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ON SOLVABILITY BY QUADRATURES CONDITIONS OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER HYPERBOLIC SYSTEMS

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Abstract. In the present work we consider boundary value problems for second order hyperbolic system with higher partial derivatives u_{xy}, v_{xy} and u_{xx}, v_{yy}. The aim of the study is to find sufficient conditions for solvability of the considered problems by quadratures. We proposed a method for finding explicit solutions for the mentioned problems based on factorization of the equations in the original systems. As a result, in terms of the coefficients of these systems, we obtain 14 conditions for solvability by quadratures for each boundary value problem.

Keywords: hyperbolic system, Goursat problem, boundary value problem, solvability by quadratures, factorization of equation.

Mathematics Subject Classification: 35L51, 35L53, 35G45

1. In works [1, 2, 3] a system having a vector matrix form

$$u_{xy} + Au_x + Bu_y + Cu = F.$$

was studied from various points of view. In particular, it is known, that the Goursat problem for this system is uniquely solvable. Here, for a certain particular case, we propose a way of finding the solution to this problem by quadratures by factorizing each equation. It turns out to be convenient to consider these equations in the form

$$\begin{aligned} u_{xy} + a_1 u_x + b_1 u_y + c_1 v_y + d_1 u + e_1 v = 0, \\ v_{xy} + a_2 u_x + b_2 v_x + c_2 v_y + d_2 u + e_2 v = 0. \end{aligned} \tag{1}$$

To realize our arguments, it is sufficient to assume that in the considered domain $D = \{x_0 < x < x_1,\ y_0 < y < y_1\}$, the inclusions

$$a_1,\ a_2,\ b_2 \in C^{(1,0)},\ b_1,\ d_1,\ d_2 \in C^{(0,1)},\ d_1,\ d_2,\ e_1,\ e_2 \in C^{(0,0)}, \tag{2}$$

hold true. We also propose a similar approach for studying some characteristic problem for the system with higher partial derivatives u_{xx}, v_{yy}.

Problem 1. Find a regular solution to systme (1) in domain D satisfying the conditions

$$\begin{aligned} u(x_0, y) = \varphi_1(y), \quad u(x, y_0) = \psi_1(x), \\ v(x_0, y) = \varphi_2(y), \quad v(x, y_0) = \psi_2(x). \end{aligned} \tag{3}$$

At that, we assume that $\varphi_1,\ \varphi_2 \in C^1(\overline{X})$, $\psi_1,\ \psi_2 \in C^1(\overline{Y})$, ($X$, Y are the sides of the characteristic rectangle D as $x = x_0$, $y = y_0$, respectively) and the matching conditions

$$\varphi_1(y_0) = \psi_1(x_0), \quad \varphi_2(y_0) = \psi_2(x_0) \tag{4}$$

hold.

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We try to find functions α_1, β_1, γ_1 such that the first equation in (1) to be of the form

$$\left(\frac{\partial}{\partial y}+\alpha_1\right)(u_x+\beta_1 u+\gamma_1 v)=0. \tag{5}$$

Making the operations in (5), we confirm that the first equation in (1) coincides with (5) if the identities

$$\begin{aligned} b_{1y}+a_1b_1-d_1\equiv 0,\\ c_{1y}+a_1c_1-e_1\equiv 0\end{aligned} \tag{6}$$

hold true and

$$\alpha_1=a_1,\ \beta_1=b_1,\ \gamma_1=c_1. \tag{7}$$

In the same way we obtain that if the identities

$$\begin{aligned} a_{2x}+a_2c_2-d_2\equiv 0,\\ b_{2x}+b_2c_2-e_2\equiv 0\end{aligned} \tag{8}$$

hold true, then the second identity in (1) can be represented as

$$\left(\frac{\partial}{\partial x}+\alpha_2\right)(v_y+\beta_2 u+\gamma_2 v)=0,$$

where

$$\alpha_2=c_2,\ \beta_2=a_2,\ \gamma_2=b_2. \tag{9}$$

Thus, Problem 1 can be reduced to the following three problems

$$w_{1y}+\alpha_1 w_1=0,\quad w_1(x,y_0)=\psi_{1x}+\beta_1\psi_1+\gamma_1\psi_2, \tag{10}$$

$$w_{2x}+\alpha_2 w_2=0,\quad w_2(x_0,y)=\varphi_{2y}+\beta_2\varphi_1+\gamma_2\varphi_2, \tag{11}$$

$$\begin{cases} u_x+\beta_1 u+\gamma_1 v=w_1,\\ v_y+\beta_2 u+\gamma_2 v=w_2,\end{cases} \tag{12}$$

$$u(x_0,y)=\varphi_1(y),\quad v(x,y_0)=\psi_2(x). \tag{13}$$

Problems (10)–(13) should be solved successively starting from the first one. The functions w_1, w_2 can be calculated by straightforward integration and in problem (10), x is considered as a parameter, while in problem (11) the role of the parameter is played by y. Thus, it remains to solve Goursat problem (12)–(13), which is uniquely solvable [4]. In order to find the condition of its explicit solvability, we employ the possibility of reducing system (12) to two equations of the form

$$\Theta_{xy}+a\Theta_x+b\Theta_y+c\Theta=f \tag{14}$$

arising by excluding from the considered system one of the sought functions. Under the inequality $\gamma_1\neq 0$ equivalent to

$$c_1\neq 0 \tag{15}$$

by (7), we arrive at (14) for $\Theta=u$. At that, the coefficients of the equation are given by the formulae

$$\begin{aligned} &a=\gamma_2-(\ln\gamma_1)_y,\quad b=\beta_1,\quad c=\beta_{1y}+\beta_1\gamma_2-\beta_2\gamma_1-\beta_1(\ln\gamma_1)_y,\\ &f=w_{1y}+\gamma_2 w_1-\gamma_1 w_2-w_1(\ln\gamma_1)_y.\end{aligned} \tag{16}$$

Under the inequality $\beta_2\neq 0$ equivalent by (9) to

$$a_2\neq 0, \tag{17}$$

we arrive at (14) for $\Theta=v$ with the coefficients

$$\begin{aligned} &a=\gamma_2,\quad b=\beta_1-(\ln\beta_2)_x,\quad c=\gamma_{2x}+\beta_1\gamma_2-\beta_2\gamma_1-\gamma_2(\ln\beta_2)_x,\\ &f=w_{2x}-\beta_2 w_1+\beta_1 w_2-w_2(\ln\beta_2)_x.\end{aligned} \tag{18}$$

The solution $u(x,y)$ to the first equation obtained for $\gamma_1\neq 0$ allows us to calculate the function $v(x,y)$ in the first equation in (12). In the same way, as $\beta_2\neq 0$, by the known solution v to the second equation, the function u can be determined by the second equation in (12). However, in order to find $\Theta = u$ or $\Theta = v$, we need to add also the values

$$u(x, y_0) = \psi_1(x), \quad v(x_0, y) = \varphi_2(y) \tag{19}$$

from (3) and matching conditions (4) to conditions (13). It is clear that the first (second) relations in (13) and (19) are the boundary conditions in the Goursat problem for the first (second) equation of form (14). At that, to obtain the solution to the original Problem 1, it is sufficient to consider a solution at least to one of the mentioned Goursat problems.

It is known [5, 614] that the solutions to the formulated Gourat problems can be written via the corresponding Riemann problems, while for latter there are various cases of explicit construction [6, 7, 8]. In the cited works these works are represented in terms of the following relations:

$$\begin{aligned}
&1)\ a_x + ab - c \equiv 0;\\
&2)\ b_y + ab - c \equiv 0;\\
&3)\ a_x \equiv b_y, \quad c - a_x - ab \equiv \xi_0(x)\eta_0(y) \neq 0;\\
&4)\ b_y - a_x \equiv a_x + ab - c \equiv \xi_1(x)\eta_1(y) \neq 0;\\
&5)\ a_x - b_y \equiv b_y + ab - c \equiv \xi_2(x)\eta_2(y) \neq 0;\\
&6)\ ma_x - b_y \equiv mb_y - a_x \equiv (m-1)(ab - c);\\
&7)\ \sigma = \frac{2s'(x)t'(y)}{(2-m)[s(x)+t(y)]^2}, \quad [s(x)+t(y)]s'(x)t'(y) \neq 0.
\end{aligned} \tag{20}$$

Here ξ_k, $\eta_k \in C^1$, $(k = \overline{0,2})$, s, t, $m \in C^2$, and m depends only on one of the variables (x, y) and does not take the value 2. There are no more conditions for these functions, that is, there should be functions in the corresponding class satisfying the mentioned conditions. The coefficients a, b, c have a smoothness ensuring the possibility of satisfying the above formulae. The smoothness classes are defined on the closed domains of the corresponding functions. Each of identities 1)–2) and 3)–5) is sufficient to obtain explicitly the Riemann function. Formulae 6)–7) should be employed simultaneously: under 6), the Riemann function can be constructed if the left hand side of one of the relations 1), 2) has the form σ mentioned in 7). In other words, there are seven options of solvability by quadratures solvability for each of the two obtained Goursat problems. For all options, the Riemann function can be found in [6]–[8]. It is clear that the total amount of the discussed solvability is equal to 14.

Employing formulae (7), (9), (16), (18), we write 1)–7) in terms of the coefficients of system (1). We begin with the first Goursat problem related to inequality (15):

$$\begin{aligned}
&1)\ b_{2x} - b_{1y} - (\ln c_1)_{xy} + a_2c_1 \equiv 0;\\
&2)\ a_2 \equiv 0;\\
&3)\ b_{2x} - b_{1y} - (\ln c_1)_{xy} \equiv 0,\ b_{1y} - b_{2x} + (\ln c_1)_{xy} - a_2c_1 \equiv \xi_0(x)\eta_0(y) \neq 0;\\
&4)\ 2[(\ln c_1)_{xy} - b_{2x} + b_{1y}] \equiv a_2c_1,\ (\ln c_1)_{xy} - b_{2x} + b_{1y} \equiv \xi_1(x)\eta_1(y) \neq 0;\\
&5)\ b_{2x} - b_{1y} - (\ln c_1)_{xy} \equiv a_2c_1 \equiv \xi_2(x)\eta_3(y) \neq 0;\\
&6)\ m[b_{2x} - (\ln c_1)_{xy}] - b_{1y} \equiv mb_{1y} - b_{2x} + (\ln c_1)_{xy} \equiv (m-1)(a_2c_1 - b_{1y});\\
&7)\ \sigma_k = \frac{2s_k'(x)t_k'(y)}{(2-m)[s_k(x)+t_k(y)]^2},\ [s_k(x)+t_k(y)]s_k'(x)t_k'(y) \neq 0,\ k = 1, 2.
\end{aligned} \tag{21}$$

In the last line we should assume that σ_1, σ_2 are equal to the left hand sides of identities 1), 2), respectively. Moreover, we take into consideration that in order to have a possibility of satisfying relations (20), we need to increase the smoothness of the coefficients of system (1) and the functions φ_i, ψ_i, $(i = 1, 2)$. Assume that $a_i, \ldots, e_i \in C^{(2,2)}$, φ_i, $\psi_i \in C^2$, $(i = 1, 2)$. Then the following theorem holds true.

Theorem 1. *Assume that under identities (6), (8) and inequality (15) either one of the identities 1), 2) in (21) holds true or there exist functions* m, ξ_k, η_k, $(k=\overline{0,2})$, s_k, t_k, $(k=1,2)$ *of the aforementioned classes such that for (21) either one of three groups of relations 3) – 5) holds true or identity 6) and representation 7) holds for one of two functions* σ_1, σ_2. *Then Problem 1 is solved by quadratures.*

The analogues of formulae (21) for the second Goursat problem (corresponding to condition (17)) are

$$\begin{aligned}
&1)\ c_1\equiv 0;\\
&2)\ -b_{2x}+b_{1y}-(\ln a_2)_{xy}+a_2c_1\equiv 0;\\
&3)\ b_{2x}-b_{1y}+(\ln a_2)_{xy}\equiv 0,\ -a_2c_1\equiv \xi_3(x)\eta_3(y)\neq 0;\\
&4)\ b_{1y}-b_{2x}-(\ln a_2)_{xy}\equiv a_2c_1\equiv \xi_4(x)\eta_4(y)\neq 0;\\
&5)\ 2[(\ln a_2)_{xy}+b_{2x}-b_{1y}]\equiv a_2c_1,\ (\ln a_2)_{xy}+b_{2x}-b_{1y}\equiv \xi_5(x)\eta_5(y)\neq 0;\\
&6)\ mb_{2x}+(\ln a_2)_{xy}-b_{1y}\equiv m(b_{1y}-(\ln a_2)_{xy})-b_{2x}\equiv (m-1)(a_2c_1-b_{2x});\\
&7)\ \sigma_k=\frac{2s'_k(x)t'_k(y)}{(2-m)[s_k(x)+t_k(y)]^2},\ [s_k(x)+t_k(y)]s'_k(x)t'_k(y)\neq 0,\ k=3,4.
\end{aligned}\tag{22}$$

In the last line σ_3, σ_4 are equal to the left hand sides of identities 1), 2) in (22), respectively.
Thus, we have the theorem.

Theorem 2. *Assume that under identities (6), (8) and inequality (17) either at least one of identities 1), 2) in (22) holds true or there exist functions* m, ξ_k, η_k $(k=\overline{3,5})$, s_k, t_k $(k=3,4)$ *in the aforementioned classes such that in (22) one of three groups of relations 3) – 5) holds true or identity 6) and representation 7) hold true for one of the functions* σ_3, σ_4. *Then Problem 1 is solved by quadratures.*

2. We apply the above described algorithm for finding the solvability by quadratures for the following problem
Problem 2. Find a regular solution to the system

$$\begin{cases} u_{xx}+a_1u_x+b_1v_x+c_1u+d_1v=0,\\ v_{yy}+a_2u_y+b_2v_y+c_2u+d_2v=0,\end{cases}\tag{23}$$

in the domain $D=\{x_0<x<x_1,\ y_0<y<y_1\}$ satisfying the conditions

$$\begin{aligned}
&u(x_0,y)=\varphi_1(y),\quad v(x,y_0)=\psi_1(x),\\
&(u_x+b_1v)(x_0,y)=\varphi_2(y),\quad (v_y+a_2u)(x,y_0)=\psi_2(x),
\end{aligned}\tag{24}$$

where $\varphi_1,\ \varphi_2\in C^1(\overline{X})$, $\psi_1,\ \psi_2\in C^1(\overline{Y})$. The smoothness for the coefficients of system (23) is determined by the belongings

$$a_1,\ b_1\in C^{(1,0)},\ a_2,\ b_2\in C^{(0,1)},\quad c_1,\ c_2,\ d_1,\ d_2\in C^{(0,0)}.\tag{25}$$

System (23) was studied, for instance, in works [9], [10]. In particular, in [9], there was obtained a solution to Problem 2 written in term of the Riemann matrix. The aim of our study is to obtain the solvability by quadratures conditions for Problem 2.

By straightforward calculations we can make sure that the first equation (23) can be represented as

$$(\frac{\partial}{\partial x})(u_x+\beta_1u+\gamma_1v)=0$$

if the identities

$$\begin{aligned}&a_{1x}-c_1\equiv 0,\\ &b_{1x}-d_1\equiv 0\end{aligned}\tag{26}$$

hold true and

$$\beta_1=a_1,\ \gamma_1=b_1.\tag{27}$$

In the same way, if

$$\begin{aligned} a_{2y} - c_2 \equiv 0, \\ b_{2y} - d_2 \equiv 0, \end{aligned} \tag{28}$$

the second equation in (23) can be written as

$$(\frac{\partial}{\partial y})(v_y + \beta_2 u + \gamma_2 v) = 0,$$

where

$$\beta_2 = a_2,\ \gamma_2 = b_2. \tag{29}$$

Thus, Problem 2 is reduced to three problems of the form

$$w_{1x} = 0, \quad w_1(x_0, y) = \varphi_2 + \beta_1\varphi_1, \tag{30}$$

$$w_{2y} = 0, \quad w_2(x, y_0) = \psi_2 + \gamma_2\psi_1, \tag{31}$$

$$\begin{cases} u_x + \beta_1 u + \gamma_1 v = w_1, \\ v_y + \beta_2 u + \gamma_2 v = w_2, \end{cases} \tag{32}$$

$$u(x_0, y) = \varphi_1(y), \quad v(x, y_0) = \psi_1(x). \tag{33}$$

Problems (30)–(33) should be solved successively starting from the first of them. The functions w_1, w_2 in (30), (31) should be calculated by straightforward integration and in problem (30), x is considered as a parameter, while in problem (31) the role of the parameter is played by y. As we know by Section 1, problem (32)–(33) is reduced to two Goursat problems for equation (14). At that, under the inequality

$$b_1 \neq 0 \tag{34}$$

we arrive at (14) for $\Theta = u$ with coefficients (16), while under

$$a_2 \neq 0 \tag{35}$$

we arrive at (14) for $\Theta = v$ with coefficients (18). The solvability conditions of these Goursat problems are given by relations (20). Employing formulae (16), (18), (27), (29), we write these relations in terms of the coefficients of system (23). For the first Goursat problem related to inequality (34) we have

$$\begin{aligned}
&1)\ b_{2x} - a_{1y} - (\ln b_1)_{xy} + a_2 b_1 \equiv 0; \\
&2)\ a_2 \equiv 0; \\
&3)\ b_{2x} - a_{1y} - (\ln b_1)_{xy} \equiv 0,\ a_{1y} - b_{2x} + (\ln b_1)_{xy} - a_2 b_1 \equiv \xi_0(x)\eta_0(y) \neq 0; \\
&4)\ 2[(\ln b_1)_{xy} - b_{2x} + a_{1y}] \equiv a_2 b_1,\ (\ln b_1)_{xy} - b_{2x} + a_{1y} \equiv \xi_1(x)\eta_1(y) \neq 0; \\
&5)\ b_{2x} - a_{1y} - (\ln b_1)_{xy} \equiv a_2 b_1 \equiv \xi_2(x)\eta_2(y) \neq 0; \\
&6)\ m[b_{2x} - (\ln b_1)_{xy}] - a_{1y} \equiv m a_{1y} - b_{2x} + (\ln b_1)_{xy} \equiv (m-1)(a_2 b_1 - a_{1y}); \\
&7)\ \sigma_k = \frac{2s'_k(x)t'_k(y)}{(2-m)[s_k(x) + t_k(y)]^2},\ [s_k(x) + t_k(y)]s'_k(x)t'_k(y) \neq 0,\ k = 1, 2.
\end{aligned} \tag{36}$$

In the last we should assume that σ_1, σ_2 are equal to the left hand sides of identities 1), 2) in (36), respectively. Moreover, we need to increase the smoothness for the coefficients of system (23) and for the functions φ_i, ψ_i $(i = 1, 2)$. Assume now that $a_i, \ldots, d_i \in C^{(2,2)}$, φ_i, $\psi_i \in C^2$ $(i = 1, 2)$. Then we have the theorem.

Theorem 3. *If identities (26), (28) and inequality (34) are satisfied and one of identities 1), 2) in (36) holds true or there exist functions m, ξ_k, η_k $(k = \overline{0, 2})$, s_k, t_k $(k = 1, 2)$ in the aforementioned classes such that in (37) either one of three groups of relations 3) – 5) holds true or identity 6) and representation 7) hold for one of the functions σ_1, σ_2, then Problem 2 is solved by quadratures.*

The analogues of formulae (36) for the second Goursat problem related to condition (35) are

$$
\begin{aligned}
&1)\ b_1 \equiv 0;\\
&2)\ a_{1y} - b_{2x} - (\ln a_2)_{xy} + a_2 b_1 \equiv 0;\\
&3)\ b_{2x} - a_{1y} + (\ln a_2)_{xy} \equiv 0,\ -a_2 b_1 \equiv \xi_3(x)\eta_3(y) \neq 0;\\
&4)\ a_{1y} - b_{2x} - (\ln a_2)_{xy} \equiv a_2 b_1 \equiv \xi_4(x)\eta_4(y) \neq 0;\\
&5)\ 2[(\ln a_2)_{xy} + b_{2x} - a_{1y}] \equiv a_2 b_1,\ (\ln a_2)_{xy} + b_{2x} - a_{1y} \equiv \xi_5(x)\eta_5(y) \neq 0;\\
&6)\ m b_{2x} + (\ln a_2)_{xy} - a_{1y} \equiv m(a_{1y} - (\ln a_2)_{xy}) - b_{2x} \equiv (m-1)(a_2 b_1 - b_{2x});\\
&7)\ \sigma_k = \frac{2 s_k'(x) t_k'(y)}{(2-m)[s_k(x) + t_k(y)]^2},\ [s_k(x) + t_k(y)] s_k'(x) t_k'(y) \neq 0,\ k = 3, 4,
\end{aligned}
\tag{37}
$$

where σ_3, σ_4 are equal to the left hand sides of identities 1), 2) in (37), respectively.

Thus, the following theorem holds true.

Theorem 4. *If identities (26), (28) and inequality (35) are satisfied and one of identities 1), 2) in (37) holds true or there exist functions m, ξ_k, η_k ($k = \overline{3,5}$), s_k, t_k ($k = 3, 4$) in the aforementioned classes such that for (37) either one of three groups of relations 3) – 5) holds true or identity 6) and representation 7) hold for one of the functions σ_3, σ_4, then Problem 2 is solved by quadratures.*

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