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SYMMETRIES AND CONSERVATION LAWS FOR A TWO-COMPONENT DISCRETE POTENTIAL KORTEWEG-DE VRIES EQUATION

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Abstract. In the work we discuss briefly a method for constructing a formal asymptotic solution to a system of linear difference equations in the vicinity of a special value of the parameter. In the case when the system is the Lax pair for some nonlinear equation on a square graph, the found formal asymptotic solution allows us to describe the conservation laws and higher symmetries for this nonlinear equation. In the work we give a complete description of a series of conservation laws and the higher symmetries hierarchy for a discrete potential two-component Korteweg-de Vries equation.

Keywords: integrable dynamical systems, equation on square graph, symmetries, conservation laws, Lax pair.

Mathematics Subject Classification: 35Q53, 37K10

1. Introduction

Considerably many works are devoted to studying the asymptotic behavior of the system of linear differential equations near the singular point, see, for instance, monograph [1]. Asymptotic representation for an eigenfunction of the Lax operators allows one to study effectively the motion integrals, higher symmetries and particular solutions of the corresponding nonlinear dynamical system [2, 3]. The method of formal asymptotic diagonalization allowed the authors of work [4] to establish a deep connection between integrable systems and affine Lie algebras.

An algorithm for solving the problem on asymptotic diagonalization of a discrete operator in the vicinity of the singular point and its applications in the theory of integrable nonlinear discrete equations was discussed in details in works [5, 6, 7]. Interesting results on non-autonomous discrete dynamical systems were obtained by using the formal diagonalization method in works [8, 9]. Alternative approaches to the problem on constructing the asymptotic expansion for an eigenfunction of the discrete Lax operator were proposed in works [10, 11, 12].

In the present work we consider the two-component discrete potential Korteweg-de Vries equation

$$
\begin{aligned}
(u-u_{1,1})(v_{1,0}-v_{0,1}) &= p^2-q^2,\\
(v-v_{1,1})(u_{1,0}-u_{0,1}) &= p^2-q^2.
\end{aligned}
$$

found in [13]. The Lax pair for this equation was provided in [13], we also mention that its explicit particular solutions were found in [14]. We recall the one-component potential Korteweg-de Vries equation

$$(u_{1,1}-u)(u_{1,0}-u_{0,1}) = 4c^2 \tag{1.1}$$

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was studied earlier in many works starting from [15, 16]. It is also known as equation H1 from the list by Adler-Bobenko-Suris, see [17]. An infinite series of conservation laws for this equation was obtained in work [18], higher symmetries were constructed in papers [19, 20, 21, 22].

In the present work, by means of the method of the formal asymptotic diagonalization of Lax pair, we describe an infinite series of conservation laws and construct the higher symmetries of the two-component discrete potential Korteweg-de Vries equation[1].

Let us clarify briefly the structure of the work. In the second section we describe the class of discrete linear equations with singular points, in the vicinities of which we construct an asymptotic solution. By an example we demonstrate the way of reducing the linear system to a special convenient form. In the third section we discuss an algorithm for reducing the Lax operator to a quasi-diagonal form. In the forth section we expose a known method of constructing the generating function for the conservation laws employing the commutation condition of the diagonalized Lax operators. In the fifth section the formal diagonalization method is applied to a particular dynamical system cdpKdV (5.1), for which we provide an infinite series of conservation laws. And, finally, in the sixth section we prove that dynamical system cdpKdV (5.1) has an infinite hierarchy of higher symmetries. First two symmetries are constructed explicitly, for the others we provide an effective way of calculating.

2. Singularities of pole kind for discrete linear system

We consider a linear discrete equation of the form

$$y(n+1,\lambda) = f(n,u(n),\lambda)y(n,\lambda), \tag{2.1}$$

where the potential $f = f(n,u,\lambda) \in \mathbb{C}^{N\times N}$ depends on an integer $n \in (-\infty,+\infty)$, functional parameter $u = u(n)$ and complex parameter λ. The potential is a meromorphic function on λ in a domain $E \subset \mathbb{C}$ and it is assumed that $\det f$ is not identically zero.

Let us define what is a singular point. We call a point $\lambda = \lambda_0$ singular for equation (2.1) if at least one of the functions $f(n,u,\lambda)$, $f^{-1}(n,u,\lambda)$ has a pole at this point. We assume that λ_0 is independent of n.

We note that some singular points can be removed by means of a transformation of the dependent variable $y(n) = r(n,\lambda)\tilde{y}(n)$, which reduces equation (2.1) to the same form

$$\tilde{y}(n+1) = \tilde{f}(n,u,\lambda)\tilde{y}(n)$$

with a new potential $\tilde{f}(n,u,\lambda) = r^{-1}(n+1,\lambda)f(n,u,\lambda)r(n,\lambda)$.

As a simple illustrative example, we consider equation (2.1) with the potential

$$f = \begin{pmatrix} \lambda g_{11} & \lambda g_{12} \\ \lambda g_{21} & \lambda g_{22} \end{pmatrix}, \quad g_{ij} = g_{ij}(n,u).$$

This equation has two singular points $\lambda = \infty$, $\lambda = 0$. Both singular points are removed by the transformation $y(n) = \lambda^n \tilde{y}(n)$. Indeed, $\tilde{f} = \lambda^{-1} f = \{g_{ij}\}$.

A less trivial example is provided the well known linear equation associated in the integrability context to equation (1.1). Its potential is of the form

$$f(n,u,\lambda) = \begin{pmatrix} -u(n+1) & 1 \\ -\lambda^{-2} - u(n)u(n+1) & u(n) \end{pmatrix}. \tag{2.2}$$

Here $u(n)$ is an arbitrary function. Equation (2.1), (2.2) has two singular points $\lambda = 0$, $\lambda = \infty$. The singular point $\lambda = 0$ can be removed by the transformation

$$y(n) = r(n,\lambda)\tilde{y}(n), \quad r(n,\lambda) = \begin{pmatrix} \lambda^{-n} & 0 \\ 0 & \lambda^{-n-1} \end{pmatrix}.$$

[1] We thank the authors of work [14], who attracted out attention to this problem.

Indeed, in this case the potential casts into the form

$$\tilde{f}(n,\lambda) = \begin{pmatrix} -u(n+1)\lambda & 1 \\ -1 - u(n)u(n+1)\lambda^2 & u(n)\lambda \end{pmatrix}$$

and has the only singular point $\lambda = \infty$.

The key step of the diagonalization algorithm is the reducing of the original system (2.1) to the special form

$$y(n+1,\lambda) = P(n,\lambda)Zy(n,\lambda) \tag{2.3}$$

in the vicinity of the singular point $\lambda = \lambda_0$. Here functions $P(n,\lambda)$ and $P^{-1}(n,\lambda)$ are analytic in the vicinity of λ_0, the principal minors of the matrix $P(n,\lambda)$ satisfy conditions (3.8) below and the matrix Z has diagonal form (3.7).

If we succeeded to reduce the system to the mentioned form, then the coefficients of the asymptotic series can be effectively calculated. By the known coefficients of the series we can construct easily the conservation laws and the symmetries. Let us discuss one of the ways of reducing the original system to form (2.3) choosing as an example the equation

$$y(n+1,\lambda) = f(n,u(n),\lambda)y(n,\lambda), \quad f = \begin{pmatrix} \lambda & -u(n) \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

It is easy to see that equation (2.4) has the unique singular point $\lambda = \infty$. First we represent f as the product $f(n,u(n),\lambda) = \alpha(n,u(n),\lambda)Z\beta(n,u(n),\lambda)$ of three matrices, a diagonal matrix Z and triangular matrices α and β:

$$\alpha = \begin{pmatrix} 1 & 0 \\ \lambda^{-1} & u(n) \end{pmatrix}, \quad Z = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & u(n)\lambda^{-1} \\ 0 & 1 \end{pmatrix}.$$

We stress that $\alpha(n,\lambda)$ and $\beta(n,\lambda)$ are analytic and non-degenerate in the vicinity of $\lambda = \infty$. After the change $\psi = \beta y$, equation (2.4) casts into the required form

$$\psi(n+1,\lambda) = P(n,u(n),\lambda)Z\psi(n,\lambda).$$

Here P is determined by the formula $P(n,u(n),\lambda) = \beta(n+1,u(n+1),\lambda)\alpha(n,u(n),\lambda)$, namely,

$$P(n,u(n),\lambda) = \begin{pmatrix} 1 - u(n+1)\lambda^{-2} & -u(n)u(n+1)\lambda^{-2} \\ \lambda^{-1} & u(n) \end{pmatrix}.$$

At that, $P(\lambda)$ is analytic at infinity and principal minors of the matrix $P(\infty)$ are non-zero.

3. Asymptotic diagonalization of the discrete operator in the vicinity of a singular point

Assume that $f(n,u,\lambda)$ has a pole at $\lambda = \lambda_0$. Then f can be expanded into the Laurent series in the vicinity of this point:

$$f(n,u,\lambda) = (\lambda - \lambda_0)^{-k} f_{(k)}(n) + (\lambda - \lambda_0)^{-k+1} f_{(k-1)}(n) + \cdots, k \geq 1. \tag{3.1}$$

The aim of the present section is to discuss sufficient conditions for the “diagonalizability” of equation (2.1) in the vicinity of the point $\lambda = \lambda_0$. In accordance with work [6], equation (2.1) is diagonalizable if there exist formal series

$$R(n,\lambda) = R_{(0)} + R_{(1)}(\lambda - \lambda_0) + R_{(2)}(\lambda - \lambda_0)^2 + \cdots, \tag{3.2}$$

$$h(n,\lambda) = h_{(0)} + h_{(1)}(\lambda - \lambda_0) + h_{(2)}(\lambda - \lambda_0)^2 + \cdots \tag{3.3}$$

with matrix coefficients $R_{(j)}, h_{(j)} \in \mathbb{C}^{N\times N}$, where $h_{(j)}$ is a diagonal (block-diagonal) matrix for all j such that the formal change of the dependent variable $y = R\varphi$ reduces equation (2.1) to the form

$$\varphi_1 = hZ\varphi. \tag{3.4}$$

Here $Z = (\lambda - \lambda_0)^d$, d is a diagonal matrix with integer entries. We assume that $\det R_{(0)} \neq 0$, $\det h_{(0)} \neq 0$. It follows from formulae (3.2)–(3.4) that equation (2.1) has a formal solution with the following asymptotic expansion

$$y(n,\lambda) = R(n,\lambda) e^{\sum_{s=n_0}^{n-1} \log h(s,\lambda)} Z^n \tag{3.5}$$

with "amplitude" $A = R(n,\lambda)$ and "phase" $\phi = n \log Z + \sum_{s=n_0}^{n-1} \log h(s,\lambda)$.

We assume that the potential $f(n,\lambda)$ can be represented as

$$f(n,u(n),\lambda) = \alpha(n,u(n),\lambda) Z \beta(n,u(n),\lambda), \tag{3.6}$$

where $\alpha(n,\lambda)$ and $\beta(n,\lambda)$ are analytic and non-degenerate in the vicinity of $\lambda = \lambda_0$, Z is a diagonal matrix of the form

$$Z = \begin{pmatrix} (\lambda-\lambda_0)^{\gamma_1} E_1 & 0 & \dots & 0 \\ 0 & (\lambda-\lambda_0)^{\gamma_2} E_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\lambda-\lambda_0)^{\gamma_N} E_N \end{pmatrix}, \tag{3.7}$$

E_j are unit matrices of the size $e_j \times e_j$, the exponents are mutually different $\gamma_1 < \gamma_2 < \ldots < \gamma_N$. We let $P(n,u,\lambda) = \beta(n+1,u(n+1),\lambda)\alpha(n,u(n),\lambda)$ and assume that the principal minors of the matrix $P(n,u,\lambda_0)$ satisfy the following condition:

$$\det_j P(n,u,\lambda_0) \neq 0 \text{ for } j = e_1, e_1+e_2, e_1+e_2+e_3, \ldots, N. \tag{3.8}$$

Theorem 1. *Assume that $\lambda = \lambda_0$ is a singular point and the potential $f(n,u(n),\lambda)$ satisfies conditions* (3.6), (3.8) *in the vicinity of λ_0 and in the varying domain of $u(n)$. Then there exists a formal series "diagonalizing" equation* (2.1) *in the vicinity of $\lambda = \lambda_0$, i.e., the formal change $y = R\varphi$ reduces* (2.1) *to form* (3.4)*, where h has the following block-diagonal structure*

$$h = \begin{pmatrix} h_{11} & 0 & \dots & 0 \\ 0 & h_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_{rr} \end{pmatrix}. \tag{3.9}$$

Here h_{jj} are square matrices of the size $e_j \times e_j$. The coefficients $R_{(i)}$ and $h_{(i)}$ depend on a finite subset of the infinite set of variables $\{u(k)\}_{k=\overline{-\infty,\infty}}$ and this subset depends on i.

The proof of Theorem 1 was given in work [7]. It should be said that in the proof of the theorem, there was constructed a formal series $T = \beta R$ satisfying the equation

$$D_n(T)h = P(n,u(n),\lambda)\overline{T}, \quad \overline{T} = ZTZ^{-1}. \tag{3.10}$$

Corollary 1. *Linear equation* (2.1) *rewritten in the following special form*

$$\psi(n+1,\lambda) = P(n,u(n),\lambda) Z \psi(n,\lambda)$$

is reduced to the block-diagonal form

$$\varphi(n+1,\lambda) = h(n,\lambda) Z \varphi(n,\lambda)$$

by the transformation $\psi(n,\lambda) = T(n,\lambda)\varphi(n,\lambda)$ if $P = D_n(\beta)\alpha$, $f = \alpha Z \beta$ and conditions (3.6), (3.8) *are satisfied.*

4. Asymptotic diagonalization of Lax operator and conservation laws

We consider the dynamical system

$$F(D_m D_n v, D_m v, D_n v, v) = 0, \tag{4.1}$$

where the sought object is an vector-valued function $v = v(n,m)$ with the coordinates $v_j(n,m)$, $j = 1,\dots,N$, depending on integer n, m. The shift operators D_n and D_m act by the rules $D_n y(n,m) = y(n+1,m)$ and $D_m y(n,m) = y(n,m+1)$. We assume that (4.1) is the compatibility condition for linear equations

$$\begin{aligned} y(n+1,m) &= P(n,m,[v],\lambda)Zy(n,m), \\ y(n,m+1) &= R(n,m,[v],\lambda)y(n,m). \end{aligned} \tag{4.2}$$

The expression $[v]$ means that the functions P and R depend on the variable v and a finite number of its shifts $D_n^k v$, $D_m^k v$. Let us introduce the discrete operators $L = D_n^{-1}PZ$ and $M = D_m^{-1}R$. Then the compatibility condition of system (4.2) can be written as

$$[L, M] = 0, \quad \text{where} \quad [L, M] = LM - ML. \tag{4.3}$$

We note that the first equation in (4.2) has the form (2.3). Assume that $P(n,m,[v],\lambda)$ satisfies the assumptions of Theorem 1: P is analytic in the vicinity of $\lambda = \lambda_0$ for arbitrary integer n, m and for all values $u = [v]$ in some domain, while principal minors (3.8) are non-zero in this domain. We also assume that the function $R(n,m,[v],\lambda)$ is meromorphic in the vicinity of the point $\lambda = \lambda_0$, when u takes the values in the considered domain.

It follows from Theorem 1 that the discrete operator $L = D_n^{-1}PZ$ is reduced to the quasi-diagonal form $L_0 = D_n^{-1}hZ$ by the transformation $L \to T^{-1}LT = L_0$, where $T(n,\lambda) = \sum_{i\geq 0}^{\infty} T_{(i)}(\lambda-\lambda_0)^i$. It follows from (4.3) that $[L_0, M_0] = 0$, where $M_0 := T^{-1}MT$. By the construction and by the above made assumption, the coefficient S in the formula $M_0 = D_m^{-1}S$ is a formal series of the form $S = (\lambda-\lambda_0)^k \sum_{i=0}^{\infty} S_i(\lambda-\lambda_0)^{-i}$.

Theorem 2. *The coefficients S_i of the series S has the same bloc-diagonal structure as the matrix h.*

By the block-diagonal structure, S commutes with Z and we find that

$$D_n(S)h = D_m(h)S. \tag{4.4}$$

Passing to the block representation $S = \{S_{ij}\}$, $h = \{h_{ij}\}$ in the identity (4.4), we obtain $D_n(S_{ii})h_{ii} = D_m(h_{ii})S_{ii}$. Now it is clear that the equation

$$(D_n - 1)\log\det S_{ii} = (D_m - 1)\log\det h_{ii}, \qquad i = 1, 2, \dots, N_0, \tag{4.5}$$

generates an infinite sequence of conservation laws for equation (4.1). Since the function $\det S = \prod_{i=1}^{N_0} \det S_{ii}$ is not identically zero, the logarithms in (4.5) are well-defined.

5. Conservation laws of two-component discrete potential Korteweg-de Vries equation

Consider the two-component discrete potential Korteweg-de Vries equation (cdpKdV)

$$\begin{aligned} (u_{n,m} - u_{n+1,m+1})(v_{n+1,m} - v_{n,m+1}) &= \delta^2 - \sigma^2, \\ (v_{n,m} - v_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) &= \delta^2 - \sigma^2. \end{aligned} \tag{5.1}$$

In this section we describe an infinite sequence of conservation laws and construct higher symmetries for system (5.1) by means of the method of asymptotic diagonalization of Lax operators. Lax pair for (5.1) was constructed in work [13] and is given by by the system of equations

$$y_{n+1,m} = f y_{n,m}, \quad y_{n,m+1} = g y_{n,m}, \tag{5.2}$$

where the potentials f and g are written as

$$f=\begin{pmatrix} 0 & -u_{n+1,m} & 0 & 1\\ -v_{n+1,m} & 0 & 1 & 0\\ 0 & -\lambda^{-1}-u_{n+1,m}v_{n,m} & 0 & v_{n,m}\\ -\lambda^{-1}-u_{n,m}v_{n+1,m} & 0 & u_{n,m} & 0\end{pmatrix},$$

$$g=\begin{pmatrix} 0 & -u_{n,m+1} & 0 & 1\\ -v_{n,m+1} & 0 & 1 & 0\\ 0 & \sigma^2-\delta^2-\lambda^{-1}-u_{n,m+1}v_{n,m} & 0 & v_{n,m}\\ \sigma^2-\delta^2-\lambda^{-1}-u_{n,m}v_{n,m+1} & 0 & u_{n,m} & 0\end{pmatrix}.$$

We represent the potentials f and g as $f=F\Omega$ and $g=G\Omega$, respectively, where

$$F=\begin{pmatrix} -u_{n+1,m} & 0 & 1 & 0\\ 0 & -v_{n+1,m} & 0 & 1\\ -\lambda^{-1}-u_{n+1,m}v_{n,m} & 0 & v_{n,m} & 0\\ 0 & -\lambda^{-1}-v_{n+1,m}u_{n,m} & 0 & u_{n,m}\end{pmatrix},$$

$$G=\begin{pmatrix} -u_{n,m+1} & 0 & 1 & 0\\ 0 & -v_{n,m+1} & 0 & 1\\ -\lambda^{-1}+\sigma^2-\delta^2-u_{n,m+1}v_{n,m} & 0 & v_{n,m} & 0\\ 0 & -\lambda^{-1}+\sigma^2-\delta^2-v_{n,m+1}u_{n,m} & 0 & u_{n,m}\end{pmatrix},$$

$$\Omega=\begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

Let $\mathcal{R}$ be the ring of matrices $X_{n\times n}$ satisfying the condition $\sigma X\sigma^{-1}=X$, where $\sigma=\mathrm{diag}(1,-1,1,-1)$. It is easy to see that the matrices F and G belong to the group $\mathcal{G}$ of the invertible elements of the ring $\mathcal{R}$.

By means of the change $\Omega^{n+m}\varphi_{n,m}=y_{n,m}$, we transform system (5.2) to the system of the equations

$$\varphi_{n+1,m}=\tilde{F}\varphi_{n,m},\quad \varphi_{n,m+1}=\tilde{G}\varphi_{n,m} \tag{5.3}$$

with new potentials

$$\tilde{F}=\Omega^{-(n+m+1)}F\Omega^{n+m+1}=\begin{pmatrix} -\tilde{p}_{n+1,m} & I\\ -\lambda^{-1}I-\tilde{p}_{n+1,m}\tilde{p}_{n,m} & \tilde{p}_{n,m}\end{pmatrix},$$

$$\tilde{G}=\Omega^{-(n+m+1)}G\Omega^{n+m+1}=\begin{pmatrix} -\tilde{p}_{n,m+1} & I\\ -\lambda^{-1}I+(\sigma^2-\delta^2)I-\tilde{p}_{n,m+1}\tilde{p}_{n,m} & \tilde{p}_{n,m}\end{pmatrix}.$$

Here I stands for the unit block $\mathrm{diag}(1,1)$, the variable $\tilde{p}_{n,m}$ is defined by the following formula:

$$\tilde{p}_{n,m}=E^{-(n+m)}\begin{pmatrix} u_{n,m} & 0\\ 0 & v_{n,m}\end{pmatrix}E^{n+m}, \tag{5.4}$$

where

$$E=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Now system (5.1) can be considered as the commutation condition of two discrete operators $\tilde{\mathcal{L}}=D_n^{-1}\tilde{F}$ and $\tilde{\mathcal{M}}=D_m^{-1}\tilde{G}$.

Let us reduce the equation $\varphi_{n+1,m}=\tilde{F}\varphi_{n,m}$ to special form (2.3). In order to do it, we represent the potential $\tilde{F}$ as the product $\tilde{F}=\tilde{\alpha}Z\tilde{\beta}$ of three matrices, block lower triangular matrix $\tilde{\alpha}$, block diagonal matrix Z and block upper triangular matrix $\tilde{\beta}$:

$$\tilde{\alpha} = \begin{pmatrix} I & 0 \\ \tilde{q_{n,m}} + \lambda^{-1}\tilde{p}_{n+1,m}^{-1} & -\tilde{p}_{n+1,m}^{-1} \end{pmatrix},$$
$$Z = \begin{pmatrix} I & 0 \\ 0 & \lambda^{-1}I \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} -\tilde{p}_{n+1,m} & I \\ 0 & I \end{pmatrix}.$$

Then the change $\psi = \tilde{\beta}\varphi$ reduces the first equation in (5.3) to the form $\psi_{n+1,m} = \tilde{P}\psi_{n,m}$, where $\tilde{P} = D_{n+1}(\tilde{\beta})\tilde{\alpha}$, namely,

$$\tilde{P}(\lambda) = \begin{pmatrix} \tilde{q}_{n,m} - \tilde{p}_{n+1,m} & -\tilde{p}_{n+1,m}^{-1} \\ \tilde{q}_{n,m} & -\tilde{p}_{n+1,m}^{-1} \end{pmatrix} + \lambda^{-1}\begin{pmatrix} \tilde{p}_{n+1,m}^{-1} & 0 \\ \tilde{p}_{n+1,m}^{-1} & 0 \end{pmatrix}.$$

We see that $\tilde{P} \in \mathcal{R}$. At that, the minors

$$\det(\tilde{q}_{n,m} - \tilde{p}_{n+1,m}), \quad \det\tilde{P}(\infty) \tag{5.5}$$

of the matrix $\tilde{P}(\infty)$ are non-zero. We recall that $\tilde{q}_{n,m}$ and $\tilde{p}_{n+1,m}$ are 2×2 blocks. By Theorem 1 there exist formal series

$$\tilde{T} = \tilde{T}_0 + \tilde{T}_1\lambda^{-1} + \cdots, \quad \tilde{h} = \tilde{h}_0 + \tilde{h}_1\lambda^{-1} + \cdots$$

such that the operator $\tilde{L}_0 := \tilde{T}^{-1}\tilde{L}\tilde{T}$, where $\tilde{L} = D_n^{-1}\tilde{P}Z$, is a diagonal operator of the form $\tilde{L}_0 = D_n^{-1}\tilde{h}Z$. We find the series $\tilde{T}$ and $\tilde{h}$ by the equation

$$D_n(\tilde{T})\tilde{h} = \tilde{P}\tilde{\tilde{T}}, \quad \tilde{\tilde{T}} = Z\tilde{T}Z^{-1}. \tag{5.6}$$

Since

$$Z\tilde{T}Z^{-1} = \begin{pmatrix} \tilde{T}_{1,1} & \lambda\tilde{T}_{1,2} \\ \lambda^{-1}\tilde{T}_{2,1} & \tilde{T}_{2,2} \end{pmatrix},$$

the identities

$$\lambda\tilde{T}_{1,2} = \tilde{\tilde{T}}_{1,2}, \quad \lambda^{-1}\tilde{T}_{2,1} = \tilde{\tilde{T}}_{2,1}, \quad \tilde{T}_{i,i} = \tilde{\tilde{T}}_{i,i}, \quad i = 1,2,$$

hold true. Substituting here the formal series

$$\tilde{T} = \tilde{T}_0 + \lambda^{-1}\tilde{T}_1 + \cdots, \quad \tilde{\tilde{T}} = \tilde{\tilde{T}}_0 + \lambda^{-1}\tilde{\tilde{T}}_1 + \cdots,$$

we obtain that

$$\begin{aligned}
&\lambda\tilde{T}_{0,1,2} + \tilde{T}_{1,1,2} + \lambda^{-1}\tilde{T}_{2,1,2} + \cdots = \tilde{\tilde{T}}_{0,1,2} + \lambda^{-1}\tilde{\tilde{T}}_{1,1,2} + \lambda^{-2}\tilde{\tilde{T}}_{2,1,2} + \cdots, \\
&\lambda^{-1}\tilde{T}_{0,2,1} + \lambda^{-2}\tilde{T}_{1,2,1} + \lambda^{-3}\tilde{T}_{2,2,1} + \cdots = \tilde{\tilde{T}}_{0,2,1} + \lambda^{-1}\tilde{\tilde{T}}_{1,2,1} + \lambda^{-2}\tilde{\tilde{T}}_{2,2,1} + \cdots, \\
&\tilde{T}_{0,i,i} + \lambda^{-1}\tilde{T}_{1,i,i} + \cdots = \tilde{\tilde{T}}_{0,i,i} + \lambda^{-1}\tilde{\tilde{T}}_{1,i,i} + \cdots, \quad i = 1,2.
\end{aligned}$$

It follows from the latter identities that $\tilde{T}_{0,1,2} = 0$, $\tilde{\tilde{T}}_{0,2,1} = 0$, i.e., the matrices $\tilde{T}$ and $\tilde{\tilde{T}}$ are block lower triangular and upper triangular, respectively, and the identities

$$\tilde{T}_{p+1,1,2} = \tilde{\tilde{T}}_{p,1,2}, \quad \tilde{T}_{p,2,1} = \tilde{\tilde{T}}_{p+1,2,1}, \quad \tilde{T}_{p,i,i} = \tilde{\tilde{T}}_{p,i,i}, \quad i = 1,2, \tag{5.7}$$

hold true. We return back to equation (5.6) and rewrite it as

$$D_n(\tilde{T}_0 + \lambda^{-1}\tilde{T}_1 + \cdots)(\tilde{h}_0 + \lambda^{-1}\tilde{h}_1 + \cdots) = (\tilde{P}_0 + \lambda^{-1}\tilde{P}_1)(\tilde{\tilde{T}}_0 + \lambda^{-1}\tilde{\tilde{T}}_1 + \cdots) \tag{5.8}$$

Comparing the coefficients at different powers of λ, we obtain the following sequence of the equations

$$D_n(\tilde{T}_0)\tilde{h}_0 = \tilde{P}_0\tilde{\overline{\overline{T}}}_0, \tag{5.9}$$

$$D_n(\tilde{T}_1)\tilde{h}_0 + D_n(\tilde{T}_0)\tilde{h}_1 - \tilde{P}_0\tilde{\overline{\overline{T}}}_1 = \tilde{P}_1\tilde{\overline{\overline{T}}}_0, \tag{5.10}$$

$$D_n(\tilde{T}_k)\tilde{h}_0 + D_n(\tilde{T}_0)\tilde{h}_k - \tilde{P}_0\tilde{\overline{\overline{T}}}_k = \tilde{P}_1\tilde{\overline{\overline{T}}}_{k-1} - \sum_{j=1}^{k-1} D_n(\tilde{T}_{k-j})\tilde{h}_j, \qquad k \geq 2. \tag{5.11}$$

Equation (5.9) is the Gauss problem on the decomposition of the matrix $\tilde{P}_0$ into the product of three matrices in the group $\mathcal{G}$, a block lower triangular matrix $D_n(\tilde{T}_0)$, a block diagonal matrix $\tilde{h}_0$ and a block upper triangular $\tilde{\overline{\overline{T}}}_0^{-1}$. The solvability of this problem is guaranteed by regularity condition (5.5). The uniqueness of the solution to this problem is ensured by letting the blocks of the matrix $\tilde{T}_0$ to be equal to the unit matrices of the size 2×2. Since the Gauss problem is solved in group $\mathcal{G}$, we obtain that the matrices $\tilde{T}_0$, $\tilde{h}_0$ and $\tilde{\overline{\overline{T}}}_0$ belong to the group $\mathcal{G}$.

We assume that the diagonal blocks of the matrices $\tilde{T}_k$ and $\tilde{\overline{\overline{T}}}_k$ are zero for all $k > 0$. We expand each of the matrices $\tilde{T}_k$ and $\tilde{\overline{\overline{T}}}_k$ into the sum of a lower block triangular matrix and an upper block triangular matrices with zero diagonal blocks

$$\tilde{T}_k = \tilde{T}_{kL} + \tilde{T}_{kU}, \qquad \tilde{\overline{\overline{T}}}_k = \tilde{\overline{\overline{T}}}_{kL} + \tilde{\overline{\overline{T}}}_{kU}. \tag{5.12}$$

The matrices $\tilde{T}_{1U}$ and $\tilde{\overline{\overline{T}}}_{1L}$ can be found easily. Indeed, employing (5.7), we have $\tilde{T}_{1,1,2} = \tilde{\overline{\overline{T}}}_{0,1,2}$, $\tilde{\overline{\overline{T}}}_{1,2,1} = \tilde{T}_{0,2,1}$. That is, there elements were found at the previous step. In order to find the unknowns $\tilde{T}_{1L}$ and $\tilde{\overline{\overline{T}}}_{1U}$, we employ equation (5.10) written as

$$\tilde{h}_1\tilde{h}_0^{-1} + D_n(\tilde{T}_0^{-1}\tilde{T}_{1L}) - \tilde{h}_0\tilde{\overline{\overline{T}}}_0^{-1}\tilde{\overline{\overline{T}}}_{1U}\tilde{h}_0^{-1} = H_1,$$

where the right hand side $H_1 = D_n(\tilde{T}_0^{-1})\tilde{P}_1\tilde{\overline{\overline{T}}}_0\tilde{h}_0^{-1} - D_n(\tilde{T}_0^{-1}\tilde{T}_{1U}) + \tilde{h}_0\tilde{\overline{\overline{T}}}_0^{-1}\tilde{\overline{\overline{T}}}_{1L}h_0^{-1}$ involves the known matrices. At that, $H_1 \in \mathcal{R}$. In order to find the unknowns $\tilde{h}_1$, $\tilde{T}_{1L}$, $\tilde{\overline{\overline{T}}}_{1U}$, we need to expand H_1 into the sum of three terms, a block diagonal matrix $\tilde{h}_1\tilde{h}_0^{-1}$, a block lower triangular matrix $D_n(\tilde{T}_0^{-1}\tilde{T}_{1L})$, and a block upper triangular matrix $-h_0\overline{T}_0^{-1}\overline{T}_{1U}h_0^{-1}$. And since this problem is solved in the ring $\mathcal{R}$, the sought terms also belong to $\mathcal{R}$, and therefore, the matrices $\tilde{T}_1$ $\tilde{h}_1$ and $\tilde{\overline{\overline{T}}}_1$ also belong to $\mathcal{R}$. Continuing this procedure, we find all coefficients $\tilde{T}_k$ and $\tilde{h}_k$ by the equation

$$\tilde{h}_k\tilde{h}_0^{-1} + D_n(\tilde{T}_0^{-1}\tilde{T}_{kL}) - \tilde{h}_0\tilde{\overline{\overline{T}}}_0^{-1}\tilde{\overline{\overline{T}}}_{kU}\tilde{h}_0^{-1} = H_k,$$

where the term H_k contains the terms found at the previous step. Thus, we have proved that all the coefficients of the series $\tilde{T}$ belong to the ring $\mathcal{R}$.

Let us write out explicitly the first elements of the formal series $\tilde{T}$ and $\tilde{h}$:

$$\tilde{T} = \begin{pmatrix} I & 0 \\ -\frac{\tilde{p}_{n-1,m}}{\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m}} & I \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\tilde{p}_{n+1,m}(\tilde{p}_{n+2,m}-\tilde{p}_{n,m})} \\ -\frac{\tilde{p}_{n+1,m}}{(\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m})^2(\tilde{p}_{n,m}-\tilde{p}_{n-2,m})} & 0 \end{pmatrix}\lambda^{-1} + \cdots,$$

$$\tilde{h} = \begin{pmatrix} \tilde{p}_{n,m} - \tilde{p}_{n+2,m} & 0 \\ 0 & -\frac{\tilde{p}_{n+2,m}}{\tilde{p}_{n+1,m}(\tilde{p}_{n+2,m}-\tilde{p}_{n,m})} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m}} & 0 \\ 0 & -\frac{\tilde{p}_{n+2,m}\tilde{p}_{n+3,m}-\tilde{p}_{n+2,m}\tilde{p}_{n+1,m}+\tilde{p}_{n,m}\tilde{p}_{n+1,m}}{\tilde{p}^2_{n+1,m}(\tilde{p}_{n+3,m}-\tilde{p}_{n+1,m})(\tilde{p}_{n+2,m}-\tilde{p}_{n,m})^2} \end{pmatrix}\lambda^{-1} + \cdots.$$

We note that $\tilde{p}$ is a diagonal matrix of the size 2×2, this is why using here the division sign to denote the operation of the inverting a matrix for reducing the formulae length, we do not spoil the meaning of the expressions.

The operator $\tilde{M} = D_m^{-1}\tilde{G}$ can be diagonalized as follows: $\tilde{M}_0 = D_m^{-1}\tilde{S}$, where $\tilde{S} = D_m(\tilde{T}^{-1}\tilde{\beta})\tilde{G}\tilde{\beta}^{-1}\tilde{T}$. We write out explicitly first terms of the series $\tilde{S}$:

$$\tilde{S} = \begin{pmatrix} -\frac{\delta^2-\sigma^2}{\tilde{p}_{n,m+1}-\tilde{p}_{n+1,m}} & 0 \\ 0 & -\frac{\delta^2-\sigma^2+\tilde{p}_{n,m}(\tilde{p}_{n,m+1}-\tilde{p}_{n+1,m})}{\tilde{p}_{n+1,m}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m}} & 0 \\ 0 & \frac{\delta^2-\sigma^2+\tilde{p}_{n,m+1}\tilde{p}_{n,m}+\tilde{p}_{n+2,m}\tilde{p}_{n+1,m}-\tilde{p}_{n+1,m}\tilde{p}_{n,m}}{\tilde{p}^2_{n+1,m}(\tilde{p}_{n,m}-\tilde{p}_{n+2,m})} \end{pmatrix}\lambda^{-1} + \cdots .$$

We write out explicitly three conservation laws in the infinite sequence obtained as a result of the diagonalization

$$\begin{aligned}
&(D_n-1)\log\frac{1}{\tilde{p}_{n+1,m}-\tilde{p}_{n,m+1}} = (D_m-1)\log(\tilde{p}_{n,m}-\tilde{p}_{n+2,m}),\\
&(D_n-1)\frac{\tilde{p}_{n+1,m}-\tilde{p}_{n,m+1}}{(\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m})(\delta^2-\sigma^2)} = (D_m-1)\frac{1}{(\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m})(\tilde{p}_{n,m}-\tilde{p}_{n+2,m})},\\
&(D_n-1)\left[-\frac{\delta^2-\sigma^2+\tilde{p}_{n,m}(\tilde{p}_{n,m+1}-\tilde{p}_{n+1,m})+\tilde{p}_{n+2,m}\tilde{p}_{n+1,m}}{\tilde{p}_{n+1,m}(\tilde{p}_{n,m}-\tilde{p}_{n+2,m})(\delta^2-\sigma^2+\tilde{p}_{n,m}(\tilde{p}_{n,m+1}-\tilde{p}_{n+1,m}))}\right]\\
&\qquad = (D_m-1)\frac{\tilde{p}_{n+2,m}\tilde{p}_{n+3,m}-\tilde{p}_{n+2,m}\tilde{p}_{n+1,m}+\tilde{p}_{n+1,m}\tilde{p}_{n,m}}{\tilde{p}_{n+1,m}\tilde{p}_{n+2,m}(\tilde{p}_{n,m}-\tilde{p}_{n+2,m})(\tilde{p}_{n+1,m}-\tilde{p}_{n+3,m})}.
\end{aligned}$$

Passing to the original variables u and v, we obtain

$$\begin{aligned}
&(D_n-1)\log\frac{1}{(u_{0,1}-u_{1,0})(v_{0,1}-v_{1,0})} = (D_m-1)\log(u-u_{2,0})(v-v_{2,0}),\\
&(D_n-1)\left[\frac{u_{0,1}-u_{1,0}}{(p^2-q^2)(u_{-1,0}-u_{1,0})}+\frac{v_{0,1}-v_{1,0}}{(p^2-q^2)(v_{-1,0}-v_{1,0})}\right]\\
&\qquad = (D_m-1)\left[-\frac{(v-v_{2,0})(u_{-1,0}-u_{1,0})}{(u_{-1,0}-u_{1,0})(v-v_{2,0})}-\frac{(u-u_{2,0})(v_{-1,0}-v_{1,0})}{(v_{-1,0}-v_{1,0})(u-u_{2,0})}\right],\\
&(D_n-1)\left[-\frac{(p^2-q^2+u\nu+v_{1,0}u_{2,0})}{v_{1,0}\rho(p^2-q^2+u\nu)}-\frac{(p^2-q^2+v\mu+u_{1,0}v_{2,0})}{u_{1,0}\sigma(p^2-q^2+v\mu)}\right]\\
&\qquad = (D_m-1)\left[\frac{v_{3,0}}{v_{1,0}\rho\sigma_{1,0}}+\frac{1}{u_{2,0}\sigma_{1,0}}+\frac{u_{3,0}}{u_{1,0}\sigma\rho_{1,0}}+\frac{1}{v_{2,0}\rho_{1,0}}\right].
\end{aligned}$$

Here we employ the notations $\rho = u-u_{2,0}$, $\sigma = v-v_{2,0}$, $\mu = u_{0,1}-u_{1,0}$, $\nu = v_{0,1}-v_{1,0}$.

We proceed to constructing the symmetries for system (5.1). The scheme of constructing the higher symmetries of a discrete dynamical system by means of the diagonalization algorithm was exposed in details in work [6].

We return back to the first of the linear equations in Lax pair (5.3) for system (5.1) and by employing the discrete operator $\tilde{\mathcal{L}} = D_n^{-1}\tilde{F}$ we rewrite it as

$$\varphi = \tilde{\mathcal{L}}\varphi.$$

As it was shown above, the change of the variables $\psi = \tilde{\beta}\varphi$ reduces this equation to the special form

$$\psi = \tilde{L}\psi$$

with the operator $\tilde{L} = D_n^{-1}\tilde{P}Z$. Then the formal series

$$\tilde{T} = \sum_{k=0}^{\infty} \tilde{T}_{(k)}\lambda^{-k}, \quad \tilde{h} = \sum_{k=0}^{\infty} \tilde{h}_{(k)}\lambda^{-k} \tag{5.13}$$

"diagonalizing" the mentioned equation in the vicinity of $\lambda = \infty$ wre found, that is, they are such that the operator $\tilde{L}_0 := \tilde{T}^{-1}\tilde{L}\tilde{T}$ is an operator with (block) diagonal coefficients $\tilde{L}_0 = D_n^{-1}\tilde{h}Z$.

Following the lines of work [6], we construct the formal series

$$\tilde{B}_0 = \sum_{k=-M}^{\infty} (\tilde{B}_0)_{(k)}\lambda^{-k},$$

with the coefficients $(\tilde{B}_0)_{(k)}$ having the same block diagonal structure as the elements of the series $\tilde{h}$ and being independent of n and such that $[\tilde{L}_0, \tilde{B}_0] = 0$. It was proved in work [6] that in this case the formal series $\tilde{B}' = \sum_{k=-M}^{\infty} \tilde{B}'_{(k)}\lambda^{-k}$ given by the formula $\tilde{B}' = \tilde{T}\tilde{B}_0\tilde{T}^{-1}$ commutes with the operator $\tilde{L}$. Then the formal series $\tilde{B} = \sum_{k=-M}^{\infty} \tilde{B}_{(k)}\lambda^{-k}$ defined by the formula

$$\tilde{B} = \tilde{\beta}^{-1}\tilde{T}\tilde{B}_0(\tilde{\beta}^{-1}\tilde{T})^{-1} \tag{5.14}$$

commutes with the operator $\tilde{\mathcal{L}}$.

Theorem 3. *The coefficients $\tilde{B}_{(k)}$ of series* (5.14) *belong to the ring $\mathcal{R}$, that is, for each k, they satisfy the relation $\sigma\tilde{B}_{(k)}\sigma^{-1} = \tilde{B}_{(k)}$, where $\sigma = \mathrm{diag}(1, -1, 1, -1)$.*

It was shown above that the coefficients $\tilde{T}_{(k)}$ of the series $\tilde{T}$ lie in the ring $\mathcal{R}$ for each k and $\tilde{\beta} \in \tilde{R}$ by the construction. Thus, the statement of Theorem 3 is implied by formula (5.14).

We choose $\tilde{B}_0$ as

$$\tilde{B}_0 = \tilde{B}_{(-M)}\lambda^{-M}, \quad \tilde{B}_{(-M)} = \mathrm{diag}(1, 1, -1, -1). \tag{5.15}$$

Then by straightforward calculations we prove that for each k, the coefficients $\tilde{B}_{(k)}$ of the series $\tilde{B}$ satisfy the condition

$$\tilde{B}_{22}^k = -\tilde{B}_{11}^k.$$

Here by $\tilde{B}_{ij}^k$ we denote 2×2 blocks of the matrix $\tilde{B}_{(k)}$:

$$\tilde{B}_{(k)} = \begin{pmatrix} \tilde{B}_{11}^k & \tilde{B}_{12}^k \\ \tilde{B}_{21}^k & \tilde{B}_{22}^k \end{pmatrix}. \tag{5.16}$$

Then we expand the series $\tilde{B}$ into the sum $\tilde{B} = \tilde{A} + (\tilde{B} - \tilde{A})$, where $\tilde{A} = \sum_{k=1}^{M} \tilde{A}_{(k)}\lambda^k = \sum_{k=-M}^{-1} \tilde{B}_{(k)}\lambda^{-k}$. Then

$$[\tilde{\mathcal{L}}, \tilde{B}] = [\tilde{\mathcal{L}}, \tilde{A}] + [\tilde{\mathcal{L}}, \tilde{B} - \tilde{A}] = 0.$$

The potential $\tilde{F}$ of the first equation in Lax pair (5.3) is a rational function of the form $\tilde{F} = \lambda^{-1}\tilde{F}_1 + \tilde{F}_0$, where

$$\tilde{F}_1 = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}, \quad \tilde{F}_0 = \begin{pmatrix} -\tilde{p}_{n+1,m} & I \\ -\tilde{p}_{n+1,m}\tilde{p}_{n,m} & \tilde{p}_{n,m} \end{pmatrix}.$$

Let us find out the form of the commutator $[\tilde{\mathcal{L}}, \tilde{A}]$:

$$[\tilde{\mathcal{L}}, \tilde{A}] = [D^{-1}\tilde{F}, \tilde{A}] = [D^{-1}(\lambda^{-1}\tilde{F}_1 + \tilde{F}_0), \sum_{k=1}^{M} \tilde{A}_{(k)}\lambda^k] = [D^{-1}\tilde{F}_1, \tilde{A}_{(1)}] + \sum_{k=1}^{M} a_k\lambda^k.$$

On the other hand,

$$[\tilde{\mathcal{L}},\tilde{A}]=-[\tilde{\mathcal{L}},\tilde{B}-\tilde{A}]=-[D^{-1}(\lambda^{-1}\tilde{F}_1+\tilde{F}_0),\sum_{k=0}^{\infty}\tilde{B}_{(k)}\lambda^{-k}]=-[D^{-1}\tilde{F}_0,\tilde{B}_{(0)}]+\sum_{k=0}^{\infty}a'_k\lambda^{-k}.$$

It follows from the latter identities that

$$[\tilde{\mathcal{L}},\tilde{A}]=R,$$

where

$$R=-[D_n^{-1}\tilde{F}_0,\tilde{B}_{(0)}]=[D_n^{-1}\tilde{F}_1,\tilde{A}_{(1)}]. \tag{5.17}$$

It follows from the second identity in (5.17) that

$$D_n(R)=\tilde{F}_1\tilde{A}_{(1)}-D_n(\tilde{A}_{(1)})\tilde{F}_1=\begin{pmatrix}-D_n\big(\tilde{A}^1_{12}\big) & 0\\ \tilde{A}^1_{11}-D_n\big(\tilde{A}^1_{11}\big) & \tilde{A}^1_{12}\end{pmatrix}. \tag{5.18}$$

Thus, $D_n(R)$ is a (block) lower triangular matrix. Employing this fact, by the first identity in (5.17), we obtain that $D_n(R)$ is of the form

$$D_n(R)=-\tilde{F}_0\tilde{B}_{(0)}+D_n\big(\tilde{B}_{(0)}\big)\tilde{F}_0=\begin{pmatrix}R_{11} & 0\\ -\tilde{p}_{n+1,m}R_{22}+\tilde{p}_{n,m}R_{11} & R_{22}\end{pmatrix}. \tag{5.19}$$

We let $\frac{\mathrm{d}}{\mathrm{d}t}D_n^{-1}\tilde{F}=R$ or $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{F}_0=D_n(R)$. This relation introduces a differential-difference equation. Indeed, the left hand side is of the form

$$\begin{pmatrix}-\frac{\mathrm{d}\tilde{p}_{n+1,m}}{\mathrm{d}t} & 0\\ -\left(\frac{\mathrm{d}\tilde{p}_{n+1,m}}{\mathrm{d}t}\tilde{p}_{n,m}+\tilde{p}_{n+1,m}\frac{\mathrm{d}\tilde{p}_{n,m}}{\mathrm{d}t}\right) & \frac{\mathrm{d}\tilde{p}_{n,m}}{\mathrm{d}t}\end{pmatrix}$$

and by (5.18) and (5.19) the right hand side is

$$D_n(R)=\begin{pmatrix}-D_n\big(\tilde{A}^1_{12}\big) & 0\\ -\left(\tilde{p}_{n+1,m}\tilde{A}^1_{12}+\tilde{p}_{n,m}D_n\big(\tilde{A}^1_{12}\big)\right) & \tilde{A}^1_{12}\end{pmatrix}.$$

Therefore,

$$\frac{\mathrm{d}\tilde{p}_{n,m}}{\mathrm{d}t}=\tilde{A}^1_{12}. \tag{5.20}$$

In what follows we provide explicitly two higher symmetries. We choose the initial series $\tilde{B}_0$ as $\tilde{B}_0=\mathrm{diag}(1,1,-1,-1)\lambda$. Now the formal series $\tilde{B}$ is of the form

$$\tilde{B}=\tilde{\beta}^{-1}\tilde{T}\tilde{B}_0\tilde{T}^{-1}\tilde{\beta}=\tilde{B}_{(-1)}\lambda+\tilde{B}_{(0)}+\tilde{B}_{(1)}\lambda^{-1}+\cdots.$$

By construction, the formal series $\tilde{A}$ consists of the elements of the series $\tilde{B}$ at the positive coefficients of the spectral parameter and in this case it has the form $\tilde{A}=\tilde{A}_{(1)}\lambda$, where $\tilde{A}_{(1)}=\tilde{B}_{(-1)}=\mathcal{A}$ and $\mathcal{A}$ stands for the following matrix:

$$\mathcal{A}=\begin{pmatrix}\frac{p_{n+1,m}+p_{n-1,m}}{p_{n+1,m}-p_{n-1,m}} & -\frac{2}{p_{n+1,m}-p_{n-1,m}}\\ \frac{2p_{n-1,m}p_{n+1,m}}{p_{n+1,m}-p_{n-1,m}} & -\frac{p_{n+1,m}+p_{n-1,m}}{p_{n+1,m}-p_{n-1,m}}\end{pmatrix}.$$

Employing (5.20), we write out the symmetries

$$\frac{\mathrm{d}\tilde{p}_{n,m}}{\mathrm{d}t}=-\frac{2}{\tilde{p}_{n+1,m}-\tilde{p}_{n-1,m}}.$$

Returning back to the original variables u and v, we obtain the symmetries

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}t}=-\frac{2}{v_{n+1,m}-v_{n-1,m}},\qquad \frac{\mathrm{d}v_{n,m}}{\mathrm{d}t}=-\frac{2}{u_{n+1,m}-u_{n-1,m}}$$

for system (5.1).

In order to find the symmetry of the next order, we choose the initial series $\tilde{B}_0$ as $\tilde{B}_0 = \operatorname{diag}(1, 1, -1, -1)\lambda^2$. Now the sought formal series is

$$\tilde{B} = \tilde{\beta}^{-1}\tilde{T}\tilde{B}_0\tilde{T}^{-1}\tilde{\beta} = \tilde{B}_{(-2)}\lambda^2 + \tilde{B}_{(-1)}\lambda + \cdots .$$

Then the formal $\tilde{A}$ is of the form $\tilde{A} = \tilde{A}_{(2)}\lambda^2 + \tilde{A}_{(1)}\lambda$,
where $\tilde{A}_{(2)} = \tilde{B}_{(-2)} = \mathcal{A}$,

$$\tilde{A}_{(1)} = \tilde{B}_{(-1)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix},$$

and

$$\begin{aligned} a_{11} &= \frac{2(\tilde{p}_{n+2,m}\tilde{p}_{n+1,m} - \tilde{p}_{n,m}\tilde{p}_{n+1,m} + \tilde{p}_{n-1,m}\tilde{p}_{n,m} - \tilde{p}_{n-1,m}\tilde{p}_{n-2,m})}{(\tilde{p}_{n+2,m} - \tilde{p}_{n,m})(\tilde{p}_{n,m} - \tilde{p}_{n-2,m})(\tilde{p}_{n+1,m} - \tilde{p}_{n-1,m})^2}, \\ a_{12} &= -\frac{2(\tilde{p}_{n+2,m} - \tilde{p}_{n-2,m})}{(\tilde{p}_{n+2,m} - \tilde{p}_{n,m})(\tilde{p}_{n,m} - \tilde{p}_{n-2,m})(\tilde{p}_{n+1,m} - \tilde{p}_{n-1,m})^2}, \\ a_{21} &= \frac{2(\tilde{p}^2_{n+1,m}\tilde{p}_{n+2,m} - \tilde{p}^2_{n+1,m}\tilde{p}_{n,m} + \tilde{p}^2_{n-1,m}\tilde{p}_{n,m} - \tilde{p}^2_{n-1,m}\tilde{p}_{n-2,m})}{(\tilde{p}_{n+2,m} - \tilde{p}_{n,m})(\tilde{p}_{n+1,m} - \tilde{p}_{n-1,m})^2(\tilde{p}_{n,m} - \tilde{p}_{n-2,m})}. \end{aligned}$$

Employing (5.20), we obtain

$$\frac{d\tilde{p}_{n,m}}{dt} = -\frac{2(\tilde{p}_{n+2,m} - \tilde{p}_{n-2,m})}{(\tilde{p}_{n+2,m} - \tilde{p}_{n,m})(\tilde{p}_{n,m} - \tilde{p}_{n-2,m})(\tilde{p}_{n+1,m} - \tilde{p}_{n-1,m})^2}$$

or, in terms of the original variables, we write the symmetries

$$\begin{aligned} \frac{du_{n,m}}{dt} &= -\frac{2(u_{n+2,m} - u_{n-2,m})}{(u_{n+2,m} - u_{n,m})(u_{n,m} - u_{n-2,m})(v_{n+1,m} - v_{n-1,m})^2}, \\ \frac{dv_{n,m}}{dt} &= -\frac{2(v_{n+2,m} - v_{n-2,m})}{(v_{n+2,m} - v_{n,m})(v_{n,m} - v_{n-2,m})(u_{n+1,m} - u_{n-1,m})^2} \end{aligned}$$

of system (5.1).

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