), we also have identities (93) and the representations for the functions $R_\pm$

$$R_\pm(\zeta) = R_\pm^0 \zeta^{-\frac{1}{2}\mp\frac{\sqrt{2\alpha}}{\hbar}} \exp(\pm\frac{\sqrt{2\alpha}}{\hbar}\zeta),$$

where $R_\pm^0$ are constants (see ODEs (89), (90)).

Such form of these functions for the case $q(t) \equiv 0$ is, obviously, the sufficient condition ensuring that functions (87) satisfy linear equations (71), (77). Changes (13), (68), (69) and (70) allow us to construct the corresponding solutions to Schrödinger equations (12) and (14) via obtained functions (87).

Among two possible options, the condition of vanishing as $\zeta \to \infty$ (that is, as $x \to -\infty$) is satisfied just by one of them:

the well known solutions [34, Ch. Ⅲ, Sect. 23, Problem 4]

$$G = Const \exp\left(\frac{it\alpha}{\hbar}\left(1 - \frac{\hbar}{2\sqrt{2\alpha}}\right)^2\right) \exp\left(\left(\frac{1}{2} - \frac{\sqrt{2\alpha}}{\hbar}\right)x\right) \exp\left(-\frac{\sqrt{2\alpha}}{\hbar}\exp(-x)\right)$$

to Schrödinger equation (14), which tend to zero as $x \to \infty$ only under the inequality $2\sqrt{2\alpha} > \hbar$,

and, respectively, the solutions

$$\Psi = Const \exp\left(\frac{it\alpha}{\hbar}\left(1 - \frac{\hbar}{\sqrt{2\alpha}}\right)\right)\zeta^{\frac{\sqrt{2\alpha}}{\hbar}} \exp\left(-\frac{\sqrt{2\alpha}}{\hbar}\zeta\right)$$

to Schrödginer equation (12). These solutions Ψ tend to zero as $\zeta \to 0$ under no additional restrictions for the positive constant α.

5.2. In the same way, together with simultaneous solutions (26) to equations (20), (24) and (25) (on the solutions to ODE (23)) in paper [16], we can consider the simultaneous solutions of the form

$$V = \exp(\pm\tau)A_\pm(z);$$

in this section we use the notations and indexation from the above cited paper. Such solutions exist only for the trivial solution $q \equiv 0$ of ODE (23). The criterion for the existence of such simultaneous solutions to equations (20), (24) and (25) is the following form for the functions $A_\pm(z)$

$$A_\pm(z) = Const \exp\left(\pm\frac{z^2}{2}\right).$$

The conditions of vanishing as $x \to \pm\infty$ for the corresponding solutions to the non-stationary Schrödinger equation for the harmonic oscillator

$$i\hbar\Psi'_t = -\frac{\hbar^2}{2}\Psi''_{xx} + \frac{x^2}{2}\Psi$$

selects just one option among two possible options, which is the known solutions of the form

$$\Psi = Const \exp\left(\frac{it}{2}\right) \exp\left(-\frac{x^2}{2\hbar}\right).$$

5.3. It was mentioned in short note [15] that by using equations (3) of IDM for the fifth Painlevé equation allows us to construct new explicit solutions to the evolution equation, which is reduced to the Schrödinger equatino

$$i\hbar\Psi'_t = -\hbar^2\frac{\Psi''_{xx}}{2} + U(x)\Psi \tag{94}$$

determined by energy Hamiltonian (21) of autonomous Hamiltonian system (16) with the smooth potential

$$U(q) = -\frac{\alpha}{\cosh^2 q} \tag{95}$$

called sometimes modified Pöschl-Teller potential [35, Ch. II, Problem 39]. These solutions to the Schrödinger equation satisfy both linear partial differential equations (19), whose coefficients K and M are determined by the solutions to Hamiltonian system (16),(95). The condition of vanishing as $x \to \pm\infty$ for these simultaneous solutions to Schrödinger equation (94),(95) and corresponding linear equation (19) selects the same solutions among real periodic solutions $(\lambda(t), \mu(t))$ of Hamiltonian system (16),(95) defining the coefficients K and M of equation (19) as those determined by the old version of Bohr-Sommerfeld formula (20).

Equations (3) of IDM for the fifth Painlevé equations can be also used for constructing explicit solutions to Schrödinger equation (94) determined by energy Hamiltonian (21) of Hamiltonian system (16) with a non-smooth Pöschl-Teller potential (see [35, Ch. II, Problem 39])

$$U(q) = Const\left[\frac{a}{\sin^2 q} + \frac{b}{\cos^2 q}\right], \tag{96}$$

where $a > 1$ and $b > 1$. These explicit solutions smooth as $-\pi < x < \pi$ satisfy also a linear partial differential equation of form (19) with the coefficients depending on the solutions to Hamiltonian system (16), (96). The condition of vanishing as $x \to \pm\pi$ for these explicit solutions to Schrödinger equation (94),(96) selects the same real periodic solutions to system (16), (96) as those determined by the old version of Bohr-Sommerfeld rule (20).

The detailed presentation of these results described briefly in this subsection is planned to be a subject for a future separate paper.

5.4 It is a topical problem to describe such independent of t potentials $U(x)$, for which Schrödinger equations (94) admit solutions like the ones discussed in Section 4 and Subsections 5.1–5.3. In view of this fact, an idea on considering the class of the potentials $U(x)$ selected in work [29] deserves an attention.

At the same time, in the opinion of the author, such restriction for the set of possible potentials $U(x)$ is not natural enough. Probably, from the point of view of more natural solutions to the problem discussed in the previous paragraph, it is more promising to use linear equation like (10) in addition to Schrödinger equation (94) and, maybe, a series of other natural restrictions. For instance, one can assume an identity like the last displayed formula in Section 3.

5.5 At present, a rather wide list of isomonodromic Hamiltonian systems with two degrees of freedom which can be found in works [36]–[39]. This list is considered to be complete [39]. The description of "quantizations" of such systems possessing the solutions written explicitly in terms of the solutions of the corresponding IDM equations was initiated just in recent works [17] and [18]. Here we still need to understand and do a lot. It seems that a key role in describing such "quantizations" should be played by changes of kind (21) in [17]. For slightly different aims, this change was used by D.P. Novikov in paper [19]; see also formula (2.3.36) in paper [40].

Unfortunately, just a few is known on the set of isomonodromic Hamiltonian systems with more than two degrees of freedom. And there are no "quantizations" of such systems with solutions written explicitly in terms of the solutions to IDM equations.

One more interesting question is on possible autonomous reductions of such isomonodromic systems and the issue on possibility of constructing new solutions to the corresponding non-stationary quantum-mechanical Schrödinger equation by using IDM equations for such systems.

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