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ASYMPTOTICS FOR THE EIGENVALUES OF A FOURTH ORDER DIFFERENTIAL OPERATOR IN A “DEGENERATE” CASE

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Abstract. In the paper we consider the operator L in $L^2[0,+\infty)$ generated by the differential expression $\mathcal{L}(y) = y^{(4)} - 2(p(x)y')' + q(x)y$ and boundary conditions $y(0) = y''(0) = 0$ in the “degenerate” case, when the roots of associated characteristic equation has different growth rate at the infinity. Assuming a power growth for functions p and q, under some additional conditions of smoothness and regularity kind, we obtain an asymptotic equation for the spectrum allowing us to write out several first terms in the asymptotic expansion for the eigenvalues of the operator L.

Keywords: differential operators, asymptotics of spectrum, turning point.

Mathematics Subject Classification: 47E05, 34L16, 34L20, 34L40, 34B40

1. Introduction

The features of the spectral problems for ordinary differential operators allows one to employ one of the most effective methods based on asymptotic estimates for a fundamental system of solutions (FSS) of the equation

$$\mathcal{L}y = \lambda y, \tag{1}$$

(see, for instance, [1, 2]). For example, if L is some self-adjoint extension of the minimal operator generated by the differential expression $\mathcal{L}y$ in $L^2(a,b)$ [1] and it has a discrete spectrum with the counting function $N(r)$, in order to study the asymptotics of $N(r)$ (as $r \to +\infty$), one can employ the Tauberian technique, but first WKB-estimates [3, Ch. III, Sect. 2] for kernel of the resolvent $(L-\lambda)^{-1}$ should be obtained for large λ far from the spectrum of L. This kernel is expressed in terms of FSS for equation (1). But if we need to find several first terms in the asymptotics for the eigenvalues λ_n taken in the ascending order counting multiplicities as $n \to +\infty$, then the Tauberian technique can not be applied and one has to “descend” to the spectrum and to study the asymptotics of the solutions to equation (1) as λ goes to infinity along a set containing the spectrum of L or a part of it. In the case, when the operator L is singular [1], this circumstance, as a rule, gives rise to the turning points [3, Ch. III, Sect. 1]; in their vicinities the WKB-estimates do not work anymore. The method of pattern equations (Langer method) [4] allows one to obtain an approximate solution to equation (1) suitable both in the vicinity of the turning point and far from this point becoming the WKB-solution. This method is both effective for self-adjoint and non-self-adjoint spectral problems [5]. At present, the spectral problems with a turning point are quite well studied for two-terms operators [6]–[11].

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In the paper we consider the operator L in $L^2[0,+\infty)$ generated by the differential expression

$$\mathcal{L}(y) = y^{(4)} - 2(p(x)y')' + q(x)y \tag{2}$$

and the boundary conditions

$$y(0) = y''(0) = 0 \tag{3}$$

in the "degenerate" case, when the roots of the associated characteristic equation have different growth rate at infinity [2, Ch. IX, Sect. 4]. Assuming a power growth for the functions p and q, under some additional smoothness and regularity assumptions we obtain the asymptotic equation for the spectrum, which allows us to write several terms in the asymptotic series for the eigenvalues L.

2. Preliminary remarks

2.1. Main condition for the coefficients. We impose the following conditions for the real-valued functions p and q:

1. As $x \geqslant x_0$ ($x_0 \geqslant 0$ is a constant), the functions p and q have absolutely continuous derivatives satisfying the inequalities

$$a_1 x^{\alpha-1} \leqslant q'(x) \leqslant A_1 x^{\alpha-1}, \quad b_1 x^{\beta-1} \leqslant p'(x) \leqslant B_1 x^{\beta-1}, \tag{4}$$

where a_1, A_1, b_1, B_1, α, β are positive constants and

$$\alpha < 2\beta; \tag{5}$$

the second derivatives of the functions p and q are sign-definite almost everywhere.

2. The functions p and q are summable on $[0, x_0]$.

Remark 1. *It follows from equation* (4) *that as $x \geqslant x_1$ ($x_1 \geqslant x_0$),*

$$ax^\alpha \leqslant q(x) \leqslant Ax^\alpha, \quad bx^\beta \leqslant p(x) \leqslant Bx^\beta, \tag{6}$$

where a, A, b, B are positive constants. Therefore [1, Sect. 24, Thm. 2]*, the spectrum of each self-adjoint extension of the minimal operator generated by expression* (2) *is discrete.*

In what follows, under additional restrictions for the functions p and q, we obtain the double asymptotics [3, Ch. 2, Sect. 7] of solutions to the equation $l(y) = \lambda y$, which, in particular, implies that the deficiency indices of the operator L_0 are equal to (2, 2). The latter fact yields the self-adjointness of the operator L.

Remark 2. *The conditions under which the deficiency indices of the minimal operator generated by expression* (2) *are equal to (2, 2) were studied by many authors* [12]–[18].

2.2. Reduction of the main equation to the canonical form. We introduce the notations. Let $\chi(x)$ be an infinitely differentiable function equalling to one on $[0, x_0]$ and vanishing on $[x_0+1, \infty)$. We let

$$p_1(x) = p(x)(1-\chi(x)), \quad q_1(x) = q(x)(1-\chi(x)), \quad f(x,\lambda,\mu) = \mu^4 - 2p_1\mu^2 + q_1 - \lambda,$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2p_1 & 0 & -1 \\ q_1-\lambda & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \chi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2p & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}.$$

Let $Y = (y, y^{[1]}, y^{[2]}, y^{[3]})^T$, where $y^{[k]}$ stands for the kth quasi-derivative [1, Ch. V, Sect. 15]. Then the equation $\mathcal{L}y = \lambda y$ is equivalent to the system of the equations

$$Y' = (A_1 + A_2)Y. \tag{7}$$

The characteristic polynomial of the matrix A_1 coincides with the function $f(x,\lambda,\mu)$. The roots of the equation $f(x,\lambda,\mu) = 0$ form two pairs

$$\mu_{1,2} = \pm\sqrt{\nu_1}, \quad \mu_{3,4} = \pm\sqrt{\nu_2},$$

where $\nu_{1,2} = p_1 \pm \sqrt{D}$, $D = p_1^2 + \lambda - q_1$, the branch of the root $\sqrt{z}$ is chosen so that $\sqrt{z} > 0$ as $z > 0$. Since $\nu_2 = (q_1 - \lambda)/\nu_1$, it follows from inequalities (4) and (5) that for each fixed $\lambda > 0$ and for each $1 \leqslant i, j \leqslant 2$

$$\mu_{j+2} = o(\mu_i), \quad x \to \infty,$$

that is, the "degenerate" case holds true.

Hereafter we assume that $\beta < \alpha + 2$.

We introduce the matrices

$$A_0 = \operatorname{diag}(A_{01}, A_{02}),$$

$$A_{01} = \sqrt{\nu_1}\operatorname{diag}(1, -1), \quad A_{02} = \begin{pmatrix} 0 & 1 \\ \nu_2 & 0 \end{pmatrix}, \tag{8}$$

$$T = D^{-1/4} \begin{pmatrix} I_2 & I_2 \\ \Lambda_1 & \Lambda_2 \end{pmatrix} \operatorname{diag}(MW, I_2), \tag{9}$$

I_2 is the unit matrix of second order,

$$\Lambda_1 = \operatorname{diag}(\nu_1, -\nu_2), \quad \Lambda_2 = \operatorname{diag}(\nu_2, -\nu_1),$$

$$W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad M_1 = \operatorname{diag}(\nu_1^{-1/4}, \nu_1^{1/4}),$$

$$B_1 = -T^{-1}T', \qquad B_2 = T^{-1}A_2T. \tag{10}$$

The entries of the matrix B_1 can be easily written:

$$B_1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{11}$$

$$B_{11} = \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix}, \qquad B_{12} = W^{-1}\operatorname{diag}(b_2, b_3),$$

$$B_{21} = \operatorname{diag}(b_3, b_2)W, \quad B_{22} = \operatorname{diag}(b_4, -b_4), \tag{12}$$

$$b_1 = -\frac{\nu_1'}{4\nu_1} - \frac{p'}{2\sqrt{D}}, \qquad b_2 = -\frac{\nu_1^{1/4}\nu_2'}{2\sqrt{D}},$$

$$b_3 = \frac{\nu_1'}{2\sqrt{D}\nu_1^{1/4}}, \qquad b_4 = \frac{p'}{2\sqrt{D}}.$$

Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

$$X_{11} = -\frac{1}{2}A_{01}^{-1}B_{11}, \tag{13}$$

$$X_{12} = -\frac{1}{2\sqrt{D}}\left(A_{01}(B_{12} + B_{12}A_{02}\right), \tag{14}$$

$$X_{21} = -\frac{1}{2\sqrt{D}}\left(B_{21}A_{01} + A_{02}B_{21}\right),$$

$$X_{22} = \begin{pmatrix} 0 & 0 \\ -b_4 & 0 \end{pmatrix}.$$

One can check easily the relations

$$T^{-1}AT = A_0, \quad XA_0 - A_0X = B_1.$$

Then the substitution

$$Y = T(I_4 + X)V$$

transforms equation (7) to the form

$$V' = (A_0 + Z_1)V, \tag{15}$$

where

$$Z_1 = (I_4 + X)^{-1}(B_1 X - X' + B_2(I_4 + X)). \tag{16}$$

3. Equation for the spectrum

3.1. Formulation of the main result. We introduce the notations:

$$\xi(x,\lambda) = \left|\frac{3}{2}\int_{a_\lambda}^{x} |\nu_2|^{1/2}\,dt\right|^{2/3} \operatorname{sgn}(x - a_\lambda),$$

$$S = (\xi'(x,\lambda))^{-1/2}, \quad K(x,\lambda) = \frac{S''}{S},$$

$$\widetilde{K}(t,\lambda) = a_\lambda^2\left[|K(a_\lambda t,\lambda)| + \left(\frac{|p_1''| + |q_1''|}{\sqrt{D}}\right)(a_\lambda t,\lambda)\right];$$

a_λ is the root of the equation $q(a_\lambda) = \lambda$.

The main result of this section is the following theorem.

Theorem 1. *Assume that for $\beta < \alpha + 2$, Conditions 1), 2) hold and moreover,*

3. The function $\widetilde{K}(t,\lambda)$ is bounded in some vicinity of the point $t = 1$ of the form $(1-\delta, 1+\delta)$ ($\delta > 0$ is independent of λ) uniformly in $\lambda \geqslant \Lambda_0$, $\Lambda_0 > 0$ is a constant.

Then the eigenvalues of the operator L are determined by the equation

$$\sin\Phi(\lambda) + K(\lambda)\cos\Phi(\lambda) + O(b(\lambda) + \lambda^{-\delta}) = 0, \tag{17}$$

where

$$\Phi(\lambda) = \int_0^{a_\lambda} |\nu_2(t,\lambda)|^{1/2}\,dt + \frac{\pi}{4},$$

$$K(\lambda) = -\frac{5}{72}\left(\Phi(\lambda) - \frac{\pi}{4}\right)^{-1} + \frac{1}{2}\int_0^{a_\lambda} |\nu_2|^{-1/2}\left(b_4^2 + b_4' - K(t,\lambda) + \frac{\nu_2'(\nu_2\nu_1' - \nu_1\nu_2')}{8D^{3/2}}\right)dt, \tag{18}$$

$$b(\lambda) = \int_0^{a_\lambda} |\nu_2|^{-1/2}\exp\left(i\int_0^t |\nu_2|^{1/2}\,dt\right)\left[\frac{(p_1')^2 + |q_1''|}{D} + \frac{|p_1''|}{\sqrt{D}} + |\chi|\,|p|\right]dt, \tag{19}$$

$$\delta = \min\left\{\frac{1}{2}, \frac{\alpha + 2 - \beta}{\alpha}, \frac{\alpha + 2 - \beta}{3\alpha} + \frac{1}{\alpha}\right\}.$$

If the function p has an absolutely continuous derivative on the entire half-line $[0,\infty)$, then the number $1/2$ in the definition of δ can be replaced by $3/2$, while in the integral $b(\lambda)$ the term $|\chi p|$ can be replaced by 0.

3.2. Pattern equation. Let

$$Q_1(x,\lambda) = \int_0^x \nu_1^{1/2}dt, \quad Q_2(x,\lambda) = \int_{a_\lambda}^x |\nu_2|^{1/2}dt.$$

We choose pattern equations as

$$V_0 = \operatorname{diag}(V_{01}, V_{02}),$$
$$V_{01} = \exp\left(\operatorname{diag}(Q_1, -Q_1)\right),$$
$$V_{02} = \begin{pmatrix} v_1 & v_2 \\ v_1' & v_2' \end{pmatrix},$$
$$v_1 = Bv(\xi(x,\lambda)), \quad v_2 = Bu(\xi(x,\lambda)),$$

where $v(\xi)$, $u(\xi)$ are real Airy functions [19, Sect. 7.4.3].

It is easy to check that V_0 satisfies the equation

$$V_0' = A_0 V_0 + Z_2 V_0,$$
$$Z_2 = \operatorname{diag}\left(0, \frac{S''}{S} J_0\right), \tag{20}$$

where

$$J_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

0 is the zero matrix of second order.

We introduce the notations

$$J = \operatorname{diag}(1, -1),$$
$$d(x,\lambda) = \begin{cases} 1, & x \geqslant a_\lambda, \\ 0, & x < a_\lambda, \end{cases} \tag{21}$$
$$D(x,\lambda) = \exp[\operatorname{diag}(Q_1 J, -dQ_2 J)], \tag{22}$$
$$T_0(x,\lambda) = \operatorname{diag}\left(1, 1, |\nu_2|^{-\sigma(x,\lambda)/4}, |\nu_2|^{\sigma(x,\lambda)/4}\right), \tag{23}$$

where $\sigma(x,\lambda)$ is the characteristic function of a set, $[0,\infty)$, $(a_\lambda(1-\delta_1), a_\lambda(1+\delta_2))$, δ_1, δ_2 are determined by the relations

$$-Q_2(a_\lambda(1-\delta_1),\lambda) = Q_2(a_\lambda(1+\delta_2),\lambda) = 1. \tag{24}$$

We let

$$\widetilde{V_0}(x,\lambda) = T_0^{-1} V_0 D^{-1}. \tag{25}$$

It follows from the asymptotic formulae for the Airy functions [3, Ch. 4, Sect. 1] that

$$\widetilde{V_0} = \operatorname{diag}(I_2, \widetilde{V_{02}}),$$
$$\widetilde{V_{02}} = \begin{pmatrix} 1/2 & 1 \\ -1/2 & 1 \end{pmatrix} \left[I_2 + O(Q_2^{-1}) + O(\nu_2' \nu_2^{-3/2})\right], \qquad Q_2 \to +\infty, \tag{26}$$
$$\widetilde{V_{02}} = \begin{pmatrix} \sin\Phi & \cos\Phi \\ -\cos\Phi & \sin\Phi \end{pmatrix} \left[I_2 - \frac{5}{72} Q_2^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right.$$
$$\left. + O(Q_2^{-2}) + O(\nu_2' |\nu_2|^{-3/2})\right], \quad Q_2 \to -\infty, \tag{27}$$
$$\Phi = -Q_2(x,\lambda) + \frac{\pi}{4}.$$

3.3. Integral equation. Applying the method of variation of constants, for FSS of system (15) we obtain the equation

$$V = V_0(x,\lambda) + \int\limits_{\Gamma(x)} V_0(x,\lambda) V_0^{-1}(t,\lambda) Z(t,\lambda) V(t,\lambda)\, dt, \tag{28}$$

where

$$Z = Z_1 - Z_2, \tag{29}$$

$\Gamma(x)$ is the matrix of the integration intervals $\gamma_{ij}(x) = (\gamma_{ij}, x)$, $0 \leqslant \gamma_{ij} \leqslant \infty$; the meaning is that each entry u_{ij} of matrix $U := V_0^{-1} Z V$ is integrated over its own interval $\gamma_{ij}(x)$ in the direction from γ_{ij} to x. Let us show that under an appropriate choice of constants γ_{ij}, we can apply the successive approximations method to equation (28) that will allows us to construct the FSS of equation (15) with a known asymptotics for small $\lambda > 0$ uniformly in $x \geqslant 0$.

We let

$$\widetilde{V} = T_0^{-1} V D.$$

Then $\widetilde{V}$ satisfies the equation

$$\widetilde{V} = \widetilde{V_0} + A(\lambda)\widetilde{V}, \tag{30}$$

where

$$(A(\lambda)\widetilde{V})(x,\lambda) = \widetilde{V_0}(x,\lambda) \int\limits_{\Gamma(x)} A(x,t,\lambda)(\widetilde{V}D)(t,\lambda)D^{-1}(x,\lambda)dt \equiv \widetilde{V_0}(x,\lambda)A_1(\lambda)\widetilde{V}, \tag{31}$$

$$A(x,t,\lambda) = D(x,\lambda)(D^{-1}\widetilde{V_0}^{-1}T_0^{-1}ZT_0)(t,\lambda).$$

Now we can define $\Gamma(x)$.

We let $\gamma_{ij} = +\infty$ as $(i,j) = (3,2), (4,2), (4,3)$ and $\gamma_{ij} = 0$ for other (i,j). The definition (22) of the matrix D implies easily that under such choice all the exponential factors in (31) are bounded.

We introduce the Banach space $\mathbf{Z}$ of matrix functions $F(x) = (f_{ij}(x))_{i,j=1}^4$ such that f_{ij} are measurable on $(0,+\infty)$ and

$$\|F(x)\|_{\mathbf{Z}} = \sup_{x>0} \|F(x)\| < \infty,$$

where

$$\|F\| = \sqrt{\sum_{1\leqslant i,j\leqslant 4} |f_{ij}|^2}.$$

It is clear that $\widetilde{V_0} \in \mathbf{Z}$ for all $\lambda > 0$. Let us show that $A(\lambda)$ is a contraction operator $\mathbf{Z}$ for sufficiently large $\lambda > 0$. In order to do it, we shall need an estimate for the norm of the matrix $G = T_0^{-1}ZT_0$. We have (see (16), (20), (29))

$$G = (I_4 + X_1)^{-1}T_0^{-1}(B_1X - X')T_0 + T_0^{-1}B_2T_0(I_4 + X_1) - T_0^{-1}Z_2T_0, \tag{32}$$

where

$$X_1 = T_0^{-1}XT_0.$$

Lemma 1. *For large $\lambda > 0$*

$$\|X_1\|_{\mathbf{Z}} = O(a_\lambda^{-1} + \lambda^{-3/4} + \lambda^{-(2+\alpha-\beta)/3\alpha}).$$

Proof. By simple calculations and (10)–(14) we have

$$\|X_1(x,\lambda)\| = O\left[\left(|\nu_2|^{-\sigma/2}|p_1'|\, D^{-1/2}\right)(x,\lambda)\right],\ \lambda \gg 1,$$

uniformly in $x \geqslant 0$. Inequalities (4), (5) sohw that for all $t \in (-\delta,\delta)$, where $\delta > 0$ is independent of λ,

$$\begin{aligned} &|\nu_2(a_\lambda(1+t),\lambda)| \geqslant c_1\lambda^{(\alpha-\beta)/\alpha}|t|, \\ &|Q_2(a_\lambda(1+t),\lambda)| \leqslant c_2\lambda^{(\alpha+2-\beta)/2\alpha}|t|^{\frac{3}{2}}, \end{aligned} \tag{33}$$

where c_1, c_2 are constants independent of λ. Then the functions $\delta_1(\lambda)$, $\delta_2(\lambda)$ defined by (24) satisfy the estimates

$$|\delta_i| \geqslant c_i^{-1}\lambda^{-(\alpha+2-\beta)/3\alpha}.$$

By (33) it implies that

$$(\nu_2^{-\sigma/2}p_1'D^{-1/2})(x,\lambda) = O\left(a_\lambda^{-1} + \lambda^{-(\alpha+2-\beta)/3\alpha}\right)$$

for all $x \in (a_\lambda(1-\delta), a_\lambda(1+\delta))$. As $x \notin (a_\lambda(1-\delta), a_\lambda(1+\delta))$, we employ again inequality (4), (5) to obtain

$$(\nu_2^{-1/2}p_1'D^{-1/2})(x,\lambda) = O\left(\lambda^{-(\alpha+2-\beta)/2\alpha} + \lambda^{-3/4} + \lambda^{-(1/4+1/2\beta)}\right).$$

The proof is complete. □

It follows from Lemma 1 that
$$\|G\| = O(\|G_0\|), \tag{34}$$
where
$$G_0 = T_0^{-1}(B_1 X - X' + B_2 - Z_2)T_0.$$
Let
$$g(t,\lambda) = \frac{(p_1')^2 + |q_1'|}{D} + \frac{|p_1''|}{\sqrt{D}} + |\chi|\,(|p| + \lambda^{-1/2}\,|q|). \tag{35}$$

Lemma 2. *For large $\lambda > 0$, the estimate*
$$\|G_0(x,\lambda)\| = O(|\nu_2(x,\lambda)|^{-\frac{1}{2}}\,(|K(x,\lambda)| + g(x,\lambda))) \tag{36}$$
holds true uniformly in $x \geqslant 0$.

Proof. Let
$$G_0 = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$
where G_{ij} are square matrices of second order. By relations (10)–(14), (16) and (29) we have
$$G_{22} = \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix}, \tag{37}$$
$$\begin{aligned} g_1 &= -\frac{1}{8}\,|\nu_2|^{\sigma/2}\,D^{-1/2}(\nu_1' D' D^{-3/2} + 8\chi p), \\ g_2 &= |\nu_2|^{-\sigma/2}\left(b_4^2 + b_4' - K + \frac{\nu_2'(\nu_2\nu_1' - \nu_1\nu_2')}{8D^{3/2}} - \frac{\chi q}{2\sqrt{D}}\right), \end{aligned} \tag{38}$$
where
$$K(t,\lambda) = -\frac{5}{36}\frac{|\nu_2|}{Q_2^2} + \frac{5}{16}\frac{(q_1')^2}{(q_1-\lambda)^2} - \frac{1}{4}\frac{q_1''}{(q_1-\lambda)} - \frac{1}{8}\frac{q_1'\nu_1'}{(q_1-\lambda)\nu_1} + \frac{1}{4}\frac{\nu_1''}{\nu^{-1}} - \frac{3}{16}\frac{(\nu_1')^2}{\nu_1^2}. \tag{39}$$
Now we see that G_{22} satisfies estimate (36). Similar calculations show that
$$\|G_{11}\| = O(\nu_1^{-1/2} g(t,\lambda)), \quad \|G_{12}\| + \|G_{12}\| = O(|q_1 - \lambda|^{-\sigma/4}\,g(t,\lambda)).$$
The proof is complete. □

Lemma 3. *Under the assumptions of Theorem 1, the operator $A(\lambda)$ is bounded and its norm $\|A(\lambda)\|_*$ can be estimated as*
$$\|A(\lambda)\|_* = O\left(\lambda^{-m}\right), \quad \lambda \to +\infty,$$
$$m = \min\left\{\frac{1}{4}, \frac{2+\alpha-\beta}{2\alpha}\right\}.$$
If in addition we assume the existence of the derivative p absolutely continuous on the half-line $x \geqslant 0$, then
$$m = \min\left\{\frac{3}{4}, \frac{1}{4} + \frac{1}{2\beta}, \frac{2+\alpha-\beta}{2\alpha}\right\}.$$

Proof. By the choice of $\Gamma(x)$, all the exponentials factors in the kernel of the operator $A(\lambda)$ (see (31)) are bounded, therefore, (see (34), (36))
$$\|A(\lambda)\|_* = O\left(\int_0^\infty |\nu_2|^{-1/2}\,(|K(t,\lambda)| + g(t,\lambda))dt\right), \ \lambda \to +\infty.$$
Then, arguing as in the proof of Lemma 1 in [10] and employing expression (39), inequalities (4), (6) and Condition 3), we obtain the desired estimate for $\|A(\lambda)\|$.

If the function p has an absolutely continuous derivative, then at each place, where the function p_1 appears, it can be replaced by the function p since the term $|Xp|$ disappears in expression (35). This implies easily the second statement of the theorem. The proof is complete. □

3.4. Proof of Theorem 1. It follows from Lemma 3 that the FSS of system (7) satisfies the asymptotic representation

$$Y(x,\lambda) = TT_0(I_4 + X_1)(\widetilde{V}_0 + A_\lambda \widetilde{V}_0 + O\left(\|A_\lambda\|_*^2\right))D(x,\lambda),$$

where T, T_0, D are defined by (9), (22), (24), and $\widetilde{V}_0$ satisfies relation (26), (27). Hence, we concluce that the deficiency indices of the operator L_0 are equal to (2.2) and the equation for the eigenvalues of the operator L is of the form:

$$\det(C_0 Y(0,\lambda)C_1^T) = 0,$$

where

$$C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $X_1(0,\lambda) = 0$, $\nu_{1,2}(0,\lambda) = \pm\sqrt{\lambda}$, then

$$C_0 Y(0,\lambda)C_1^T = \lambda^{-3/8}\mathrm{diag}(1,\sqrt{\lambda})C_2\widetilde{V}(0,\lambda)(I_4 + A_1(\lambda)\widetilde{V})(0,\lambda) + O\left(\|A_\lambda\|_*^2\right)C_1^T, \tag{40}$$

where

$$C_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\left(A_1(\lambda)\widetilde{V}_0\right)(0,\lambda)C_1^T = \left(A_1(\lambda)\widetilde{V}_0 C_1^T\right)(0,\lambda),$$

by the definition of $A_1(\lambda)$ (see (31)) we get

$$\left(A_1(\lambda)\widetilde{V}_0 C_1^T\right)(0,\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha_{31} & 0 \\ \alpha_{41} & \alpha_{42} \end{pmatrix}, \tag{41}$$

and

$$\alpha_{ij} = O\left(\int_0^\infty |\nu_2|^{-1/2}\left(|K(t,\lambda)| + g(t,\lambda)\right)\exp(-\delta Q_1)dt\right), \quad \lambda \to +\infty,$$

for all $(i,j) = (3,1)$ and $(4,1)$. Here $\delta > 0$ is a constant independent of λ. Taking into consideration that as $t < x_0$,

$$|K(t,\lambda)| + g(t,\lambda) = \frac{5}{36}|\nu_2|\,Q_2^{-2}(t,\lambda) + |\chi|\left(|p| + \lambda^{-1/2}|q|\right),$$

we obtain

$$\alpha_{ij} = O\left(\lambda^{-1/4}\int_0^{x_0+1} |\chi p|\exp(-\delta_0 Q_1)dt\right) + O\left(\lambda^{-3/4}\right) + O\left(Q_2^{-2}(0,\lambda)\right), \quad (i,j) \neq (4,2), \tag{42}$$

where δ_0 is a positive constant. In order to calculate α_{42}, we note that (see (27))

$$\det(\widetilde{V_{02}}(x,\lambda)) = \det(\widetilde{V_{02}}(\infty,\lambda)) = 1,$$

therefore,

$$\widetilde{V_{02}}^{-1} = \begin{pmatrix} \omega_{22} & -\omega_{12} \\ -\omega_{21} & \omega_{11} \end{pmatrix},$$

where ω_{ij} are the entries of the matrix $\widetilde{V}_{02}$. Then (see (37), (38))

$$\alpha_{42} = \int_0^\infty (g_1\omega_{12}\omega_{21} - g_2\omega_{11}^2) \exp(-2d(t,\lambda)Q_1(t,\lambda))dt + O(\alpha(\lambda)),$$
$$\alpha(\lambda) = \int_0^\infty \|G - G_0\| \exp(-2d(t,\lambda)Q_2(t,\lambda))dt.$$

The straightforward calculations give

$$\alpha(\lambda) = O\left(\int_0^\infty |q_1 - \lambda|^{-1/2} \left(\left((p_1')^2 + |q_1''|\right) D^{-1/2} + |p_1''|\right) p_1' D^{-1/2} dt\right).$$

Employing inequalities (4), (6), Condition 3) and the sign-definiteness of the second derivatives of p and q, by simple calculations we obtain

$$\alpha_{42}(\lambda) = \frac{1}{2}\int_0^{a_\lambda} (g_1 - g_2)dt + O(b(\lambda)) + O\left(\lambda^{-m_1}\right),$$

where $b(\lambda)$ is defined by formula (19),

$$m_1 = \min\left\{\frac{2+\alpha-\beta}{3\alpha} + \frac{1}{\alpha}, \frac{3}{2}, \frac{1}{2} + \frac{2}{\alpha}\right\}.$$

Then, taking into consideration (42), by (40), (41) we get

$$\lambda^{-1/8}\det(C_0 Y(0,\lambda)C_1^T) = \omega_{11}(0,\lambda) + \alpha_{42}(\lambda)\omega_{12}(0,\lambda) + O\left(\beta(\lambda) + \|A_\lambda\|_*^2 + Q_2^{-2}(0,\lambda)\right).$$

Replacing in the latter expression $\omega_{11}(0,\lambda)$ and $\omega_{12}(0,\lambda)$ by their asymptotics in accordance with (27), we obtain (17). The proof is complete.

4. Asymptotics of the spectrum

In this section we obtain the asymptotics for the eigenvalues of the operator L, when p and q have the form

$$p(x) = x^\beta, \quad q(x) = x^\alpha, \quad 0 < \frac{\alpha}{2} < \beta < \alpha + 2. \tag{43}$$

We shall show that the leading term in the asymptotic series for λ_k depends on the value of $\operatorname{sgn}(\beta - 2)$.

4.1. First approximation for the solution to equation (17)**.** It follows from equation (17) that

$$\Phi(\lambda_k) = \nu_k\pi + o(1), \quad k \to \infty$$

for each fixed pair (α, β) satisfying (43). We shall show below (Lemma 6) that $\nu_k = k$.

Lemma 4. *The asymptotics of the spectrum of the operator L in the case $p(x) = x$, $q(x) = x^2$ is of the form*

$$\lambda_k = \left[\frac{3}{2}\pi\left(k - \frac{1}{4}\right)\right]^{4/3} - \frac{1}{16}\left[3/2\pi\left(k - \frac{1}{4}\right)\right]^{-2/3} + O(k^{-1}).$$

Proof. It is easy to see that under the assumptions of the lemma, $L = L_1^2$, where L_1 is the Sturm-Liouville operator generated by the expression $-y'' + xy$ and the boundary condition $y(0) = 0$ in $L(0, +\infty)$. The asymptotics of the eigenvalues of L_1 is known [11, Lm. 5]. It implies the statement of the lemma. ∎

Under the assumptions of Lemma 4,

$$\lim_{x\to\infty} \frac{\mu_i(x,\lambda)}{\mu_{i+2}(x,\lambda)} = 1, \quad i = 1, 2,$$

that is, we deal with the case of the asymptotically multiple roots [3, Ch. V, Sect. 4]. In this case $D(x,\lambda) = \lambda$ and the integral

$$\int_0^\infty |\nu_2|^{-1/2} \left(|K(t,\lambda)| + g(t,\lambda)\right) dt$$

diverges. Nevertheless, up to some minor changes, Theorem 1 can be extended to the case $q(x) = p^2(x)$.

Lemma 5. *Let $q(x) = x^\alpha$, $p(x) = x^\beta$, where $(\alpha,\beta) \in \Omega$,*

$$\Omega = \left\{(\alpha,\beta) : 0 < \alpha/2 < \beta < \alpha + 2 \ \text{or}\ 0 < \alpha/2 = \beta < 2\right\}.$$

Then

$$\sin\Phi(\lambda) = o(1), \lambda \to \infty, \tag{44}$$

uniformly over each compact set $K \subset \Omega$.

Proof. Let $b_\lambda = (1+\delta)a_\lambda$, $\delta > 0$ be independent of λ. Under the mentioned α, β, Theorem 7 remains true if in the definition of the space $\mathbf{Z}$ and operator $A(\lambda)$ the half-line $[0,\infty)$ is replaced by the segment $[0, b_\lambda]$. Therefore, the FSS of system (7) has the asymptotics

$$Y = TT_0\widetilde{V}_0(I_4 + o(1))D(x,\lambda), \quad \lambda \to \infty, \tag{45}$$

uniformly in $x \in [0, b_\lambda]$.

By the WKB-method one can show easily that as $0 < \alpha < 4$, relation (45) is true on the half-line $[b_\lambda, \infty)$. It implies (44). The proof is complete. □

Lemma 6. *Let p and q are of form* (43). *Then*

$$\Phi(\lambda_k) = k\pi + o(1), \quad k \to \infty. \tag{46}$$

Proof. By Lemma 5 we have

$$\Phi(\lambda_k) = \nu_k\pi + o(1), \quad k \to \infty.$$

But $\lambda_k = \lambda_k(\alpha,\beta)$ is continuous on Ω [20, Example 1]. Therefore, $\nu_k(\alpha,\beta)$ is also continuous on Ω. Since $\nu_k(2,1) = k$ (Lemma 9), then $\nu_k(\alpha,\beta) = k$ on Ω. The proof is complete. □

Remark 3. *In the proof of Lemma 4 we have employed a special form of boundary conditions (2.2). In the case of arbitrary self-adjoint boundary conditions one can proceed as in the proof of Lemma 5 in* [11] *observing that as $p(x) = x$, $q(x) = x^2$, the substitution*

$$Y = TU,$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Y = (y_1, y_2)^T, \quad y_1 = \sqrt{\lambda}y, \quad y_2 = -y'' + xy,$$

transforms the equation $\mathcal{L}(y) = \lambda y$ to the system $-V'' + xV = \sqrt{\lambda}\mathrm{diag}(1,-1)V$, whose solutions are expressed directly in terms of the Airy functions.

4.2. Asymptotics of the spectrum.

Theorem 2. *Assume that the functions p and q are of form (2.40). Then as $1<\beta<2$, the eigenvalues of the operator L have the asymptotics*

$$\lambda_k = m_k^{\frac{2\alpha}{2+\alpha-\beta}} - \frac{2\alpha}{2+\alpha-\beta}C_0\left\{C_1 m_k^{\frac{2\alpha}{2+\alpha-\beta}-\frac{(2\beta-\alpha)(2-\beta)}{2\beta(2+\alpha-\beta)}} + C_2 m_k^{-\frac{(\alpha+2\beta)(2-\beta)}{2\beta(2+\alpha-\beta)}}\right\} + O\left(k^{-\frac{4(\beta-\alpha)}{2+\alpha-\beta}}\right),$$

where

$$m_k = C_0\pi\left(k-\frac{1}{4}\right), \quad C_0 = \frac{\sqrt{2}\alpha\Gamma\left(\frac{3}{2}+\frac{2-\beta}{2\alpha}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{-\beta+2}{2\alpha}\right)},$$

$$C_1 = \frac{1}{\sqrt{2}}\int_0^\infty t^{-\beta/2}\left(t^\beta+\sqrt{t^{2\beta}+1}\right)^{-3/2}\left(\sqrt{2}t^{\beta/2}+\left(t^\beta+\sqrt{t^{2\beta}+1}\right)^{1/2}\right)^{-1}dt,$$

$$C_2 = \frac{\beta}{8}\int_0^1 t^{-1/\beta}(1-t)^{-5/4+1/2\beta}(1+t)^{-3/4+1/2\beta}\left((\beta-1)/\beta+3t/4-2t^3+t^4\right)dt.$$

As $0<\beta\leqslant 1$, similar formulae hold true.

In order to obtain the asymptotics for λ_k by (46) and (17), we need to study the behavior of the functions $Q_2(0,\lambda)$, $K(\lambda)$, $\beta(\lambda)$ for large $\lambda>0$, where p and q are of form (43).

Lemma 7. *If $0<\beta<2$, then*

$$-Q_2(0,\lambda) = \lambda^{(2+\alpha-\beta)/2\alpha}\left(C_0^{-1}+\sum_{k=1}^{n-1}a_k\lambda^{-(2\beta-\alpha)k/\alpha}\right) + O\left(\lambda^{-(2\beta-\alpha)n/\alpha}\rho(\lambda)\right),$$

as $\lambda\to+\infty$, where $n=n(\beta)\in\mathbb{N}$ are defined by inequalities (48), *$a_k=f_kI_k$, f_k and I_k are determined by the formulae* (47) *and* (50),

$$\rho(\lambda) = \begin{cases} \ln\lambda, & \beta = 2/(4n+1), \\ \lambda^{-\{1/4-1/2\beta\}}, & \text{for other } \beta, \end{cases}$$

$\{x\}$ denotes the fractional part of a number x.

Proof. The substitution $x=a_\lambda t$ transforms $Q_2(0,\lambda)$ to the form

$$-Q_2(0,\lambda) = \lambda^{(2+\alpha-\beta)/2\alpha}\int_0^1(1-t^\alpha)^{1/2}\left[t^\beta+\left(t^{2\beta}+\varepsilon(1-t^\alpha)\right)^{1/2}\right]^{-1/2}dt \equiv \lambda^{(2+\alpha-\beta)/2\alpha}I(\varepsilon).$$

where $\varepsilon=\lambda^{-(2\beta-\alpha)/\alpha}$.

Since the function

$$f(z) = \left(1+(1+z)^{1/2}\right)^{-1/2}$$

in analytic in the unit circle, then

$$f(z) = \sum_{k=1}^\infty f_kz^k, \quad |z|<1, \tag{47}$$

and $f_0=1/\sqrt{2}$. Let n be a natural number satisfying the inequalities

$$\frac{1}{2\beta}-\frac{1}{4}\leqslant n<\frac{1}{2\beta}+\frac{3}{4}. \tag{48}$$

We let

$$R_n(\varepsilon) = I(\varepsilon)-\sum_{k=0}^{n-1}f_kI_k\varepsilon^k, \tag{49}$$

where

$$I_k = \int_0^1(1-x^\alpha)^{k+1/2}x^{-(4k+1)\beta/2}dx.$$

We have [22, Ch. XII, No. 855.42]

$$I_k = \frac{\Gamma\left(\frac{2k+3}{2}\right)\Gamma\left(\frac{-(4k+1)\beta+2}{2\alpha}\right)}{\alpha\Gamma\left(\frac{2k+3}{2}+\frac{2-(4k+1)\beta}{2\alpha}\right)}. \tag{50}$$

Since

$$\left[t^\beta+\sqrt{t^{2\beta}+\varepsilon(1-t^\alpha)}\right]^{-1/2} = t^{-\beta/2} f\left(\varepsilon(1-t^\alpha)/t^{2\beta}\right),$$

by the uniform convergence of series (47) in the circle $|z| \leqslant r < 1$, the function

$$r_n(t,\varepsilon) = \left[t^\beta+\sqrt{t^{2\beta}+\varepsilon(1-t^\alpha)}\right]^{-1/2} - \sum_{k=0}^{n-1} f_k\cdot(1-t^\alpha)^k t^{-(2k+1/2)\beta}\varepsilon^k$$

satisfies the estimate

$$|r_n(t,\varepsilon)| \leqslant C_n\varepsilon^n t^{-(2n+1/2)\beta}, \quad t\in[(1+\delta)\varepsilon^{1/2\beta},1],$$

where M, δ are positive constants independent of ε. Then, since as $t\in(0,(1+\delta)\varepsilon^{1/2\beta})$,

$$|r_n(t,\varepsilon)| \leqslant M_n'\varepsilon^{n-1}t^{-(2n-3/2)\beta},$$

where $M'>0$ is independent of t and ε, then

$$R_n(\varepsilon) \leqslant M\varepsilon^n\int_{(1+\delta)\varepsilon^{1/2\beta}}^1 t^{-(2n+1/2)\beta}dt + M'\varepsilon^{n-1}\int_0^{(1+\delta)\varepsilon^{1/2\beta}} t^{-(2n-3/2)\beta}dt.$$

According to (48),

$$-2\beta < -(2n+1/2)\beta+1 \leqslant 0, \quad 0 < -(2n-3/2)\beta \leqslant 2\beta,$$

so that

$$R_n(\varepsilon) = \begin{cases} O\left(\varepsilon^n\ln\varepsilon\right), & \beta = 2/(4n+1), \\ O\left(\varepsilon^{n-\{1/4-1/2\beta\}}\right), & 1/2\beta-1/4\notin\mathbb{N}, \end{cases} \quad \varepsilon\to+0.$$

By relations (49), (50) it implies the statement of the lemma. □

Lemma 8. *As $1<\beta<2$ and $\lambda\to+\infty$, the estimate*

$$K(\lambda) = C_2\lambda^{-(\beta+2)/4\beta} + O\left(\lambda^{-(2+\alpha-\beta)/2\alpha}+\lambda^{-3/4}\right) \tag{51}$$

holds true.

Proof. Since $\beta>1$, then p' is absolutely continuous on $[0,+\infty)$ and this is why in expressions (18) and (12) we can take p and q instead of p_1 and q_1, respectively, and $\chi\equiv 0$. Hence, by (18) and (12) we obtain

$$K(\lambda) = K_1(\lambda)+K_2(\lambda)+O\left(\lambda^{-3/4}\right),$$

$$K_1(\lambda) = -\frac{1}{2}\int_0^{a_\lambda}\nu_2^{-1/2}K(t,\lambda)dt,$$

$$K_2(\lambda) = \frac{1}{16}\int_0^{a_\lambda}\nu_2^{-1/2}\left(\frac{2p'^2}{D}+4\left(\frac{p'}{\sqrt{D}}\right)'+\frac{\nu_2'(\nu_2\nu_1'-\nu_1\nu_2')}{8D^{3/2}}\right)dt.$$

By straightforward calculations we confirm that (see (39))

$$\begin{aligned} K_1(\lambda) =& -\frac{1}{8}\int_0^{a_\lambda}\nu_2^{-1/2}\frac{\nu_1''}{\nu_1}dt+\frac{3}{32}\int_0^{a_\lambda}\nu_2^{-1/2}\left(\frac{\nu_1'}{\nu_1}\right)^2dt+O(\lambda^{-(2+\alpha-\beta)/2\alpha}) \\ =& K_{11}(\lambda)+K_{12}(\lambda)+O(\lambda^{-(2+\alpha-\beta)/2\alpha}). \end{aligned}$$

The integrals $K_{11}(\lambda)$, $K_{12}(\lambda)$, $K_2(\lambda)$ are of the same nature and one can find easily their asymptotics. We have

$$K_{11}(\lambda) = -\frac{\beta}{8}\lambda^{-1/2}\int_0^{a_\lambda} k(t,\lambda)dt + O\left(\lambda^{-(2+\alpha-\beta)/2\alpha}\right),$$
$$k(t,\lambda) = t^{\beta/2}\left(t^\beta+\sqrt{t^{2\beta}+\lambda}\right)^{-1/2}\left[\beta-1+(2\beta-1)t^\beta(t^{2\beta}+\lambda)^{-1/2}-\beta t^{2\beta}(t^{2\beta}+\lambda)^{-1}\right].$$

Making the change of the variables $t\longmapsto s=(1+t^{-2\beta})^{-1/2}$, we obtain

$$K_{11}(\lambda) = -\frac{\beta}{8}\lambda^{-(\beta+2)/4\beta}\int_0^{a_\lambda} s^{-1/\beta}(1-s)^{-(5\beta+2)/4\beta}\left((\beta-1)/\beta+s(1-s)\right)ds + O\left(\lambda^{-(2+\alpha-\beta)/2\alpha}\right).$$

Making similar calculations for $K_{12}(\lambda)$, $K_2(\lambda)$ as $1<\beta<2$, we obtain (51). The proof is complete. □

Completion of the proof of Theorem 2. According to (19),

$$b(\lambda) = b_1(\lambda) + \exp\left(i\int_0^{a_\lambda}|\nu_2|^{1/2}\,dt\right)b_2(\lambda),$$

where

$$b_1(\lambda) = \int_0^{a_\lambda/2}\exp\left(i\int_0^t|\nu_2|^{1/2}\,dt\right)B(t,\lambda)dt,$$
$$b_2(\lambda) = \int_{a_\lambda/2}^{a_\lambda}\exp\left(-i\int_t^{a_\lambda}|\nu_2|^{1/2}\,dt\right)B(t,\lambda)dt,$$
$$B(t,\lambda) = |\nu_2|^{-1/2}\left[\left(\frac{p'}{\sqrt{D}}\right)^2+\frac{p''+|q_1''|}{\sqrt{D}}\right].$$

Integrating by parts, we have

$$b_1(\lambda) = O\left(\lambda^{-\delta_1}\right),\quad \lambda\to+\infty,$$

where

$$\delta_1 = \min\left\{\frac{\alpha+2-\beta}{2\alpha},\frac{4-\beta}{4\beta},\frac{4+2\beta-\alpha}{2\alpha}\right\}.$$

Let us estimate b_2. We let $Q(x,\lambda)=\int_t^{a_\lambda}|\nu_2|^{1/2}\,dt$. Since

$$Q(x,\lambda) = -3/2\int_t^{a_\lambda}(q')^{-1}\nu_1^{-1/2}d\left((\lambda-q)^{3/2}\right),$$

and the functions q' and ν_1 are monotonous, then

$$A_1a_\lambda^{1-\alpha-\beta/2}\leqslant\frac{Q(x,\lambda)}{(\lambda-q(x))^{3/2}}\leqslant A_2a_\lambda^{1-\alpha-\beta/2},\quad x\in[a_\lambda/2,a_\lambda],\quad \lambda\gg1,\tag{52}$$

where A_1, A_2 are positive constants independent of λ. For each $k=0,1,\dots$

$$\frac{\partial^k}{\partial x^k}\left[\left(\frac{(p')^2+q_1''}{\sqrt{D}}\right)^2+\frac{p''}{\sqrt{D}}\right](x,a_\lambda)=O\left(x^{-2-k}\right),\ x\geqslant a_\lambda/2,$$

uniformly in $\lambda\geqslant\Lambda_0\gg1$. Then

$$(\lambda-q(t))^{-1}B(t,\lambda)=Q(t,\lambda)^{-2/3}\lambda^{-\gamma}\psi(t,\lambda),\quad x\in[a_\lambda/2,a_\lambda],\quad \lambda\gg1,$$

where $\gamma=2(\alpha+2-\beta)/3\alpha$, the function ψ and its derivatives in x satisfy the estimates

$$\psi^{(k)}(x,\lambda)=O(x^{-k}),\quad x\geqslant a_\lambda/2,\tag{53}$$

uniformly in $\lambda \geqslant \Lambda_0 \gg 1$. Making the change $x \mapsto Q = Q(x,\lambda)$ in the integral g_2, we obtain

$$b_2(\lambda) = \lambda^{-\gamma} \int_0^{A(\lambda)} e^{-iQ} Q^{-2/3} \Psi(Q,\lambda) dQ,$$

where $A(\lambda) = Q(a_\lambda/2, \lambda)$, and by (53) the function $\Psi(Q,\lambda) = \psi(t(Q),\lambda)$ satisfies the estimates

$$\left| \frac{\partial^k}{\partial Q^k} \Psi(Q(t,\lambda),\lambda) \right| \leqslant \frac{B_k}{|\nu_2(t,\lambda)|^{k/2} a_\lambda^k}, \quad t \in [a_\lambda/2, a_\lambda], \quad \lambda \gg 1, \tag{54}$$

$B_k > 0$, $k \in \mathbb{N}$, are independent of t, λ. It follows that

$$b_{21}(\lambda) := \int_0^1 e^{-iQ} Q^{-2/3} \left(\lambda^{-\gamma} \Psi(Q,\lambda) \right) dQ = O\left(\lambda^{-\gamma}\right), \quad \lambda \to +\infty.$$

By inequalities (52) we see that as $Q(t,\lambda) \geqslant 1$, $|\nu_2(t,\lambda)t| \geqslant C\lambda^{(\alpha+2-\beta)/3\alpha}$, where $C > 0$ is independent of t, λ. By (54) it follows that for some $n \in \mathbb{N}$

$$\int_1^{A(\lambda)} q^{-2/3} \left| \frac{\partial^n}{\partial Q^n} \Psi(Q(t,\lambda),\lambda) \right| dQ = O\left(\lambda^{-\gamma}\right), \quad \lambda \to +\infty, \quad i = 1, 2.$$

This is why, integrating by parts n times in $b_{22} := b_2 - b_{21}$ and taking into consideration inequalities (54), we obtain

$$\begin{aligned} &b_2(\lambda) = O\left(\lambda^{-\gamma}\right), \ \lambda \to +\infty, \\ &b(\lambda) = O\left(\lambda^{-(2+\alpha-\beta)/2\alpha} + \lambda^{-1+\beta/4}\right), \quad \lambda \to +\infty. \end{aligned} \tag{55}$$

Substituting the obtained expression for $K(\lambda)$, $Q_2(0,\lambda)$, $b(\lambda)$ into equation (17), solving it w.r.t λ_k and taking into consideration (46), we arrive at the statement of the theorem.

Remark 4. *In the case* $0 < \beta \leqslant 1$, *we can not neglect the terms in formulae* (18) *and* (12) *involving the cut-off function* χ. *It produces additional terms in formulae* (51) *and* (55). *Moreover, additional difficulties arise related to the non-integrability of the functions* p'^2 *and* p'' *at zero and this is why the expansions like* (49) *for the integrals in the expression for* $K(\lambda)$ *depend on a particular value of* β. *This is exactly the reason why we have restricted ourselves by the case* $1 < \beta < 2$ *in Theorem 1.*

As $\beta \geqslant 2$, the asymptotics of the integrals $Q_2(0,\lambda)$, $K(\lambda)$ can be studied in the same way as for $\beta < 2$. As $\beta > 2$, the formulae for the mentioned integrals are similar to the case $\beta < 2$, a formal difference is due to the fact that

$$\frac{\beta+2}{4\beta} - \frac{2+\alpha-\beta}{2\alpha} = \frac{(\beta-2)(2\beta-\alpha)}{4\alpha\beta} > 0$$

as $\beta > 2$. In the case $\beta = 2$, the formulae for $Q_2(0,\lambda)$ and $K(\lambda)$ has its own features. Omitting intermediate steps similar to the case $\beta < 2$, we provide the final form of the asymptotic formulae as $\lambda \to +\infty$:

$$-Q_2(0,\lambda) = \begin{cases} \widetilde{C}_1 \lambda^{(\beta+2)/4\beta} - \widetilde{C}_2 \lambda^{(2+\alpha-\beta)/2\alpha} + O\left(\lambda^{(2\alpha+2-3\beta)/4\beta}\right), & \beta > 2, \\ \widetilde{C}_3 \lambda^{1/2} \ln\lambda + \widetilde{C}_4 \lambda^{1/2} + O\left(\lambda^{(\alpha-2)/4}\right) + O\left(\lambda^{(3\alpha-8)/2\alpha}\right), & \beta = 2, \end{cases} \tag{56}$$

$$K(\lambda) = \begin{cases} O\left(\lambda^{(\alpha+2-\beta)/2\alpha}\right), & \beta > 2, \\ O(\lambda^{-1/2}\ln\lambda), & \beta = 2, \end{cases} \tag{57}$$

$$b(\lambda) = o(K(\lambda)),$$

where

$$\widetilde{C}_1 = \int_0^\infty \frac{dt}{\sqrt{t^\beta + \sqrt{t^{2\beta}+1}}}, \quad \widetilde{C}_2 = \frac{\sqrt{2}}{\beta-2} + \frac{1}{\sqrt{2}}\int_0^1 \frac{t^{\alpha-\beta/2}dt}{1+\sqrt{1-t^\alpha}}, \quad \widetilde{C}_3 = \frac{4-\alpha}{4\sqrt{2}\alpha}, \tag{58}$$

$$\widetilde{C}_4 = \int_0^\infty \left[\frac{1}{\sqrt{t^2+\sqrt{t^4+1}}} - \frac{1}{\sqrt{2}(t+1)}\right] dt - \frac{1}{\sqrt{2}}\int_0^1 \frac{t^{\alpha-1}dt}{1+\sqrt{1-t^\alpha}}. \tag{59}$$

Theorem 3. *Assume that the functions p and q are of form (43). Then the spectrum of the operator L has the asymptotics*

a) as $\beta > 2$,

$$\lambda_k = m_k^{\frac{4\beta}{2+\beta}} + \frac{4\beta}{2+\beta}\frac{\widetilde{C}_2}{\widetilde{C}_1} m_k^{\frac{4\beta\alpha-(\beta-2)(2\beta-\alpha)}{(2+\beta)\alpha}} + O(k^\delta),$$

$$\delta = \max\left\{\frac{(\beta-2)(2\beta+\alpha)}{\alpha(\beta+2)}, \frac{2\alpha}{\beta+2}\right\},$$

b) as $\beta = 2$,

$$\lambda_k = \exp\left(-\frac{\widetilde{C}_4}{\widetilde{C}_3}\right)\Psi^2(\mu_k)\left[1 + O\left(k^{-\frac{4-\alpha}{2}}(\ln k)^{-1+\frac{4-\alpha}{2}} + \left(\frac{k}{\ln k}\right)^{-\frac{8}{\alpha}+2}\right)\right],$$

where

$$m_k = \frac{\pi(4k-1)}{4\widetilde{C}_1}, \quad \mu_k = \frac{\pi(4k-1)\exp\left(\widetilde{C}_4/2\widetilde{C}_3\right)}{8\widetilde{C}_3},$$

the constants $\widetilde{C}_i$, $i=\overline{1,4}$, are defined by (58), (59), $\Psi(\mu)$ is the inversion function for $\varphi(\mu) = \mu\ln\mu$ satisfying asymptotic expansion (60) for large μ.

Proof. Let $\beta = 2$. In view of (56), (57), by equations (17), (46) we find

$$\widetilde{C}_3\lambda_k^{1/2}\ln\lambda_k + \widetilde{C}_4\lambda_k^{1/2} + O\left(\lambda_k^{\alpha-2/4} + \lambda_k^{(3\alpha-8)/2\alpha}\right) = \pi\left(k - \frac{1}{4}\right),$$

which implies

$$\lambda_k = \exp\left(-\frac{\widetilde{C}_4}{\widetilde{C}_3}\right)\Psi^2\left(\mu_k\left(1 + O\left(\frac{\lambda_k^{(\alpha-4)/4}}{\ln\lambda_k} + \frac{\lambda_k^{(\alpha-4)/\alpha}}{\ln\lambda_k}\right)\right)\right).$$

For large $\mu > 0$, the function $\Psi(\mu)$ has asymptotic expansion [21, Ch. I, Sect. 5]:

$$\Psi(\mu) = \frac{\mu}{\ln\mu}\left(1 + \sum_{m=0,\,k=1} c_{km}(\ln\ln\mu)^k(\ln\mu)^{-k-\mu}\right), \tag{60}$$

where the coefficients c_{km} can be found explicitly. It implies Statement b).

Statement a) can be proved in the same way. □

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