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BEHAVIOR OF SOLUTIONS TO ELLIPTIC EQUATIONS WITH NON-POWER NONLINEARITIES IN UNBOUNDED DOMAINS

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Abstract. We establish estimates characterizing the decay rate as $|x| \to \infty$ of solutions to the Dirichlet problems in unbounded domains for a certain class of elliptic equations with non-power nonlinearities.

Keywords: anisotropic elliptic equations, non-power nonlinearity, Sobolev-Orlicz space, unbounded domain.

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Introduction

Let Ω be an arbitrary unbounded domain of the space $\mathbb{R}_n = \{x = (x_1, x_2, \ldots,\ x_n)\}$, $\Omega \subset \mathbb{R}_n$, $n \geqslant 2$. We consider the Dirichlet problem for quasilinear anisotropic second order elliptic equations

$$\sum_{\alpha=1}^{n} (a_\alpha(x, u, \nabla u))_{x_\alpha} - a_0(x, u, \nabla u) = 0, \quad x \in \Omega; \tag{0.1}$$

$$u\Big|_{\partial\Omega} = 0. \tag{0.2}$$

We assume that the functions $a_\alpha(x, s_0, s)$, $\alpha = 0, \ldots, n$, are measurable in $x \in \Omega$ for $\mathbf{s} = (s_0, s) = (s_0, s_1, \ldots, s_n) \in \mathbb{R}_{n+1}$ and continuous in $\mathbf{s} \in \mathbb{R}_{n+1}$ for almost each $x \in \Omega$. Suppose that there exist positive numbers $\widehat{a}$, $\widehat{A}$ and measurable nonnegative functions $\psi(x)$, $\Psi(x)$ such that the inequalities

$$\sum_{\alpha=0}^{n} a_\alpha(x, s_0, s) s_\alpha \geqslant \overline{a} \sum_{\alpha=0}^{n} B_\alpha(s_\alpha) - \psi(x); \tag{0.3}$$

$$\sum_{\alpha=0}^{n} \overline{B}_\alpha(a_\alpha(x, s_0, s)) \leqslant \widehat{A} \sum_{\alpha=0}^{n} B_\alpha(s_\alpha) + \Psi(x); \tag{0.4}$$

$$\sum_{\alpha=0}^{n} (a_\alpha(x, s_0, s) - a_\alpha(x, t_0, t))(s_\alpha - t_\alpha) > 0 \tag{0.5}$$

hold true for almost each $x \in \Omega$ and $\mathbf{s} = (s_0, s)$, $\mathbf{t} = (t_0, t) \in \mathbb{R}_{n+1}$, $\mathbf{s} \neq \mathbf{t}$.

Here $B_0(z)$, $B_1(z)$, ..., $B_n(z)$ are N-functions satisfying the Δ_2-condition and $\overline{B}_0(z)$, $\overline{B}_1(z)$, ..., $\overline{B}_n(z)$ are dual functions, see Section 1.

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As an example we can consider the equation

$$\sum_{\alpha=1}^{n}(B'_{\alpha}(u_{x_\alpha}) + f_{\alpha}(x))_{x_\alpha} - B'_0(u) - f_0(x) = 0 \qquad (0.6)$$

with continuously differentiable N-functions $B_0(z)$, $B_1(z)$, $\dots$, $B_n(z)$ (see Lemma 4).

Starting from the 70s of the last century (see [1]–[4]) and till the present, the qualitative properties are intensively studied for the solutions to ellptic equations with non-power nonlinearities both of second and higher orders. The solutions to boundary values problems for the equations of form (0.1) with the functions $a_0(x,\mathbf{s})$, $a_1(x,\mathbf{s})$, $\dots$, $a_n(x,\mathbf{s})$ having not only polynomial growth in the variables s_0, s_1, $\dots$, s_n were considered mostly in bounded domains. For example, in work [5], the Dirichlet problem in a bounded domain Ω was studied for a nonlinear elliptic equation with a vector function $\mathbf{a}(x,s) = (a_1(x,s),\dots,a_n(x,s))$ satisfying non-standard growth conditions described in terms of N- depending on x. The existence the renormalized solution was proved, while under the strong monotonicity condition the uniqueness was established.

Boundary value problems in unbounded domains for quasilinear elliptic equations with power nonlinearities were also studied in works [6], [7]. It should be noted that a solution to an elliptic problem in an unbounded domain with non-summable data belongs to the corresponding space of locallly summable functions. As a rule, to ensure the uniqueness of the solution of the corresponding boundary value problem in an unbounded domain, one has to impose a restriction for the growth of the solution at the infinity, while for the existence of a solution in the selected class one usually needs the restrictions for the growth of the input data [8].

In 1984,by the semi-linear equation

$$-\Delta u + |u|^{p_0-2}u = f(x), \quad x\in\mathbb{R}_n, \quad p_0>2,$$

H. Brezis showed [9] that there exist elliptic equations for which there exist unique solutions to the boundary value problems with no conditions for their behavior and the growth of the input data at the infinity. Namely, H. Brezis established the existence and the uniqueness of the solution $u\in L_{p_0-1,\mathrm{loc}}(\mathbb{R}_n)$ as $f\in L_{1,\mathrm{loc}}(\mathbb{R}_n)$. The results by H. Brezis were generalized for the equations of higher order by F. Bernis [10].

In work [11], J.I. Diaz and O.A. Oleinik employed the energy integral method and established apriori estimates for a solution to prove the unique solvability for the boundary value problem with homogeneous boundary condition of first and second type (in particular, for the Dirichlet and Neumann problems) for semi-linear equations with variable coefficients

$$-\sum_{i,j=1}^{n}\left(a_{ij}(x)u_{x_j}\right)_{x_i} + a_0(x)|u|^{p_0-2}u = f(x), \quad x\in\Omega, \quad p_0>2, \qquad (0.7)$$

$a_{ij}(x)\in L_{\infty,\mathrm{loc}}(\Omega)$, $a_0(x)\in L_{1,\mathrm{loc}}(\Omega)$, $a_0(x)\geqslant a_0>0$ with no conditions at the infinity. Moreover, in [11] the authors studied the asymptotic behavior at infinifty for the solution to equation (0.7). Under the condition$f(x)=0$, $x\in\Omega\setminus\overline{\Omega}(r_0)$, $\overline{\Omega}(r_0)=\{x\in\Omega\,|\,|x|\leqslant r_0\}$, $r_0>0$, for a solution to equation (0.7) the estimate

$$|u(x)|\leqslant C_1|x|^{-2/(p_0-2)}, \quad x\in\Omega\setminus\overline{\Omega}(r_0). \qquad (0.8)$$

was obtained. Under an additional restriction for the geometry of the unbounded domain Ω the inequality

$$|u(x)|\leqslant C_2e^{-\alpha|x|}, \quad x\in\Omega\setminus\overline{\Omega}(r_0), \quad \alpha>0, \qquad (0.9)$$

was established.

In work [12], M.M. Bokalo and E.V. Domanska studied boundary value problems in unbounded domains for elliptic anisotropic equations with variable nonlinearity exponents. At that, the well-posedness of the boundary value problems was proved with no restrictions for the growth of the solutions and data at the infinity.

The authors of the present work succeeded to select some class of elliptic equations having not only power nonlinearities and to obtain the results close to the cited above. For instance, in work [13] by L.M. Kozhevnikova, A.A. Khadzhi, the solvability of the Dirichlet problem in unbounded domains was established with no restrictions for the growth of the data at infinity. Under additional restrictions for the stucture of the equation, in [14] the uniqueness of the solution to problem (0.1), (0.2) was proved with no restrictions for the growth of the solution at the infinity.

Here we obtain the estimates characterizing the behavior as$|x| \to \infty$ of the solution to problem (0.1), (0.2) in unbounded domains Ω. A power estimate was established for the solutions to anisotropic equations in arbitrary unbounded domains (Theorem 2). For "non-wide" unbounded domains we obtained an exponential estimates for the solutions to isotropic equations (Theorem 3).

1. N-functions and Sobolev-Orlicz spaces

Let us provide necessary information from the theory of N-functions and Sobolev-Orlicz spaces [15]. A non-negative continuous convex function $M(z)$, $z \in \mathbb{R}$, is called N-function if it is even and $\lim\limits_{z\to 0} M(z)/z = 0$, $\lim\limits_{z\to\infty} M(z)/z = \infty$. We note that $M(\epsilon z) \leqslant \epsilon M(z)$ as $0 < \epsilon \leqslant 1$. For an N-function $M(z)$ the integral representation $M(z) = \int_0^{|z|} m(\theta)d\theta$ holds, where $m(\theta)$ is positive as $\theta > 0$, is non-decreasing and right continuous as $\theta \geqslant 0$ and $m(0) = 0$, $\lim\limits_{\theta\to\infty} m(\theta) = \infty$.

For an N-function $M(z)$ and its dual N-function

$$\overline{M}(z) = \sup_{y\geqslant 0}(y|z| - M(y))$$

the Young inequality

$$|zy| \leqslant M(z) + \overline{M}(y), \quad z, y \in \mathbb{R} \tag{1.1}$$

is satisfied [15, Ch. I, Sect. 2, Ineq. (2.6)].

Given N-functions $P(z)$, $M(z)$, we write $P(z) \prec M(z)$ if there exist numbers $l > 0$, $z_0 \geqslant 0$ such that

$$P(z) \leqslant M(lz), \quad |z| \geqslant z_0.$$

N-functions $P(z)$, $M(z)$ are called comparable if one of the relations $P(z) \prec M(z)$ or $M(z) \prec P(z)$ is satisfied. N-function $P(z)$ and $M(z)$ are called equivalent if $P(z) \prec M(z)$ and $M(z) \prec P(z)$.

An N-function $P(z)$ grows slowly than an N-function $M(z)$ ($P(z) \prec\prec M(z)$) if

$$\lim_{z\to\infty} P(z)/M(lz) = 0$$

for each number $l > 0$.

An N-function $M(z)$ satisfies the Δ_2-condition for large z if there exist numbers $c > 0$, $z_0 \geqslant 0$ such that $M(2z) \leqslant cM(z)$ for all $|z| \geqslant z_0$. The Δ_2-condition is equivalent to the inequality

$$M(lz) \leqslant c(l)M(z) \tag{1.2}$$

as $|z| \geqslant z_0$, where l is an arbitrary number larger than one, $c(l) > 0$.

In each class of equivalent N-function obeying the Δ_2-condition there exist N-functions satisfying inequality (1.2) for all z. Hereafter we assume that the Δ_2-condition is satisfied for the considered N-functions for all values $z \in \mathbb{R}$ (i.e., $z_0 = 0$).

Due to the convexity and estimate (1.2), the N-function $M(z)$ satisfies the inequality

$$M(y + z) \leqslant cM(z) + cM(y), \quad z, y \in \mathbb{R}. \tag{1.3}$$

Let Q be an arbitrary domain in the space $\mathbb{R}_n$. The Orlicz class $K_M(Q)$ associated with an N-function $M(z)$ is the set of functions v measurable in Q such that

$$\int\limits_Q M(v(x))dx < \infty.$$

The Orlicz space $L_M(Q)$ is the linear span of $K_M(Q)$. we shall consider the Orlicz space $L_M(Q)$ with the Luxembourg norm

$$\|v\|_{L_M(Q)} = \|v\|_{M,Q} = \inf\left\{k > 0 \;\middle|\; \int\limits_Q M\left(v(x)/k\right) dx \leqslant 1\right\}.$$

The Orlicz class $K_M(Q)$ coincides with the Orlicz space $L_M(Q)$ if and only if $M(z)$ satisfies the Δ_2-condition [15, Ch. II, Sect. 8, Thm. 8.2].

Given a function $v \in L_M(Q)$, the estimate

$$\|v\|_{M,Q} \leqslant \int\limits_Q M(v)dx + 1 \tag{1.4}$$

hold true [15, Ch. II, Sect. 9, Ineq. (9.12)]. Functions $u \in L_M(Q)$, $v \in L_{\overline{M}}(Q)$ saitsfy the Hölder inequality [15, Ch. II, Sect. 9, Ineq. (9.24), (9.27)]:

$$\left|\int_Q u(x)v(x)dx\right| \leqslant 2\|u\|_{M,Q}\|v\|_{\overline{M},Q}. \tag{1.5}$$

Given N-functions $B_1(z), \ldots, B_n(z)$, we introduce the Sobolev-Orlicz space $\mathring{H}^1_B(Q)$ as the completion of $C_0^\infty(Q)$ w.r.t. the norm

$$\|v\|_{\mathring{H}^1_B(Q)} = \sum_{\alpha=1}^n \|v_{x_\alpha}\|_{B_\alpha,Q}.$$

The norms in the spaces $L_1(Q)$, $L_\infty(Q)$ are denoted by $\|\cdot\|_{1,Q}$, $\|\cdot\|_{\infty,Q}$, respectively.

We let

$$h(t) = t^{-1/n}\left(\prod_{\alpha=1}^n B_\alpha^{-1}(t)\right)^{1/n}$$

and assume that the integral $\int\limits_0^1 h(t)/tdt$ converges. Then we can define an N-function $B^*(z)$ by the formula

$$(B^*)^{-1}(z) = \int\limits_0^{|z|} h(t)/tdt.$$

We provide the embedding theorem by A.G. Korolev [16] proven for bounded domains Q.

Lemma 1. *Let $v \in \mathring{H}^1_B(Q)$.*

1) If

$$\int\limits_1^\infty h(t)/tdt = \infty, \tag{1.6}$$

then $\mathring{H}^1_B(Q) \subset L_{B^*}(Q)$ and

$$\|v\|_{B^*,Q} \leqslant A_1\|v\|_{\mathring{H}^1_B(Q)};$$

2) If

$$\int_{1}^{\infty} h(t)/t dt < \infty, \tag{1.7}$$

then $\mathring{H}^1_B(Q) \subset L_\infty(Q)$ *and*

$$\|v\|_{\infty,Q} \leqslant A_2 \|v\|_{\mathring{H}^1_B(Q)}.$$

Here $A_1 = \frac{n-1}{n}$, $A_2 = \int_0^\infty \frac{h(t)}{t} dt$.

Thanks to the Δ_2-condition, the convergence in the norm is equivalent to the mean convergence [15, Ch. II, Sect. 9, Thm. 9.4]. Moreover, in [17] the following lemma was proved.

Lemma 2. *If an N-function $M(z)$ satisfies the Δ_2-condition, $v(x)$, $v^i(x) \in L_M(Q)$, $i = 1, 2, \ldots$, $v^i(x) \to v(x)$ in $L_M(Q)$, then*

$$\int_Q |M(v^i) - M(v)| dx \to 0, \quad i \to \infty. \tag{1.8}$$

2. Formulation of theorems

Assume that N-functions $B_0(z)$, $B_1(z)$, ..., $B_n(z)$ and their dual N-functions $\overline{B}_0(z)$, $\overline{B}_1(z)$, ..., $\overline{B}_n(z)$ satisfy the Δ_2-condition.

By $\mathbf{L}_{\overline{\mathbf{B}}}(\Omega)$ we denote the space $L_{\overline{B}_0}(\Omega) \times L_{\overline{B}_1}(\Omega) \times \ldots \times L_{\overline{B}_n}(\Omega)$ with the norm

$$\|\mathbf{g}\|_{\mathbf{L}_{\overline{\mathbf{B}}}(\Omega)} = \|g_0\|_{\overline{B}_0,\Omega} + \|g_1\|_{\overline{B}_1,\Omega} + \ldots + \|g_n\|_{\overline{B}_n,\Omega}, \quad \mathbf{g} = (g_0, g_1, \ldots, g_n) \in \mathbf{L}_{\overline{\mathbf{B}}}(\Omega).$$

We introduce a Sobolev-Orlicz space $\mathring{W}^1_{\mathbf{B}}(\Omega)$ as the completion of the space $C_0^\infty(\Omega)$ w.r.t. the norm

$$\|v\|_{\mathring{W}^1_{\mathbf{B}}(\Omega)} = \|v\|_{B_0,\Omega} + \|v\|_{\mathring{H}^1_B(\Omega)}.$$

If condition (1.6) is satisfied, we assume that

$$B_0(z) \prec B^*(z), \tag{2.1}$$

while in the case of (1.7), $B_0(z)$ is an arbitrary N-function.

We define $L_{1,\mathrm{loc}}(\overline{\Omega})$, $\mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega})$ as the spaces consisting of the functions $v(x)$ defined in Ω and such that for each bounded $Q \subset \Omega$ there exists a function in the space $L_1(\Omega)$, $\mathring{W}^1_{\mathbf{B}}(\Omega)$, respectively, coinciding with the function $v(x)$ in Q. For nonnegative functions we assume that $\psi(x), \Psi(x) \in L_{1,\mathrm{loc}}(\overline{\Omega})$. In the same way we define the space $\mathbf{L}_{\overline{\mathbf{B}},\mathrm{loc}}(\overline{\Omega})$.

We define the operator $\mathbf{B} : \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega}) \to L_{1,\mathrm{loc}}(\overline{\Omega})$ by the formula

$$\mathbf{B}(v) = B_0(v) + \sum_{\alpha=1}^{n} B_\alpha(v_{x_\alpha}), \quad v \in \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega}).$$

We denote

$$\mathbf{a}(x, \mathbf{s}) = (a_0(x, \mathbf{s}), a_1(x, \mathbf{s}), \ldots, a_n(x, \mathbf{s})).$$

Employing (1.4), by condition (0.4) we get the estimate

$$\begin{aligned} \|\mathbf{a}(x, u, \nabla u)\|_{\mathbf{L}_{\overline{\mathbf{B}}}(Q)} &= \sum_{\alpha=0}^{n} \|a_\alpha(x, u, \nabla u)\|_{\overline{B}_\alpha,Q} \\ &\leqslant \sum_{\alpha=0}^{n} \int_Q \overline{B}_\alpha(a_\alpha(x, u, \nabla u)) dx + n + 1 \leqslant \widehat{A} \|\mathbf{B}(u)\|_{1,Q} + \|\Psi\|_{1,Q} + n + 1 \end{aligned} \tag{2.2}$$

for $u \in \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega})$ and each bounded $Q \subset \Omega$.

Given an element $\mathbf{a}(x,u,\nabla u) \in \mathbf{L}_{\overline{\mathbf{B}},\mathrm{loc}}(\overline{\Omega})$, for compactly supported $v(x) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ we introduce a functional $\mathbf{A}(u)$ by the identity:

$$\langle \mathbf{A}(u), v\rangle = \int\limits_{\Omega} \left(\sum_{\alpha=1}^{n} a_\alpha v_{x_\alpha} + a_0 v\right) dx. \tag{2.3}$$

Employing Hölder inequality (1.5), for functions $u(x) \in \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega})$, $v(x) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ (supp $v = \overline{Q}_v$) we obtain the inequalities

$$\begin{aligned} |\langle \mathbf{A}(u), v\rangle| \leqslant 2\sum_{\alpha=1}^{n} \|a_\alpha\|_{\overline{B}_\alpha, Q_v} \|v_{x_\alpha}\|_{B_\alpha, Q_v} + 2\|a_0\|_{\overline{B}_0, Q_v}\|v\|_{B_0,Q_v} \leqslant \\ \leqslant 2\|\mathbf{a}(x,u,\nabla u)\|_{\mathbf{L}_{\overline{\mathbf{B}}}(Q_v)}\|v\|_{\mathring{W}^1_{\mathbf{B}}(\Omega)}. \end{aligned} \tag{2.4}$$

Thus, estimates (2.2), (2.4) imply the boundedness of the functional $\mathbf{A}(u)$ in the space of compactly supported functions in $\mathring{W}^1_{\mathbf{B}}(\Omega)$.

Definition 1. *A generalized solution to problem* (0.1), (0.2) *is a function* $u(x) \in \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\overline{\Omega})$ *satisfying the integral identity*

$$\langle \mathbf{A}(u), v\rangle = 0 \tag{2.5}$$

for each compactly supported function $v(x) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$.

We suppose that there exists $0 < \epsilon < 1$ such the conditions

$$B_\alpha(z^{1+\epsilon}) \prec B_0(z), \quad \alpha = 1, 2, \ldots, n, \tag{2.6}$$

hold true.

In work [13], the solvability of problem (0.1), (0.2) was proven in arbitrary unbounded domains Ω. Namely, the following theorem was established.

Theorem 1. *Assume that conditions* (0.3) – (0.5), (2.6) *are satisfied. Then there exists a generalized solution* $u(x)$ *to problem* (0.1), (0.2).

A power estimate for the decay rate of the solution was obtained under the condition that

$$B_\alpha(z) = c_\alpha |z|^{p_\alpha}, \quad |z| < 1, \quad p_\alpha > 1, \quad c_\alpha > 0, \quad \alpha = 0, 1, \ldots, n. \tag{2.7}$$

We note that given an arbitrary N-function $\widetilde{B}(z)$, such N-function can be constructed easily:

$$B(z) = \begin{cases} \widetilde{B}(1)|z|^p, \ |z| < 1; \\ \widetilde{B}(z), \ |z| \geqslant 1, \end{cases} \qquad p = \frac{\widetilde{B}'(1)}{\widetilde{B}(1)} > 1.$$

At that, the functions $\widetilde{B}(z)$, $B(z)$ are equivalent.

We assume that the exponents p_α, $\alpha = 1, \ldots, n$, are ordered: $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n$ and obey the conditions:

$$p_0 > p_1, \quad \sum_{\alpha=1}^{n} \frac{1}{p_\alpha} > 1. \tag{2.8}$$

Then then numbers $q_\alpha = \frac{p_0 p_\alpha}{p_0 - p_\alpha}$, $\alpha = 1, \ldots, n$, are ordered as well: $q_1 \geqslant q_2 \geqslant \ldots \geqslant q_n$. We assume that

$$q_n > n. \tag{2.9}$$

Theorem 2. *Assuem that conditions* (0.3)–(0.5), (2.6)–(2.8) *are satisfied. Then there exists a positive number* $\mathcal{M}_1$ *such that a generalized solution to problem* (0.1), (0.2) *satisfies the estimate*

$$\|\mathbf{B}(u)\|_{1,\Omega(r/2)} \leqslant \mathcal{M}_1 \left(r^{n-q_n} + \|\psi + \Psi\|_{1,\Omega(r)}\right), \quad r \geqslant 1, \tag{2.10}$$

where $\Omega(r) = \{x \in \Omega \mid |x| < r\}$.

For example, the assumption of Theorem 2 are satisfied for equation (0.6) with the functions

$$B_\alpha(z) = \begin{cases} |z|^{p_\alpha}, \ |z| < 1; \\ |z|^{p_\alpha - 1}(\ln|z| + 1), \ |z| \geqslant 1 \end{cases}$$

under an appropriate choice of $p_\alpha > 2$, $\alpha = 0, 1, \dots, n$ (see Example 1).

For unbounded domains located along a selected axis, in works [17], [18] the authors established exponential estimates for the decay rate of a solution to problem (0.1), (0.2) with compactly supported data in terms of a special geometric characteristics. Here we succeeded to obtain an exponential estimate for the isotropic case

$$B_\alpha(z) = B(z), \quad \alpha = 1, 2, \dots, n, \tag{2.11}$$

for unbounded domains satisfying the only condition

$$d(r) = \operatorname{diam}\gamma(r) \leqslant D, \quad D > 0, \quad \gamma(r) = \{x \in \Omega \mid |x| = r\}, \quad r \geqslant r_1. \tag{2.12}$$

Theorem 3. *Assume that conditions (0.3)–(0.5), (2.6),* (2.11), (2.12) *hold true. Then there exist positive numbers* κ, $\mathcal{M}_2$, r_0 *such that a solution* $u(x)$ *to problem* (0.1), (0.2) *satisfies the estimate*

$$\|\mathbf{B}(u)\|_{1,\Omega(r/2)} \leqslant \mathcal{M}_2 \left(\exp(-\kappa r)\, r^{n-1} + \|\psi + \Psi\|_{1,\Omega(2r)}\right) \tag{2.13}$$

for all $r \geqslant r_0$.

It should be mentioned that estimates (2.10), (2.13) obtained in this work are in an agreement with the results of paper [11].

3. Preliminaries

Lemma 3. *Assume that* N*-functions* $B_0(z), B_1(z), \dots, B_n(z)$ *satisfy conditions* (2.6), *then*

$$B_\alpha(z) \prec\prec B_0(z), \quad \alpha = 1, 2, \dots, n. \tag{3.1}$$

For the proof of Lemma see [13, Rem. 6].

Lemma 4. *If the functions* $b_\alpha(s_\alpha) = B'_\alpha(s_\alpha)$, $s_\alpha \geqslant 0$, $\alpha = 0, 1, \dots, n$, *are continuous and strictly monotonous,* $\mathbf{f} = (f_0, f_1, \dots, f_n) \in \mathbf{L}_{\overline{\mathbf{B}},\mathrm{loc}}(\overline{\Omega})$, *then the functions*

$$a_\alpha(x, s_\alpha) = B'_\alpha(s_\alpha) + f_\alpha(x) = b_\alpha(|s_\alpha|)\operatorname{sign} s_\alpha + f_\alpha(x), \quad \alpha = 0, \dots, n,$$

satisfy conditions (0.3) – (0.5).

For the proof of the lemma see [13, Rem. 5].

Hereinafter in this section by C_i we denote positive constants.

Lemma 5. *Assume that* N*-functions* $B_0(z)$, $B_1(z)$, ..., $B_n(z)$ *satisy conditions* (2.6), *then for the* N*-functions* $T_\alpha(z) = B_\alpha\left(\overline{M}_\alpha(z)\right)$, $(M_\alpha(z) = B_\alpha^{-1}(B_0(z))$ *there exist numbers* $c > 0$, $\tau \geqslant q_n$ *such that the inequalities*

$$T_\alpha(z) \leqslant c|z|^\tau, \quad |z| \geqslant 1, \quad \alpha = 1, 2, \dots, n, \tag{3.2}$$

hold true.

For the proof of the lemma see [14, Lm. 3.3].

Lemma 6. *Assume that N-functions $B_0(z), B_1(z), \ldots, B_n(z)$ satisfy conditions (2.7), (2.8), then for the N-functions $T_\alpha(z) = B_\alpha\left(\overline{M}_\alpha(z)\right)$ there exists a number $c > 0$ such that the inequalities*

$$T_\alpha(z) \leqslant c|z|^{q_\alpha}, \quad |z| \leqslant 1, \quad \alpha = 1, 2, \ldots, n, \tag{3.3}$$

hold.

For the proof of the lemma see [14, Lm. 3.4].

Lemma 7. *Let $\Sigma_{R,d}$ be a spherical segment of a diameter d on the sphere of a radius R, $d \leqslant R/8$, in the space $\mathbb{R}_n$, $n \geqslant 2$. If an N-function $B(z)$ satisfies the Δ_2-condition, then there exists a constant $c(n) > 0$ such that the function $v(x) \in C_0^\infty(\mathbb{R}_n)$, $v\big|_{\Sigma_{R,d}} \in C_0^\infty(\Sigma_{R,d})$ satisfies the inequality*

$$\int\limits_{\Sigma_{R,d}} B(v)dS \leqslant c \int\limits_{\Sigma_{R,d}} B(d|\nabla v|)dS. \tag{3.4}$$

For the proof of the lemma see [19].

4. Proof of Theorem 2

Proof. Let ξ be an absolutely continuous nonnegative compactly supported function. Letting $v = \xi^p u$, $p \geqslant \tau$, in identity (2.5) (see Lemma 5), we obtain the inequality

$$\begin{aligned}&\int\limits_\Omega \xi^p \left(\sum_{\alpha=1}^n a_\alpha(x, u, \nabla u)u_{x_\alpha} + a_0(x, u, \nabla u)u\right) dx \\ &\qquad \leqslant p\sum_{\alpha=1}^n \int\limits_\Omega |a_\alpha(x, u, \nabla u)||u||\xi_{x_\alpha}(x)|\xi^{p-1}dx = p \cdot J_1.\end{aligned} \tag{4.1}$$

Applying (1.1), for $\varepsilon \in (0, 1)$ we get

$$\begin{aligned}J_1 &\leqslant \sum_{\alpha=1}^n \int\limits_\Omega \xi^p \left(\overline{B}_\alpha(\varepsilon a_\alpha(x, u, \nabla u)) + B_\alpha\left(\frac{u}{\varepsilon}\frac{\xi_{x_\alpha}}{\xi}\right)\right) dx \\ &\leqslant \sum_{\alpha=1}^n \int\limits_\Omega \xi^p \varepsilon \overline{B}_\alpha(a_\alpha(x, u, \nabla u))dx + \sum_{\alpha=1}^n \int\limits_\Omega \xi^p B_\alpha\left(\frac{u}{\varepsilon}\frac{\xi_{x_\alpha}}{\xi}\right) dx = J_{11} + J_{12}.\end{aligned} \tag{4.2}$$

Let us estimate integral J_{12}. Since according to Lemma 3, relations (3.1) hold true, then the N-functions $B_0(z) = B_\alpha(M_\alpha(z))$ can be represented as compositions of two N-functions $M_\alpha(z)$, $B_\alpha(z)$, $\alpha = 1, \ldots, n$. Applying (1.1), (1.3), we establish that

$$\begin{aligned}J_{12} &\leqslant \sum_{\alpha=1}^n \int\limits_\Omega \xi^p B_\alpha \left\{M_\alpha(\varepsilon u) + \overline{M}_\alpha\left(\frac{1}{\varepsilon^2}\frac{|\nabla\xi|}{\xi}\right)\right\} dx \\ &\leqslant \sum_{\alpha=1}^n \int\limits_\Omega \xi^p \left(\varepsilon C_1 B_0(u) + C_1 B_\alpha\left(\overline{M}_\alpha\left(\frac{1}{\varepsilon^2}\frac{|\nabla\xi|}{\xi}\right)\right)\right) dx \\ &= C_1\left(\varepsilon n \int\limits_\Omega \xi^p B_0(u)dx + J_2\right),\end{aligned} \tag{4.3}$$

where

$$J_2 = \sum_{\alpha=1}^{n} \int_\Omega \xi^p T_\alpha \left(\frac{1}{\varepsilon^2} \frac{|\nabla \xi|}{\xi} \right) dx, \quad T_\alpha(z) = B_\alpha \left(\overline{M}_\alpha(z) \right). \tag{4.4}$$

Combining (4.2), (4.3) and employing condition (0.4), we get

$$\begin{aligned} J_1 \leqslant \int_\Omega \xi^p \varepsilon \left(\widehat{A} \sum_{\alpha=1}^{n} B_\alpha(u_{x_\alpha}) + (C_1 n + \widehat{A}) B_0(u) \right) dx \\ + \int_\Omega \xi^p \Psi dx + C_1 J_2 \leqslant \varepsilon C_2 \int_\Omega \xi^p \mathbf{B}(u) dx + \int_\Omega \xi^p \Psi dx + C_1 J_2. \end{aligned} \tag{4.5}$$

Employing (0.3), by (4.1), (4.5) we obtain the estimate

$$\overline{a} \int_\Omega \xi^p \mathbf{B}(u) dx \leqslant \varepsilon p C_2 \int_\Omega \xi^p \mathbf{B}(u) dx + p \int_\Omega \xi^p \left\{ \Psi + \psi \right\} dx + C_1 p J_2.$$

Choosing ε sufficiently small, we have the inequality

$$\|\xi^p \mathbf{B}(u)\|_1 \leqslant C_3 \int_\Omega \xi^p \left\{ \Psi + \psi \right\} dx + C_4 J_2. \tag{4.6}$$

Let r_0 be an arbitrary positive number. We fix $r > r_0$ and consider the cut-off function $\xi(x) = \frac{1}{r}(r^2 - |x|^2)$ as $|x| < r$, $\xi(x) = 0$ as $|x| \geqslant r$. Let us justify the finiteness of the integral J_2. It is obvious that $|\nabla \xi| \leqslant 2$. Employing (3.2), (3.3), we get the inequalities

$$J_2 \leqslant \sum_{\alpha=1}^{n} \int_{\Omega(r)} \xi^p T_\alpha \left(\frac{C_5}{\xi} \right) dx \leqslant C_6 \int_{\Omega(r) \cap \{x \,|\, C_5/\xi(x) < 1\}} \xi^{p - q_n} dx + C_7 \int_{\Omega(r) \cap \{x \,|\, C_5/\xi(x) \geqslant 1\}} \xi^{p - \tau} dx. \tag{4.7}$$

As a result we have

$$J_2 \leqslant C_8 r^{n - q_n + p}, \quad r \geqslant 1, \; r > r_0. \tag{4.8}$$

It is obvious that $\xi(x) \geqslant r - r_0$ as $|x| \leqslant r_0$ and this is why by (4.6), (4.8) we get the inequality

$$\|\mathbf{B}(u)\|_{1, \Omega(r_0)} \leqslant C_9 \left(\frac{r}{r - r_0} \right)^p \left(\|\Psi + \psi\|_{1, \Omega(r)} + r^{n - q_n} \right). \tag{4.9}$$

Letting $r_0 = r/2$ in (4.9), we arrive at estimate (2.10). □

Corollary 1. *Assume that conditions* (0.3)–(0.5) *hold with* $\psi = \Psi = 0$ *in* Ω *and conditions* (2.6)–(2.9). *Then a generalized solution* $u(x)$ *to problem* (0.1), (0.2) *vanishes in* Ω.

Indeed, letting $\psi(x) = \Psi(x) = 0$, $x \in \Omega$, in (4.9) letting r tend to infinity, we obtain that $\|\mathbf{B}(u)\|_{1, \Omega(r_0)} = 0$ for each $r_0 > 0$. It follows that $B_0(u) = 0$ in Ω and hence $u = 0$ in Ω.

5. Proof of Theorem 3

We fix $r \geqslant \max(2r_1, r_2, 32D)$ (r_1 is from condition (2.12), while r_2 will be detemined below). Let $\theta(x)$, $x > 0$, be an absolutely continuous function equallting to one as $x \leqslant r/2$, vanishing as $x \geqslant 2r$, being linear as $x \in [r, 2r]$ and satisfying the equation

$$\theta'(x) = -\delta \theta(x), \quad x \in (r/2, r), \tag{5.1}$$

the constant δ will be determined later. Solving this equation, we find

$$\theta(x) = \exp\left(-\delta (x - r/2) \right), \quad x \in (r/2, r),$$

then

$$\theta'(x) = \frac{\theta(r)}{r} = \frac{1}{r}\exp\left(-\delta r/2\right), \quad x \in (r, 2r). \tag{5.2}$$

Letting $\xi(x) = \theta^p(|x|)$, $p \geqslant \tau$, in (4.1) and applying (5.1), (5.2), we obtain

$$\begin{aligned}\int\limits_{\Omega} \theta^p \left(\sum_{\alpha=1}^n a_\alpha(x,u,\nabla u)u_{x_\alpha} + a_0(x,u,\nabla u)u\right) dx \leqslant p \sum_{\alpha=1}^n \int\limits_{\Omega(r)\setminus\Omega(r/2)} |u||a_\alpha(x,u,\nabla u)|\delta\theta^p dx \\ + p\sum_{\alpha=1}^n \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^{p-1}|u||a_\alpha(x,u,\nabla u)|\frac{\theta(r)}{r}dx = pI_1 + pI_2.\end{aligned} \tag{5.3}$$

Employing (1.1), be means of (0.4), (2.11) we estimate the first integral ($\varepsilon_1 \in (0,1)$):

$$\begin{aligned} I_1 &\leqslant \sum_{\alpha=1}^n \int\limits_{\Omega(r)\setminus\Omega(r/2)} \theta^p\left(\overline{B}(\varepsilon_1 a_\alpha(x,u,\nabla u)) + B\left(u\frac{\delta}{\varepsilon_1}\right)\right)dx \\ &\leqslant \int\limits_{\Omega(r)\setminus\Omega(r/2)} \theta^p\left(\varepsilon_1\widehat{A}\sum_{\alpha=1}^n B(u_{x_\alpha}) + \Psi\right)dx + I_{12}, \\ I_{12} &= \int\limits_{\Omega(r)\setminus\Omega(r/2)} \theta^p B\left(u\frac{\delta}{\varepsilon_1}\right)dx. \end{aligned} \tag{5.4}$$

We choose $\varepsilon_1 \leqslant \frac{\overline{a}}{8\widehat{A}p}$ and δ so that $\delta \leqslant \varepsilon_1$.

Thanks to the inclusion $\gamma(\rho) \subset \Sigma_{\rho,2d(\rho)}$, the inequality

$$I_{12} \leqslant \frac{\delta}{\varepsilon_1}\int\limits_{r/2}^{r} \theta^p(\rho) \int\limits_{\Sigma_{\rho,2d(\rho)}} B(u)dSd\rho$$

holds true for a function $u(x) \in C_0^\infty(\Omega)$.

Employing inequality (3.4) and condition (2.12) as well as (1.2), we get

$$I_{12} \leqslant \frac{\delta}{\varepsilon_1}c\int\limits_{r/2}^{r} \theta^p(\rho) \int\limits_{\Sigma_{\rho,2d(\rho)}} B(2d(\rho)|\nabla u|)dSd\rho \leqslant C_1\frac{\delta}{\varepsilon_1}\int\limits_{r/2}^{r} \theta^p(\rho)\int\limits_{\gamma(\rho)} B(|\nabla u|)dSd\rho.$$

By (1.3) we obtain the inequality

$$I_{12} \leqslant C_2\frac{\delta}{\varepsilon_1}\sum_{\alpha=1}^n \int\limits_{\Omega(r)\setminus\Omega(r/2)} \theta^p(|x|)B(u_{x_\alpha})dx. \tag{5.5}$$

Employing Lemma 2 and passing to a limit, we get inequality (5.5) for a function $u(x) \in \mathring{W}^1_{\mathbf{B},\mathrm{loc}}(\Omega)$. Combining (5.4), (5.5), choosing $\delta C_2 \leqslant \varepsilon_1\frac{\overline{a}}{8p}$, we obtain

$$I_1 \leqslant \frac{\overline{a}}{4p}\int\limits_{\Omega(r)\setminus\Omega(r/2)} \theta^p\mathbf{B}(u)dx + \|\Psi\|_{1,\Omega(r)\setminus\Omega(r/2)}. \tag{5.6}$$

Let us estimate the integral I_2. Employing (1.1), for $\varepsilon_2 \in (0,1)$ we get

$$I_2 \leqslant \sum_{\alpha=1}^n \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p\overline{B}(\varepsilon_2 a_\alpha(x,u,\nabla u))dx + n\int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p B\left(\frac{u}{\varepsilon_2}\frac{\theta(r)}{r\theta(|x|)}\right)dx = I_{21} + I_{22}. \tag{5.7}$$

Let us estimate the integral I_{22}. Thanks to relations (3.1), we can represent the N-function $B_0(z) = B(M(z))$ as a composition of two N-functions $M(z)$, $B(z)$. Employing (1.1), (1.3), we obtain

$$\begin{aligned} I_{22} \leqslant & n \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p B\left\{ M(\varepsilon_2 u) + \overline{M}\left(\frac{1}{\varepsilon_2^2}\frac{\theta(r)}{r\theta(|x|)}\right)\right\} dx \\ \leqslant & n \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p \left(\varepsilon_2 C_3 B_0(u) + C_3 B\left(\overline{M}\left(\frac{1}{\varepsilon_2^2}\frac{\theta(r)}{r\theta(|x|)}\right)\right)\right) dx \\ = & C_3 n \left(\varepsilon_2 \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p B_0(u) dx + I_3\right), \end{aligned} \tag{5.8}$$

where

$$I_3 = \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p T\left(\frac{1}{\varepsilon_2^2}\frac{\theta(r)}{r\theta(|x|)}\right) dx, \quad T(z) = B\left(\overline{M}(z)\right). \tag{5.9}$$

Combining (5.7), (5.8) and employing condition (0.4), we get

$$\begin{aligned} I_2 \leqslant & \varepsilon_2 \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p \left(\widehat{A}\sum_{\alpha=1}^n B_\alpha(u_{x_\alpha}) + (nC_3 + \widehat{A})B_0(u)\right) dx \\ & + \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p \Psi dx + nC_3 I_3 \leqslant \varepsilon_2 C_4 \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p \mathbf{B}(u) dx + \int\limits_{\Omega(2r)\setminus\Omega(r)} \Psi dx + C_4 I_3. \end{aligned} \tag{5.10}$$

We choose $\varepsilon_2 \leqslant \frac{\overline{a}}{4C_4 p}$ to obtain

$$I_2 \leqslant \frac{\overline{a}}{2} \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p \mathbf{B}(u) dx + \int\limits_{\Omega(2r)\setminus\Omega(r)} \Psi dx + C_4 I_3. \tag{5.11}$$

Substituting estimates (5.6), (5.11) into (5.3) and employing condition (0.3), we get

$$\|\mathbf{B}(u)\|_{1,\Omega(r/2)} \leqslant C_5 \int\limits_{\Omega(2r)} \{\Psi + \psi\} dx + C_5 I_3. \tag{5.12}$$

Let us estimate the integral I_3. We let $r_2 = 1/\varepsilon_2^2$, then as $r \geqslant r_2$, due to the convexity of the function $T(z)$, the inequality

$$I_3 \leqslant \frac{r_2}{r} \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^p T\left(\frac{\theta(r)}{\theta(|x|)}\right) dx$$

holds true. As $|x| \in (r, 2r)$, the inequality $\theta(|x|) \leqslant \theta(r)$ is satisfied and applying Lemma 5, we obtain the estimate

$$I_3 \leqslant \frac{C_6}{r} \int\limits_{\Omega(2r)\setminus\Omega(r)} \theta^{p-\tau}(|x|)\theta^\tau(r) dx \leqslant C_7 r^{n-1} \exp(-\delta p r/2). \tag{5.13}$$

Combining (5.12), (5.13), we get (2.13).

6. Examples

Example 1. Let $n = 3$, $p_1 = 11/3$, $p_2 = 11/4$, $p_3 = 11/5$,

$$B_\alpha(z) = \begin{cases} |z|^{p_\alpha}, & |z| < 1 \\ |z|^{p_\alpha - 1}(\ln|z| + 1), & |z| \geqslant 1, \end{cases}, \quad \alpha = 1, 2, 3.$$

Since $|z|^{p_\alpha-1} \leqslant B_\alpha(z) \leqslant |z|^{p_\alpha}$ as $|z| \geqslant 1$, then $t^{1/(p_\alpha-1)} \geqslant B_\alpha^{-1}(t) \geqslant t^{1/p_\alpha}$ as $t \geqslant 1$, $\alpha = 1, 2, 3$. It follows that

$$h(t) = t^{1/33},\ 0 < t < 1, \quad \int_0^1 t^{-1}h(t)dt < \infty,$$
$$t^{131/504} \geqslant h(t) \geqslant t^{1/33},\ t \geqslant 1, \quad \int_1^\infty t^{-1}h(t)dt = \infty,$$

and hence, we can define the functions $(B^*)^{-1}(t)$, $B^*(z)$ and to satisfy the relations

$$(B^*)^{-1}(t) = 33t^{1/33},\ 0 < t < 1, \quad 504/131t^{131/504} \geqslant (B^*)^{-1}(t) \geqslant 33t^{1/33},\ t \geqslant 1,$$
$$B^*(z) = (|z|/33)^{33},\ |z| < 33, \quad (131/504|z|)^{504/131} \leqslant B^*(|z|) \leqslant (|z|/33)^{33},\ |z| \geqslant 33.$$

We take $B_0(z) = |z|^{42/11}$. Such choise of the functions $B_\alpha(z)$, $\alpha = 0, 1, 2, 3$, ensures conditions (2.1), (2.6).

We consider the functions $a_0(x, z) = |z|^{20/11}z + f_0(x)$,

$$a_\alpha(x, z) = f_\alpha(x) + B'_\alpha(z) = f_\alpha(x) + \begin{cases} p_\alpha|z|^{p_\alpha-2}z, & |z| < 1 \\ |z|^{p_\alpha-3}z\left((p_\alpha - 1)\ln|z| + p_\alpha\right), & |z| \geqslant 1 \end{cases},$$

$f_\alpha \in L_{\overline{B}_\alpha,\mathrm{loc}}(\overline{\Omega})$, $\alpha = 0, 1, 2, 3$. According to Lemma 4, conditions (0.3)–(0.5) are satisfied. Thus, by Theorem 1, there exists a generalized solution to problem (0.1), (0.2).

Since $1/p_1 + 1/p_2 + 1/p_3 = 12/11 > 1$, $q_3 = \frac{p_0 p_3}{p_0 - p_3} = 462/89 > 3$, conditions (2.8), (2.9) are satisfied as well. According to Theorem 2, generalized solution to problem (0.1), (0.2) obeys the estimate

$$\|\mathbf{B}(u)\|_{1,\Omega(r/2)} \leqslant M\left(r^{-195/89} + \|\psi + \Psi\|_{1,\Omega(r)}\right), \quad r \geqslant 1. \tag{6.1}$$

Example 2. Let $n > 2$, $2 < p < n$,

$$B(z) = \begin{cases} |z|^{p-1}\left(-\ln|z| + \frac{p+1}{p-1}\right), & |z| < 1 \\ \frac{2}{p-1} + |z|^{p-1}(\ln|z| + 1), & |z| \geqslant 1. \end{cases}$$

Since $B(z) \geqslant \frac{p+1}{p-1}|z|^{p-1}$ as $|z| < 1$, then $B^{-1}(t) \leqslant \left(\frac{p-1}{p+1}t\right)^{1/(p-1)}$ as $0 < t < \frac{p+1}{p-1}$. Moreover, $|z|^{p-1} \leqslant B(z) \leqslant \frac{p+1}{p-1}|z|^p$ as $|z| \geqslant 1$, and this is why $\left(\frac{p-1}{p+1}t\right)^{1/p} \leqslant B^{-1}(t) \leqslant t^{1/(p-1)}$ as $t \geqslant \frac{p+1}{p-1}$. It follows that

$$h(t) \leqslant C_1 t^{\frac{n-p+1}{n(p-1)}}, \qquad 0 < t < \frac{p+1}{p-1}, \qquad \int_0^1 t^{-1}h(t)dt < \infty,$$
$$C_2 t^{\frac{n-p}{np}} \leqslant h(t) \leqslant t^{\frac{n-p+1}{n(p-1)}}, \qquad t \geqslant \frac{p+1}{p-1}, \qquad \int_1^\infty t^{-1}h(t)dt = \infty,$$

and we can define functions $(B^*)^{-1}(t)$, $B^*(z)$ and to satisfy the inequalities

$$C_3|z|^{\frac{n-p}{np}} \leqslant (B^*)^{-1}(z) \leqslant C_4|z|^{\frac{n-p+1}{n(p-1)}},\ |z| \geqslant \frac{p+1}{p-1},$$
$$C_5|z|^{\frac{n(p-1)}{n-p+1}} \leqslant B^*(z) \leqslant C_6|z|^{\frac{np}{n-p}},\ |z| \geqslant C_7.$$

We take $p_0 = \frac{n(p-1)}{n-p+1}$,

$$B_0(z) = \begin{cases} |z|^{p_0-1}\left(-\ln|z| + \frac{p_0+1}{p_0-1}\right), & |z| < 1 \\ \frac{2}{p_0-1} + |z|^{p_0-1}(\ln|z| + 1), & |z| \geqslant 1. \end{cases}$$

Such choice of the functions $B_0(z)$, $B(z)$ as $p > (1+\sqrt{1+4n})/2$ ensures conditions (2.1), (2.6).

We consider the functions

$$a_\alpha(x,z) = f_\alpha(x) + B'(z) = f_\alpha(x) + |z|^{p-3}z\left((p-1)|\ln|z|| + p\right), \ \alpha = 1,\ldots,n,$$

$a_0(x,z) = f_0(x) + B_0'(z) = |z|^{p_0-3}z\left((p_0-1)|\ln|z|| + p_0\right)$. According to Lemma 4, conditions (0.3)–(0.5) are satisfied. Thus, by Theorem 1, there exist a generalized solution to problem (0.1), (0.2).

According to Theorem 3, a generalized solution to problem (0.1), (0.2) in the domains satisfying condition (2.12) obeys the estimate

$$\|\mathbf{B}(u)\|_{1,\Omega(r/2)} \leqslant \mathcal{M}_2\left(\exp(-\kappa r)\, r^{n-1} + \|\psi + \Psi\|_{1,\Omega(2r)}\right), \quad r \geqslant r_0.$$

BIBLIOGRAPHY

1. T. Donaldson. *Nonlinear elliptic boundari value problems in Orlicz-Sobolev spaces* // J. Diff. Equat. **10**:3, 507–528 (1971).
2. V.S. Klimov. *Imbedding theorems for Orlicz spaces and their applications to boundary value problems* // Sibir. Matem. Zhurn. **13**:2, 334–348 (1972). [Sib. Math. J. **13**:2, 231–240 (1972).]
3. A. Fougeres. *Opérateurs elliptiques du calcul des variations à coefficients tres fortement non linéaires* // C.R. Acad. Sci. Paris Ser. A. **274**, 763–766 (1972).
4. J.P. Gossez. *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients* // Trans. Amer. Math. Soc. **190**, 163–206 (1974).
5. P. Gwiazda, P. Wittbold, A. Wróblewska, A. Zimmermann. *Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces*. Nonlinear Anal.: Theory Methods Appl. **129**, 1-36 (2015).
6. R.Kh. Karimov, L.M. Kozhevnikova. *Behavior on infinity of decision quasilinear elliptical equations in unbounded domain* // Ufimskij Matem. Zhurn. **2**:2, 53–66 (2010). (in Russian).
7. L. M. Kozhevnikova, A. A. Khadzhi. *Solutions of anisotropic elliptic equations in unbounded domains* // Vestn. Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki. 1(30):90–96 (2013) (in Russian).
8. A.L. Gladkov. *The Dirichlet problem for quasilinear degenerating elliptic equations in unbounded domains* // Differ. Uravn. **29**:2, 267–273 (1993). [Differ. Equat. **29**:2, 217–223 (1993).]
9. H. Brezis. *Semilinear equations in R_N without condition at infnity* // Appl. Math. Optim. **12**:3, 271–282 (1984).
10. F. Bernis. *Elliptic and parabolic semilinear problems without conditions at infnity* // Arch. Rational Mech. Anal. **106**:3, 217–241 (1989).
11. J.I. Diaz, O.A. Oleinik. *Nonlinear elliptic boundary-value problems in unbounded domains and the asymptotic behaviour of its solution* // C.R. Acad. Sci. Paris. Ser. I Math. **315**:1, 787–792 (1992).
12. M. Bokalo, O. Domanska. *On well-posedness of boundary problems for elliptic equations in general anisotropic Lebesgue-Sobolev spaces* // Mat. Stud. **28**:1, 77–91 (2007).
13. L.M. Kozhevnikova, A.A. Khadzhi. *Existence of solutions of anisotropic elliptic equations with nonpolynomial nonlinearities in unbounded domains* // Matem. Sborn. **206**:8, 99–126 (2015). [Sb. Math. **206**:8, 1123–1149 (2015).]
14. L.M. Kozhevnikova, A.A. Khadzhi. *Uniqueness of solutions to anisotropic elliptic equations with nonpower nonlinearities in unbounded domains* // Probl. Matem. Anal. **85**, 153–163 (2016). [J. Math. Sci. **219**:1, 160–171 (2016).]
15. A. Krasnosel'skiy, Ya.B. Rutitskiy. *Convex functions and Orlicz spaces*. Fizmatlit, Moscow (1958). [P. Noordhoff Ltd., Groningen, (1961).]

16. A.G. Korolev. *Embedding theorems for anisotropic SobolevOrlicz spaces* // Vestn. Mosk. Univ. Ser 1. 1, 32–37 (1983). [Mosc. Univ. Math. Bull., 38:1, 37–42 (1983).]
17. L.M. Kozhevnikova, A.A. Khadzhi. *On solutions of elliptic equations with nonpower nonlinearities in unbounded domains* // Vestn. Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki. **19**:1, 44–62 (2015) (in Russian).
18. L.M. Kozhevnikova, R.Kh. Karimov, A.A. Khadzhi. *On behavior of solutions to elliptic equations with non-power nonlinearities in unbounded domains* // Aktual'nye problemy gumanitarnych i estestevennych nauk. 9, 3–17 (2015) (in Russian).
19. A.A. Khadzhi. *On Fridrichs type inequality* // Nauchno-tekhnicheskii vestnik Povolzh'ya. 9, 30–33 (2015) (in Russian).

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