ISSN 2304-0122 Ufa Mathematical Journal. Vol. 8. No 3 (2016). P. 8-21.

doi:10.13108/2016-8-3-8 UDC 517.538.2 + 517.984.26 + 517.547

ON 2-GENERATENESS OF WEAKLY LOCALIZABLE SUBMODULES IN THE MODULE OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND POLYNOMIAL GROWTH ON THE REAL AXIS

N.F. ABUZYAROVA

Abstract. In the work we consider a topological module $\mathcal{P}(a;b)$ of entire functions, which is the isomorphic image under the Fourier-Laplace transform of the Schwarz space of distributions with compact supports in a finite or infinite interval $(a;b)\subset\mathbb{R}$. We prove that each weakly localizable module in $\mathcal{P}(a;b)$ is either generated by its two elements or is equal to the closure of two submodules of special form. We also provide dual results on subspaces in $C^\infty(a;b)$ invariant w.r.t. the differentiation operator.

Keywords: entire functions, subharmonic functions, Fourier-Laplace transform, finitely generated submodules, description of submodules, local description of submodules, invariant subspaces, spectral synthesis.

Mathematics Subject Classification: 30D15, 30H99, 42A38, 47E05

1. Introduction

Let $[a_1;b_1]\Subset[a_2;b_2]\Subset\ldots$ be a sequence of a finite of infinite interval $(a;b)$ of the real axis, P_k be a Banach space formed by all entire functions φ having a finite norm

$$\|\varphi\|_k=\sup_{z\in\mathbb{C}}\frac{|\varphi(z)|}{(1+|z|)^k\exp(b_ky^+-a_ky^-)},\quad y^\pm=\max\{0,\pm y\},\quad z=x+\mathrm{i}y. \tag{1.1}$$

We denote by $\mathcal{P}(a;b)$ the inductive limit of the sequence $\{P_k\}$. Each of the embeddings $P_k\subset P_{k+1}$ is completely continuous and hence, $\mathcal{P}(a;b)$ is a locally convex space of type (LN^*) (see [1]). It is known (see, for instance, [2, Sect. 16.1]) that each element φ of space $\mathcal{P}(a;b)$ is an entire function of completely regular growth at order 1, and its indicator diagram is a segment of the imaginary axis $\mathrm{i}[c_\varphi;d_\varphi]\subset\mathrm{i}(a;b)$.

By $\mathcal{P}_0(a;b)$ we denote a linear subspace of space $\mathcal{P}(a;b)$ formed by all functions φ which decay fast on the real axis:

$$|\varphi(x)|=o(|x|^n),\qquad n\in\mathbb{N}.$$

In the space $\mathcal{P}(a;b)$, the multiplication by the independent variable z is a continuous mapping, and this is why $\mathcal{P}(a;b)$ is a topological module over the ring of polynomials $\mathbb{C}[z]$. For the sake of brevity, if else is not said, we shall use the word "submodule" for a closed submodule of module $\mathcal{P}(a;b)$, i.e., a closed subspace invariant w.r.t. the multiplication by z.

We denote by $\mathcal{J}_{\varphi_1,\ldots,\varphi_m}$ a submodule *generated by functions* $\varphi_1,\ldots,\varphi_m\in\mathcal{P}(a;b)$ (or *m-generated*):

$$\mathcal{J}_{\varphi_1,\ldots,\varphi_m}=\overline{\{p_1\varphi_1+\cdots+p_m\varphi_m,\quad p_1,\ldots,p_m\in\mathbb{C}[z]\}}, \tag{1.2}$$

N.F. Abuzyarova, On 2-generateness of weakly localizable submodules in the module of entire functions of exponential type and polynomial growth on the real axis.

The work is supported by the grant no. 01201456408 of the Ministry of Education and Science of Russia.
Submitted May 31, 2016.

Functions $\varphi_1, \ldots, \varphi_m$ are called *generators* of submodule $\mathcal{J}_{\varphi_1,\ldots,\varphi_m}$. A submodule with a single generator is called *principle.*

In what follows we provide the definition of notions characterizing the properties of the submodules and used in the issues on the local description (see [3] – [6]).

Given a submodule $\mathcal{J} \subset \mathcal{P}(a;b)$, we let $c_{\mathcal{J}} = \inf\limits_{\varphi\in\mathcal{J}} c_\varphi$, $d_{\mathcal{J}} = \sup\limits_{\varphi\in\mathcal{J}} d_\varphi$. The set $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ is called the *indicator segment* of submodule $\mathcal{J}$.

The divisor of a function $\varphi \in \mathcal{P}(a;b)$ is defined by the formula

$$n_\varphi(\lambda) = \begin{cases} 0, & \text{if} \quad \varphi(\lambda) \neq 0, \\ m, & \text{if } \lambda \text{ is a zero of multiplicity } m, \end{cases}$$

for all $\lambda \in \mathbb{C}$.

The divisor of a submodule $\mathcal{J} \subset \mathcal{P}(a;b)$ is defined by the formula $n_{\mathcal{J}}(\lambda) = \min\limits_{\varphi\in\mathcal{J}} n_\varphi(\lambda)$. Then we introduce *the zero set* Λ_φ *of a function* φ:

$$\Lambda_\varphi = \{(\lambda_k; m_k) : \ n_\varphi(\lambda_k) = m_k > 0\},$$

and *the zero set* $\Lambda_{\mathcal{J}}$ *of submodule* $\mathcal{J}$:

$$\Lambda_{\mathcal{J}} = \{(\lambda_k; m_k) : \ n_{\mathcal{J}}(\lambda_k) = m_k > 0\}.$$

Submodule $\mathcal{J}$ *is weakly localizable* if it contains *all* functions $\varphi \in \mathcal{P}(a;b)$ satisfying the conditions:

1) $n_\varphi(z) \geqslant n_{\mathcal{J}}(z)$, $z \in \mathbb{C}$;

2) the indicator diagramm of function φ is contained in the set $\mathrm{i}[c_{\mathcal{J}}; d_{\mathcal{J}}]$. In the case $c_{\mathcal{J}} = a$ and $d_{\mathcal{J}} = b$ the weak localizable property of $\mathcal{J}$ means that this submodule is *localizable* (*ample*).

Let $\varphi \in \mathcal{P}(a;b)$, $c,\ d \in \overline{\mathbb{R}}$ and

$$a \leqslant c \leqslant c_\varphi \leqslant d_\varphi \leqslant d \leqslant b.$$

We denote by $\mathcal{J}(\varphi, \langle c; d\rangle)$ a submodule in $\mathcal{P}(a;b)$ formed by all functions $\psi \in \mathcal{P}(a;b)$ with the set of zeroes $\Lambda_\psi \supset \Lambda_\varphi$ and the indicator diagram $\mathrm{i}[c_\psi; d_\psi] \subset \mathrm{i}\langle c; d\rangle$; hereinafter symbol "$\langle$" stands for the bracket "[" or "(" subject to which of the relations $a = c$ or $a < c$ holds true. In the same way we treat the bracket "$\rangle$". It is clear that the submodule $\mathcal{J}(\varphi, \langle c; d\rangle)$ is weakly localizable. For the submodule $\mathcal{J}(\varphi, [c_\varphi; d_\varphi])$ we shall employ a shorter notation $\mathcal{J}(\varphi)$.

A submodule $\mathcal{J}$ is called *stable at a point* $\lambda \in \mathbb{C}$ if conditions $\varphi \in \mathcal{J}$ and $n_\varphi(\lambda) > n_{\mathcal{J}}(\lambda)$ imply the belonging $\frac{\varphi}{z-\lambda} \in \mathcal{J}$. A submodule $\mathcal{J}$ is *stable* if it is stable at each point $\lambda \in \mathbb{C}$.

It is easy to see that a *stability of submodule* $\mathcal{J}$ *is a necessary condition for its weak localizable property.* However, not each stable submodule $\mathcal{P}(a;b)$ is weakly localizable. Indeed, it follows from the results of work [7, Sect. 4] that each principle submodule in $\mathcal{P}(a;b)$ is stable. It can be also checked straightforwardly by employing the definition of stability and the description of the topology in $\mathcal{P}(a;b)$. On the other hand, an example constructed in [8] as well as Theorem 3 of work [9] show that not all principle submodules in the submodule $\mathcal{P}(a;b)$ are weakly localizable. Thus, the statement that each stable finite generated submodule in $\mathcal{P}(a;b)$ is weakly localizable is wrong.

In the present work we prove the inverse statement: each weakly localizable submodule $\mathcal{J} \subset \mathcal{P}(a;b)$ either is generated by two (probably, coinciding) elements or is equal to the closure of the sum of two (probably, coinciding) submodules of the form $\mathcal{J}(\varphi, \langle c; d\rangle)$. In [3, Thms. 4, 5] we announced less general statements.

The issue on 2-degenerateness in a wide sense was studied earlier for localizable (ample) submodules in the module of entire functions of finite order determined by restrictions for the indicator [10], [11], for localized (ample) submodules in abstract weighted submodules of holomorphic functions [12], for submodules with a finite zero set in the module $\mathcal{P}(a;b)$ [4]. One of the results of work [12] is the theorem stating that localizable (ample) submodules of the module $\mathcal{P}(a;b)$ are generated by two submodules of the form $\mathcal{J}(\varphi,\ (a;b))$. We note that by the abstract part of the paper [12] one can get Statement 1) of Theorem 1 in the present for the case $c_{\mathcal{J}} = a$ or (and) $d_{\mathcal{J}} = b$. Other statements on 2-generateness on weakly localizable submodules in $\mathcal{P}(a;b)$ proven here, namely, Statement 2) of Theorem 1, Theorem 3 and Statement 1) of Theorem 1 in the general formulation, can not be obtained by means of the results of the work [12].

The further presentation is as follows. The second section contains theorems on 2-generateness in a wide sense of an arbitrary localizable submodule $\mathcal{J}$ in $\mathcal{P}(a;b)$ (Theorems 1 and 3). In the third section by these theorems we obtain dual statements on the structure of closed subspaces in space $C^\infty(a;b)$ invariant w.r.t. the differentiation operator.

2. Structure of weakly localizable submodules

Theorem 1. *Let $\mathcal{J} \subset \mathcal{P}(a;b)$ be a weakly localizable submodule.*

1) If $\mathcal{J}$ contains functions in $\mathcal{P}_0(a;b)$, then for each function $\varphi_1 \in \mathcal{J} \bigcap \mathcal{P}_0(a;b)$ there exists infinitely many functions $\varphi_2 \in \mathcal{J} \bigcap \mathcal{P}_0(a;b)$ possessing the property

$$\mathcal{J} = \overline{\mathcal{J}\left(\varphi_1, \langle c_{\mathcal{J}}; d_{\mathcal{J}}\rangle\right) + \mathcal{J}\left(\varphi_2, \langle c_{\mathcal{J}}; d_{\mathcal{J}}\rangle\right)}. \tag{2.1}$$

2) If $\mathcal{J} \bigcap \mathcal{P}_0(a;b) = \emptyset$, there exists a function $\varphi_0 \in \mathcal{J}$ such that

$$\mathcal{J} = \mathcal{J}_{\varphi_0} = \{p\varphi_0,\quad p \in \mathbb{C}[z]\}. \tag{2.2}$$

Proof. 1) The first of the formulate statements can be proved in the same way as Theorem 2 in the work [4], where stable submodules with a finite set of zeroes were considered.

Without loss of generality we can assume that $0 \notin \Lambda_{\mathcal{J}}$ and $\varphi_1(0) = 1$. Let $\Lambda_{\varphi_1} = \{\lambda_j\}$, $|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots$, and each zeroes is taken counting the multiplicity.

We choose and fix two numbers a', $b' \in \mathbb{R}$ satisfying the conditions

$$a \leqslant a' < c_{\varphi_1} \leqslant d_{\varphi_1} < b' \leqslant b, \quad a' \leqslant c_{\mathcal{J}}, \quad d_{\mathcal{J}} \leqslant b',$$

where $c_{\varphi_1} = h_{\varphi_1}(-\pi/2)$, $d_{\varphi_1} = h_{\varphi_1}(\pi/2)$, h_{φ_1} is the indicator of the function φ_1. We also choose and fix a sequence $\widetilde{\Gamma} = \{\tilde{\gamma}_k\}$, $0 \notin \widetilde{\Gamma}$ close to Λ_{φ_1} so that both sequences Λ_{φ_1} and $\widetilde{\Gamma}$ satisfy the condition

$$\sum_{j=1}^{\infty} \frac{|\lambda_j - \tilde{\gamma}_j|}{1 + |\operatorname{Im}\lambda_j| + |\operatorname{Im}\tilde{\gamma}_j|} < +\infty. \tag{2.3}$$

We let

$$\widetilde{C} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \tilde{\gamma}_j|}{1 + |\operatorname{Im}\lambda_j|}, \quad \widetilde{A}_m = e^{2\widetilde{C}} \|s_1^{(m+1)}\|_{L^1(a';b')},$$

where $s_1 \in C_0^\infty(a';b')$ is the image of the function φ_1 under the Fourier-Laplace transform $\mathcal{F}$. The convergence of the series in the definition of the quantity $\widetilde{C}$ is implied by condition (2.3) (see the proof of Theorem 5.1.2 in [13]).

Let us consider an arbitrary sequence $\Gamma = \{\gamma_k\}$, $0 \notin \Gamma$, for which

$$|\gamma_k - \lambda_k| \leqslant |\tilde{\gamma}_k - \lambda_k|, \quad k = 1, 2, \ldots \tag{2.4}$$

In accordance with Proposition 3 and Remark 1 in the work [4], a function φ_2 defined in terms of the function φ_1 and the sequence Γ by the identity

$$\varphi_2(z) = e^{-\mathrm{i}cz} \lim_{R\to\infty} \prod_{|\gamma_k|<R} \left(1 - \frac{z}{\gamma_k}\right), \quad \text{where} \quad c = \frac{c_{\varphi_1} + d_{\varphi_1}}{2}, \tag{2.5}$$

is the Fourier-Laplace transform of some function $s_2 \in C_0^\infty(a';b') \subset C_0^\infty(a;b)$, at that, $\operatorname{ch}\operatorname{supp} s_2 = [c_{\varphi_1}; d_{\varphi_1}]$ and

$$|s_2^{(m)}(t)| \leqslant \tilde{A}_m, \quad t \in (a;b), \quad m = 0, 1, \dots \tag{2.6}$$

Here $\operatorname{ch}\operatorname{supp} s_2$ is the closure of the convex hull of the support of the function s_2.

Let $\{r_k\}_{k=0}^\infty$ be an increasing sequence of real numbers greater than 2 such that

$$|\varphi_1(x)| \leqslant |x|^{-k}, \quad x \in \mathbb{R}, \quad |x| \geqslant r_k. \tag{2.7}$$

We let

$$R_k = \max\{r_k, \tilde{A}_{k+1}(b' - a')\}, \quad k = 0, 1, 2, \dots \tag{2.8}$$

By the relation $\varphi_2 = \mathcal{F}(s_2)$, $s_2 \in C_0^\infty(a';b')$ and estimates (2.6), for the function φ_2 we have

$$|\varphi_2(x)| \leqslant \frac{\tilde{A}_{k+1}(b' - a')}{|x|^{k+1}} \leqslant \frac{1}{|x|^k}, \quad |x| \geqslant R_k, \quad k = 0, 1, \dots \tag{2.9}$$

We observe that the latter estimates are valid *for all* functions φ_2 defined by formula (2.5) via the function φ_1 and the sequence Γ provided Γ satisfies (2.4).

The sequences Λ and Γ have the same density; we denote it by Δ_0. For arbitrary fixed numbers $\Delta > \Delta_0$, $\delta > 0$ we let $R_j^* = \mu(\delta/\Delta) \max\{|\lambda_j|, |\gamma_j|\}$, where a function $\mu(\chi)$ is the inverse one for the function

$$\chi(\mu) = \frac{1}{\mu} \ln(1 + \mu) + \ln\left(1 + \frac{1}{\mu}\right). \tag{2.10}$$

Let us make use of the following statement valid for functions $\varphi_1, \varphi_2 \in \mathcal{F}(C_0^\infty(a;b))$ satisfying the conditions

$$\varphi_1(0) = \varphi_2(0) = 1, \qquad h_{\varphi_1}(\theta) = h_{\varphi_2}(\theta), \quad \theta \in [0; 2\pi).$$

Theorem A [4, Thm. 1]. *Assume that for some numbers $\Delta > \Delta_0$, $\delta > 0$ and an increasing sequence $R_k \geqslant 2$, $k = 1, 2, \dots$, such that*

$$|\varphi(x)| \leqslant \frac{1}{|x|^k}, \quad |\psi(x)| \leqslant \frac{1}{|x|^k}, \quad x \in \mathbb{R}, \ |x| \geqslant R_k, \quad k = 1, 2, \dots,$$

the relation

$$\limsup_{k\to\infty} \frac{\ln \frac{1}{S_{k+1}}}{\max\{R_k, R_k^*\}} > \delta \tag{2.11}$$

holds true, where

$$S_k = \sum_{j \geqslant k} \left| \frac{1}{\lambda_j} - \frac{1}{\gamma_j} \right|.$$

Then the submodule $\mathcal{J}_{\varphi_1,\varphi_2}$ generated by functions φ_1 and φ_2 in the module $\mathcal{P}(a;b)$ is stable.

We fix an *arbitrary* sequence Γ satisfying, apart of (2.4), additional requirements: the intersection $\Gamma \bigcap \Lambda$ is $\Lambda_{\mathcal{J}}$ and the sequences Λ and Γ satisfy relation (2.11). Since $\mathcal{J}$ is a weakly localizable submodule, the function φ_2 defined by formula Relations (2.7), (2.9) and (2.11) mean that the assumptions of Theorem A hold true with the numbers R_k defined by formula (2.8). Hence, according to this theorem, 2-generated submodule $\mathcal{J}_{\varphi_1,\varphi_2}$ is stable or, that is equivalent in our case (see [7, Prop. 4.9]), *the identical zero can be approximated by the functions of the form $(p\varphi_1 - q\varphi_2)$ in the topology of $\mathcal{P}(a;b)$, where p, q are polynomials and $p(0) = q(0) = 1$.* Due to [7, Prop. 4.8], this fact is a sufficient condition for the stability of the submodule

$$\widetilde{\mathcal{J}} := \overline{\mathcal{J}(\varphi_1, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle) + \mathcal{J}(\varphi_2, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle)}.$$

A stable submodule $\widetilde{\mathcal{J}}$ contains a weakly localizable submodule $\mathcal{J}(\varphi_1)$. Theorem 1 in [3] states that then $\widetilde{\mathcal{J}}$ is a weakly localizable submodule. Taking into consideration that submodules $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ have the same indicator segments and zero sets, we conclude that $\mathcal{J} = \widetilde{\mathcal{J}}$.

2) It is easy to check that if the submodule $\mathcal{J}$ contains no functions in the subspace $\mathcal{P}_0(a;b)$, then $[c_{\mathcal{J}};d_{\mathcal{J}}]\subset(a;b)$ and $2\rho_{\Lambda_{\mathcal{J}}}=d_{\mathcal{J}}-c_{\mathcal{J}}$. Moreover, in this case the set $\Lambda_\psi\setminus\Lambda_{\mathcal{J}}$ is finite for each function $\psi\in\mathcal{J}$. Indeed, if this is not the case, letting

$$\omega(z)=\prod_{j=1}^{\infty}\left(1-\frac{z}{\mu_j}\right),$$

where sequence $\{\mu_j\}\subset\Lambda_\psi\setminus\Lambda_{\mathcal{J}}$ is "sparse", i.e., $\lim\limits_{j\to\infty}|\mu_{j+1}|/|\mu_j|=+\infty$, we obtain that $\frac{\psi}{\omega}\in\mathcal{J}\bigcap\mathcal{P}_0(a;b)$.

It follows from the above that for some $c\in\mathbb{R}$ the function

$$\varphi_0(z)=e^{icz}\lim_{R\to+\infty}\prod_{\lambda_j\in\Lambda_{\mathcal{J}},|\lambda_j|<R}\left(1-\frac{z}{\lambda_j}\right)$$

is contained in $\mathcal{J}$ and generates this submodule, more precisely, relation (2.2) holds true. □

In the rest of this section we prove the following fact: if the indicator segment of a weakly localizable submodule $\mathcal{J}$ is the proper set of the interval $(a;b)$, then this submodule is either principal or 2-generated in the sense of (1.2).

Let a function $\Phi\in\mathcal{P}(a;b)$ be such that

$$\mathcal{J}(\Phi)=\mathcal{J}_\Phi=\{p\Phi,\quad p\in\mathbb{C}[z]\}. \tag{2.12}$$

Then $\mathcal{J}_\Phi$ is weakly localizable submodule and in accordance with Theorem 2 [9], $\Phi\notin\mathcal{P}_0(a;b)$. As we shall see later in the proof of Theorem 3, in each weakly localizable submodule there exists a function with such properties.

Let us consider an arbitrary sequence $\{\mu_j\}\subset\Lambda_\Phi\setminus\{0\}$, for which

$$\liminf_{j\to\infty}\frac{|\mu_{j+1}|}{|\mu_j|}=\alpha_0>1. \tag{2.13}$$

We define the functions

$$\omega(z)=\prod_{j=1}^{\infty}\left(1-\frac{z}{\mu_j}\right),\qquad \varphi=\frac{\Phi}{\omega}.$$

For $z\in\mathbb{C}$, $M\subset\mathbb{C}$, by the symbol $\operatorname{dist}(z,M)$ we denote the distance from a point z to a set M.

Theorem 2. *Function φ belongs to $\mathcal{P}_0(a;b)$ and generates a weakly localizable principle submodule $\mathcal{J}_\varphi$.*

In order to prove this theorem, we shall make use of the three lemmata.

Lemma 1. *1) For each natural number n there exists a representation of function ω as a product of two entire functions $\omega_{1,n}$ and $\omega_{2,n}$ such that for all z, $\operatorname{dist}(z,\Lambda_\omega)\geqslant\delta>0$ the inequality*

$$\big|\ln|\omega_{1,n}(z)|-2^{-n}\ln|\omega(z)|\big|\leqslant A\ln(e+|z|) \tag{2.14}$$

holds true, where A is a positive constant dependent only on the function ω and the quantity δ, $\Lambda_\omega=\{\mu_j\}$ is the zero set of function ω.

2) There exists a sequence $\{\omega_{2,n_k}\varphi\}_{k=1}^{\infty}$ converging to a function $\widetilde{\Phi}$ in the sense of the topology in the space $\mathcal{P}(a;b)$ and $\left(\Phi/\widetilde{\Phi}\right)$ is a polynomial.

Proof. 1) Let

$$\widetilde{\mathcal{M}} = \{\mu_j \in \Lambda_\omega : \ |\operatorname{Im}\mu_j| < 1\}, \quad \widehat{\mathcal{M}} = \Lambda_\omega \setminus \widetilde{\mathcal{M}},$$
$$\tilde{\omega}(z) = \prod_{\mu_j \in \widetilde{\mathcal{M}}} \left(1 - \frac{z}{\mu_j}\right), \qquad \hat{\omega}(z) = \prod_{\mu_j \in \widehat{\mathcal{M}}} \left(1 - \frac{z}{\mu_j}\right).$$

It is clear that $\omega = \tilde{\omega}\hat{\omega}$.

In order to obtain the representation $\tilde{\omega} = \tilde{\omega}_{1,n}\tilde{\omega}_{2,n}$, we employ the following theorem.

Theorem B [15, Thm. 2]. *Let $\{z_k\}$, $k \in \mathbb{Z}$, be the zeroes of an entire function v taken so that* $\operatorname{Re} z_k$ *is an ascending sequence and*

$$\operatorname{Re} z_0 = \min_k \{\operatorname{Re} z_k, \ \operatorname{Re} z_k \geqslant 0\}.$$

If all points z_k are located in the strip $|\operatorname{Im} z| < 1$, and $|\operatorname{Re} z_k| > 1$, and each square

$$\Pi_j = \{z : \quad |\operatorname{Im} z| < 1, \ 2j - 1 \leqslant \operatorname{Re} z < 2j + 1\}, \quad j \in \mathbb{Z},$$

contains at most one point z_k, then function v can be represented as the product of entire functions v_1, v_2 so that

$$|\ln|v_1(z)| - \ln|v_2(z)|| \leqslant C_1 \ln^+|z| + C_2 \ln^+ \frac{1}{d(z)} + C_3,$$

where $d(z)$ is the distance from point z to the set of zeroes of the function v, and $C_i > 0$ are absolute constants independent of the function v.

Neglecting if needed finitely many zeroes of the function $\tilde{\omega}$ and reordering the remaining zeroes so that their real parts ascend, we see that the sequebce $\widetilde{\mathcal{M}} = \{\tilde{\mu}_k\}$, $k \in \mathbb{Z}$, satisfies the assumptions of Theorem B. According to this theorem, for all z, $\operatorname{dist}(z, \widetilde{\mathcal{M}}) \geqslant \delta > 0$, the function

$$\tilde{\omega}_{1,n}(z) = \prod_{k \in \mathbb{Z}} \left(\left(1 - \frac{z}{\tilde{\mu}_{2^{n+1}k}}\right) \left(1 - \frac{z}{\tilde{\mu}_{2^{n+1}k+1}}\right) \right), \quad n \in \mathbb{N}, \tag{2.15}$$

satisfies the relation

$$\Big|\ln|\tilde{\omega}_{1,n}(z)| - 2^{-n}\ln|\tilde{\omega}(z)|\Big| \leqslant \tilde{A}\ln(e + |z|), \quad n \in \mathbb{N}, \tag{2.16}$$

where constant $\tilde{A} > 0$ depends only on δ, while the choice of indices $2^{n+1}k$, $2^{n+1}k+1$ in formula (2.15) is made in accordance with the arguments in the proof of Theorem B [15, Thm. 2]. We shall obtain the same statement for the function $\hat{\omega}$ by employing one more result of work [15]. In order to do it, we recall needed notations. Let

$$P_k = \{z : \ 1 \leqslant \operatorname{Im} z \leqslant 2^k + 1, \ 0 \leqslant \operatorname{Re} z \leqslant 2^k\}, \quad k = 0, 1, 2, \ldots$$

Then the difference $P_k \setminus P_{k-1}$, $k = 1, 2, \ldots$, consists of three squares congruent to P_{k-1}. By the symbols P_k^m, $m = 1, 2, \ldots, 12$, we denote these three squares as well as those symmetric w.r.t. both the axes and the origin. We locate the boundary segments and vertices so that the squares P_k^m are mutually disjoint and cover the set $\{z : \ |\operatorname{Im} z| \geqslant 1\}$.

Theorem C [15, Thm. 3]. *Let $\{z_k\}$, $k \in \mathbb{Z}$, be the zeroes of an entire function v taken so that $|z_k|$ ascend. Assume that $|\operatorname{Im} z| \geqslant 1$, and each square P_k^m contains at most one zero of function v. Then function v is represented as the product of two entire functions v_1, v_2 so that*

$$|\ln|v_1(z)| - \ln|v_2(z)|| \leqslant C_1 \ln^+|z| + C_2 \ln^+ \frac{1}{d(z)} + C_3,$$

where $d(z)$ is the distance from a point z to the set of zeroes of the function v and C_i are absolute constants independent of the function v.

We fix a number $\alpha \in (1;\alpha_0)$. Neglecting if needed finitely many zeroes $\hat{\mu}_k$ of the function $\hat{\omega}$ and then reordering the remaining zeroes so that $|\hat{\mu}_k|$ ascend, in view of condition (2.13) we have

$$|\hat{\mu}_{k+1}| > \alpha|\hat{\mu}_k|, \quad k = 1, 2, \dots$$

We let

$$m = \left[\log_\alpha \sqrt{5}\right] + 1.$$

It is easy to confirm that all functions

$$\hat{\omega}_j(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{\hat{\mu}_{mk+j}}\right), \quad j = 1, \dots, m,$$

satisfy the assumption of Theorem C. Applying this theorem n times to each function $\hat{\omega}_j$, $j = 1, \dots, m$, we obtain the representation

$$\hat{\omega}_j = \hat{\omega}_{j,1,n}\hat{\omega}_{j,2,n},$$

at that,

$$\big|\ln|\hat{\omega}_{j,1,n}(z)| - 2^{-n}\ln|\hat{\omega}_j(z)|\big| \leqslant \widehat{A}\ln(e+|z|), \quad \mathrm{dist}(z, \widehat{\mathcal{M}}) \geqslant \delta, \tag{2.17}$$

where constant $\widehat{A} > 0$ depends only on δ and ω. Letting

$$\hat{\omega}_{1,n} = \hat{\omega}_{1,1,n} \dots \hat{\omega}_{m,1,n}, \quad \hat{\omega}_{2,n} = \frac{\hat{\omega}}{\hat{\omega}_{1,n}},$$

we obtain the required factorization

$$\hat{\omega} = \hat{\omega}_{1,n}\hat{\omega}_{2,n}.$$

By estimates (2.16), (2.17) we see that the functions

$$\omega_{1,n} = \tilde{\omega}_{1,n}\hat{\omega}_{1,n}, \quad \omega_{2,n} = \frac{\omega}{\omega_{1,n}}$$

satisfy the first statement of the lemma.

2) It follows from the relations $\omega = \omega_{1,n}\omega_{2,n}$ and (2.14) that for all natural n and all $z \in \mathbb{C}$, $\mathrm{dist}(z, \Lambda_\omega) \geqslant \delta$, the estimates

$$|\omega_{2,n}(z)\varphi(z)| \leqslant (e + |z|)^{[A]+1}|\Phi(z)|$$

hold true. By the topological properties of the space $\mathcal{P}(a;b)$, the sequence $\{\omega_{2,n}\varphi\}_{n=1}^{\infty}$ is relatively compact in this space. Hence, there exists a sequence $\{\omega_{2,n_k}\varphi\}_{k=1}^{\infty}$ converging to some function $\widetilde{\Phi}$ in the topology of $\mathcal{P}(a;b)$ and the indicator of this function is equal to with to the coinciding indicator of the functions Φ and φ. The corresponding sequence of entire functions of minimal type at order 1

$$\omega_{1,n_k} = \frac{\Phi}{\omega_{2,n_k}\varphi}$$

converges to the entire function $(\Phi/\widetilde{\Phi})$ having a minimal type at order 1. Passing to the limit, by means of estimates (2.14) we obtain the upper polynomial bound for $|\Phi/\widetilde{\Phi}|$ on the real axis. Applying the corollary of Phragmén-Lindelöf theorem [2, Sect. 6.1], we conclude that $(\Phi/\widetilde{\Phi})$ is a polynomial. □

Let $n(r) = \sum\limits_{|\mu_j|<r} 1$ be the counting function of the sequence Λ_ω, $N(r) = \int\limits_0^r \frac{n(\tau)}{\tau}\mathrm{d}\tau$, $M(r) = \max\limits_{|z|=r}|\omega(z)|$, $m(r) = \min\limits_{|z|=r}|\omega(z)|$.

Condition (2.13) for the sequence Λ_ω implies that

$$n(r) = C_0\ln(1+r), \quad r \geqslant 0, \tag{2.18}$$

where C_0 is a positive constant. By Lemma 3.5.8 in the monograph [22], in view of (2.18) and Jensen formula (see, for instance, [22, Sect. 1.2]) we obtain the double inequality

$$N(r) \leqslant M(r) \leqslant N(r) + C_0 \ln(1+r). \tag{2.19}$$

Lemma 2. *1) For all $z \in \mathbb{C}$ the estimate from above*

$$\ln|\omega(z)| \leqslant N(|z|) + C_0 \ln(1+|z|) \tag{2.20}$$

holds true.

2) For all $\varepsilon > 0$ and $\delta > 0$ and all $z \in \mathbb{C}$, $\operatorname{dist}(z, \Lambda_\omega) \geqslant \delta$ the estimate from below

$$\ln|\omega(z)| \geqslant (1-\varepsilon)N(|z|) - C_1 \ln(1+|z|) - C_{2,\varepsilon} \tag{2.21}$$

holds true, where constant $C_{2,\varepsilon} > 0$ depends on Λ_ω, δ and ε, while constant $C_1 > 0$ depends only on Λ_ω.

Proof. 1) Required estimate (2.20) is implied by the right inequality in (2.19).
2) It is known that for an entire function, whose zero set satisfies condition (2.18), the relation $\ln m(r) \sim \ln M(r)$ holds true as $r \to \infty$ over a set of unit relative measure [22, Thm. 3.6.1]. An exceptional set of values of r can be covered by a countable set of segments disjoint thanks to (2.13)and centred together with the set $\{|\mu_j|\}$ (i.e., each interval contains exactly one point $|\mu_j|$). This set of the intervals has a zero relative length. Without loss of generality we can assume that there exists a decreasing sequence of positive numbers δ_j, $j = 1, 2, \dots$, such that for each $\varepsilon > 0$ the inequality

$$\ln m(r) \geqslant (1-\varepsilon)\ln M(r), \quad r > r_\varepsilon, \quad rt \in \bigcup_{j=1}^{\infty} ((1-\delta_j)|\mu_j|; (1+\delta_j)|\mu_j|)$$

holds true. By (2.13) and (2.18) one can get easily that

$$N(r) \leqslant N((1-\delta_j)r) + (C_0 \ln 2 + 1)\ln(1+r) + \widetilde{C}_{2,\varepsilon}, \quad r > 0,$$

where constant $\widetilde{C}_{2,\varepsilon} > 0$ depends only on Λ_ω, δ and ε.

Desired lower bound (2.21) is obtained by standard methods by two last estimates and the left inequality in (2.19). □

Lemma 3. *For each natural n function $\omega_{2,n}\varphi$ is contained in the submodule $\mathcal{J}_\varphi$.*

Proof. For a fixed $n \in \mathbb{N}$ by (2.14) we have

$$\ln|\omega_{2,n}(z)| \leqslant (1-2^{-n})\ln|\omega(z)| + A \ln(e+|z|), \quad \operatorname{dist}(z, \Lambda_\omega) \geqslant \delta. \tag{2.22}$$

In view of (2.13) and (2.20) it implies the estimate

$$\ln|\omega_{2,n}(z)| \leqslant (1-2^{-n})N(|z|) + \tilde{A} \ln(e+|z|), \quad z \in \mathbb{C}. \tag{2.23}$$

We consider the weight function $\widetilde{V}(x) = (e+|x|)^{\tilde{A}+1} \exp((1-2^{-n})N(|x|)) \geqslant 1$, $x \in \mathbb{R}$. This function is even, convex in $\ln|x|$, and for all $k = 0, 1, \dots$ the relation

$$|x|^k = o(\widetilde{V}(x)), \quad |x| \to +\infty,$$

holds true. By estimate (2.23) it yields for the function $\omega_{2,n}$ that

$$\frac{|\omega_{2,n}(x)|}{\widetilde{V}(x)} \to 0, \quad |x| \to +\infty.$$

Arguing as in the proof of Lemma 3 in the work [9], we obtain that there exists a sequence of polynomials $\{p_j\}$ converging to the function $\omega_{2,n}$ in the weighted norm $\|\cdot\| = \sup\limits_{x\in\mathbb{R}} \frac{|\cdot|}{V(x)}$, where $V(x) = (1+|x|)^2 \widetilde{V}(x)$.

We let $v(x) = \ln V(x)$,

$$P_v(z) = \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{v(\tau)}{(\tau - x)^2 + y^2} \mathrm{d}\,\tau$$

is the Poisson integral of a function v, $z = x + \mathrm{i}y$. By condition (2.13) it is easy to obtain that the function v belongs to the class *of slowly varying canonical weights* introduced in the monograph [19, §1.3]. This is why (see [19, Sect. 1.4]) the function P_v is harmonic in the upper and lower half-planes, is continuous and subharmonic in the whole complex plane and satisfies the estimate

$$P_v(z) \geqslant v(|z|), \quad z \in \mathbb{R},$$

and the relation

$$\limsup_{z\to\infty} \frac{P_v(z)}{v(|z|)} = 1. \tag{2.24}$$

Since $\mathcal{P}(a;b)$ is a locally convex space of type (LN^*), the sequence $p_j\varphi$ in this space if and only if it is bounded in one of norms (1.1) (see [1]). Taking into account estimate (2.21), the definition of the weight V, relation (2.24) and the properties of the function $N(r)$ implied by condition (2.13), and employing the Phragmén-Lindelöf we establish

$$|p_j(z)\varphi(z)| \leqslant (e + |z|)^{\text{const}} \exp(d_\varphi y^+ - c_\varphi y^-),$$

where d_φ (c_φ) is the value of the indicator of the function φ at the point $\pi/2$ (respectively, at the point $-\pi/2$). The last estimate is equivalent to the boundedness of the sequence $\{p_j\varphi\}$ in one of norms (1.1).

Employing once again the properties of locally convex space of type (LN^*) (see [1]), by the above fact we obtain that there exists a subsequence of this sequence converging to the function $\omega_{2,n}\varphi$ in $\mathcal{P}(a;b)$. □

Proof of Theorem 2. The belonging $\varphi \in \mathcal{P}_0(a;b)$ is obvious. It follows from Statement 2) of Lemma 1 and Lemma 3 that

$$\Phi \in \mathcal{J}_\varphi. \tag{2.25}$$

By (2.12) we have $\mathcal{J}(\Phi) \subset \mathcal{J}_\varphi$. As it is stated in Theorem 1 of work [3], thanks to the stability of submodule $\mathcal{J}_\varphi$, this relation is equivalent to the weak localizable property of $\mathcal{J}_\varphi$. □

Theorem 3. *Assume that the submodule $\mathcal{J}$ is weakly localizable and $[c_\mathcal{J}; d_\mathcal{J}] \subset (a;b)$. Then either $\mathcal{J}$ is the principal submodule or $\mathcal{J} = \mathcal{J}_{\varphi_1,\varphi_2}$, where $\varphi_1,\ \varphi_2 \in \mathcal{J} \bigcap P_0(a;b)$.*

Proof. If $\mathcal{J} \bigcap \mathcal{P}_0(a;b) = \emptyset$, then, as it was shown in Statement 2) of Theorem 1, $\mathcal{J}$ is the principal submodule. This is why we shall argue assuming that $\mathcal{J} \bigcap \mathcal{P}_0(a;b) \neq \emptyset$.

Let us show first that the submodule $\mathcal{J}$ contains a function $\varphi_1 \in \mathcal{P}_0(a;b)$ with the properties: $c_{\varphi_1} = c_\mathcal{J}$, $d_{\varphi_1} = d_\mathcal{J}$, the principle submodule $\mathcal{J}_{\varphi_1}$ is weakly localizable.

In order to do it, we consider an arbitrary function $\tilde{\varphi} \in \mathcal{J} \bigcap \mathcal{P}_0(a;b)$ and we let

$$\varphi = \left(e^{\mathrm{i}(c_{\tilde{\varphi}} - c_\mathcal{J})z} + e^{\mathrm{i}(d_\mathcal{J} - d_{\tilde{\varphi}})z}\right) \tilde{\varphi}.$$

It is clear that the function φ belongs to the set $\mathcal{J} \bigcap \mathcal{P}_0(a;b)$ and its indicator diagram is $\mathrm{i}[c_\mathcal{J}; d_\mathcal{J}]$. If the principle submodule $\mathcal{J}_\varphi$ is weakly localizable, we let $\varphi_1 = \varphi$. Otherwise we consider the maximal subharmonic minorant $v(z)$ of the function $(H(z) - \ln|\varphi(z)|)$, where $H(z) = d_\mathcal{J}(\operatorname{Im} z)^+ - c_\mathcal{J}(\operatorname{Im} z)^-$.

The function v satisfies the relation $vt \equiv -\infty$. Indeed, by the inclusion $\varphi \in \mathcal{P}_0(a;b)$, for each $k = 0, 1, 2, \dots$ we have $M_k = \max\limits_{x\in\mathbb{R}} |\varphi(x)x^k| < +\infty$, as well as

$$\varphi(z) = \int\limits_a^b s(t)e^{-\mathrm{i}tz}\mathrm{d}\,t, \quad s \in C_0^\infty(a;b).$$

The class $\mathcal{C}_{(a;b)}(\{M_k\})$ (see, for instance, [21, Sect. IV.A]), contains a non-zero function s, and therefore, it is not quasi-analytic. In accordance with Carleman criterion, it is equivalent to the relation

$$\int^{\infty} \frac{\ln T(r)}{1+r^2}\mathrm{d}\, r < +\infty,$$

where $T(r) = \sup\limits_{k\geqslant 0} \frac{r^k}{M_k}$ is the trace function of the sequence $\{M_k\}$, (see, for instance, [21, Sect. IV.A]). Thus, $\ln T(e^t)$ is a finite function convex in $t \in \mathbb{R}$. Therefore, function $u(z) = \ln T(|z|)t \equiv -\infty$ is subharmonic in $\mathbb{C}$ [23, Thm. 2.1.2]. The definition of u yields the estimate

$$u(x) + \ln|\varphi(x)| \leqslant 0, \quad x \in \mathbb{R}. \tag{2.26}$$

Function φ, as well as all the elements of the space $\mathcal{P}(a;b)$ has a completely regular growth in the entire plane, while the function u depends only on $|z|$. Applying the theorem on summation of indicators [24, Thm. 1], by (2.26) we obtain that u has the minimal type at order 1. This fact, estimate (2.26), Phragmén-Lindelöf theorem for subharmonic functions [2, Sect. 7.3] imply the estimate

$$u(z) + \ln|\varphi(z)| \leqslant H(z), \quad z \in \mathbb{C},$$

which yields the inequality $u(z) \leqslant v(z)$, $z \in \mathbb{C}$. Hence, $vt \equiv -\infty$.

Let ω be an entire function (of a minimal exponential type) satisfying the relation

$$|\ln|\omega(z)| - v(z)| \leqslant C \ln(1+|z|), \quad zt \in E, \tag{2.27}$$

with some constant $C > 0$ whose exceptional set E can be covered by a countable union of circles with a finite sum of radii. The existence of such function is implied by Theorem 5 of the work [14]. We let $\Phi = \omega\varphi$. It is clear that $\Phi \in \mathcal{J}$. The fact that the function v is the *maximal* subharmonic minorant function $(H - \ln|\varphi|)$ and estimate (2.27) imply relations (2.12) for the function Φ. We choose a sequence $\{\mu_j\} \subset \Lambda_\Phi \setminus \Lambda_{\mathcal{J}}$ satisfying conditions (2.13), $\mu_j \neq 0$. Let

$$\varphi_1 = \frac{\Phi}{\prod\limits_{j=1}^{\infty}\left(1 - \frac{z}{\mu_j}\right)}.$$

For this function, Theorem 2 holds true and therefore, J_{φ_1} is a weakly localizable submodule.

Now we argue as in the proof of Statement 1 of Theorem 1. We introduce the function φ_2 by formula (2.5), where the sequence Γ satisfies the condition $\Gamma \bigcap \Lambda_{\varphi_1} = \Lambda_{\mathcal{J}}$ and is so close to the sequence Λ_{φ_1} that the submodule $\mathcal{J}_{\varphi_1,\varphi_2}$ is stable. Moreover, this stable submodule contains a weakly localizable submodule $\mathcal{J}(\Phi)$. Theorem 1 in the work [3] states that then the submodule $\mathcal{J}_{\varphi_1,\varphi_2}$ is weakly localizable. The indicator segment and the zero set of the submodule $\mathcal{J}_{\varphi_1,\varphi_2}$ are the same as for the original submodule $\mathcal{J}$. Therefore, $\mathcal{J} = \mathcal{J}_{\varphi_1,\varphi_2}$. □

3. Representation for invariant subspaces admitting a weak spectral synthesis

We consider the Schwarz space $\mathcal{E}(a;b) = C^\infty(a;b)$ equipped with the metrizable topology of the projective limit of the Banach spaces $C^k[a_k;b_k]$, where $[a_1;b_1] \Subset [a_2;b_2] \Subset \ldots$ is some sequence of the segments exhausting the interval $(a;b)$. It is known that $\mathcal{E}(a;b)$ is the reflexive Fréchet space. By W is a closed subspace of this space invariant w.r.t. the differentiation operator $D = \frac{\mathrm{d}}{\mathrm{d}\, t}$ (shortly, *D-invariant*). If else is not said, in what follows we consider only closed subspaces in $\mathcal{E}(a;b)$.

Let $\operatorname{Exp} W$ be all root elements of the operator D (exponential monomials $t^j e^{-\mathrm{i}\lambda t}$) contained in W. For a non-trivial not coinciding with the entire space $\mathcal{E}(a;b)$) subspace W, the set $\operatorname{Exp} W$ is at most countable.

We let

$$W_I = \{f \in \mathcal{E} : \ f^{(k)}(t) = 0, \ t \in I, \ k = 0, 1, 2, \dots\}, \tag{3.1}$$

where $I \subset (a; b)$ is a relatively closed non-empty segment and denote by I_W the minimal relatively closed in $(a; b)$ non-empty segment satisfying the condition $W_I \subset W$ (the existence of such segment is implied by Theorem 4.1 in [16]).

The Fourier-Laplace transform $\mathcal{F}$ acting in the strongly dual space $\mathcal{E}'(a; b)$ by the rule

$$\mathcal{F}(S)(z) = (S, e^{-itz}), \quad S \in (C^\infty(a; b))',$$

is a linear topological isomorphism of space $(C^\infty(a; b))'$ and $\mathcal{P}(a; b)$ [17, Thm. 7.3.1]. We have the following

Duality principle *There exists a one-to-one correspondence between the set $\{\mathcal{J}\}$ of weakly localizable submodules of the module $\mathcal{P}(a; b)$ and the set $\{W\}$ of D-invariant subspaces of the space $\mathcal{E}(a; b)$ determined by rule $\mathcal{J} \longleftrightarrow W$ if and only if $\mathcal{J} = \mathcal{F}(W^0)$, where a closed subspace $W^0 \subset \mathcal{E}'(a; b)$ is formed by all the distributions $S \in \mathcal{E}'(a; b)$ annulating W; at that,*

$$\overline{I}_W = [c_{\mathcal{J}}; d_{\mathcal{J}}], \quad \operatorname{Exp} W = \{t^j e^{-i\lambda_k t}, \ j = 0, \dots m_k - 1, \quad (\lambda_k, m_k) \in \Lambda_{\mathcal{J}}\},$$

where $\Lambda_{\mathcal{J}}$ is the set of the zeroes of the submodule $\mathcal{J}$ ([3, Duality principle], [4, Prop. 1]).

It is known (see [16, Thm. 2.1]) that given a nontrivial D-invariant subspace W, the spectrum σ_W of the operator $D : W \to W$ either coincides with the entire complex plane or is discrete; in the second case $\sigma_W = \Lambda_{\mathcal{J}}$ according the **duality principle**.

A nontrivial D-invariant subspace admits a *weak spectral synthesis* if

$$W = \overline{W_{I_W} + \mathcal{L}(\operatorname{Exp} W)}, \qquad \mathcal{L}(\,\cdot\,) \quad \text{is the linear span of a set.} \tag{3.2}$$

It is clear that a D-invariant subspace W admitting a weak spectral synthesis is minimal among all D-invariant subspaces $\widetilde{W}$ satisfying

$$I_{\widetilde{W}} = I_W, \quad \operatorname{Exp} \widetilde{W} = \operatorname{Exp} W.$$

By the **duality principle**, the *annulating submodule* $\mathcal{J} = \mathcal{F}(W^0)$ of such subspace is the maximal one among all maximal submodules $\widetilde{\mathcal{J}} \subset \mathcal{P}(a; b)$, with the zero set and the indicator diagram satisfying the conditions:

$$\Lambda_{\widetilde{\mathcal{J}}} = \Lambda_{\mathcal{J}}, \quad [c_{\widetilde{\mathcal{J}}}; d_{\widetilde{\mathcal{J}}}] = [c_{\mathcal{J}}; d_{\mathcal{J}}].$$

Therefore, $\mathcal{J}$ is a weakly localizable submodule. It is clear that the opposite is true as well: if an annulating submodule of a D-invariant subspace is weakly localizable, then this subspace admits a weak spectral synthesis.

We recall that the *completeness radius* $\rho(\Lambda)$ of a sequence of multiple points $\Lambda = \{(\lambda_j, m_j)\}$ is defined as the infimum of the radii of open intervals $I \subset \mathbb{R}$, for which the system of the exponential monomials $\{t^k e^{-\mathrm{i}\lambda_j t}, \ k = 0, \dots, m_j - 1, \ j \in \mathbb{N}\}$, is incomplete in the spaces $\mathcal{E}(I)$, $C(I)$, $L^p(I)$, $1 \leqslant p < \infty$ (see [18]).

Given arbitrary subsets $A, \ B \subset \mathbb{R}$, we denote by $A \dotdiv B$ their *geometric difference*, i.e., the set of all $x \in \mathbb{R}$ obeying $x + B \subset A$. Let $S \in \mathcal{E}'(a; b)$ and $h \in (a; b) \dotdiv \operatorname{ch} \operatorname{supp} S$, where $\operatorname{ch} \operatorname{supp} S$ is the convex hull of $\operatorname{supp} S$. We define a functional $S_h \in \mathcal{E}'(a; b)$ by the formula

$$(S_h, f) = (S, f(t + h)), \quad f \in \mathcal{E}(a; b).$$

Given a distribution $S \in \mathcal{E}'(a; b)$ and a non-empty relatively closed in $(a; b)$ segment $\langle c; d\rangle$ satisfying the condition

$$\operatorname{ch} \operatorname{supp} S \Subset \langle c; d\rangle, \tag{3.3}$$

we let

$$W(S, \langle c; d\rangle) = \{f \in \mathcal{E}(a; b) : \ (S * f)(h) = (S_h, f) = 0, \ \forall h \in \langle c; d\rangle \ \dotdiv \operatorname{ch} \operatorname{supp} S\}.$$

It is clear that $W(S, \langle c; d\rangle)$ is a D-invariant subspace.

Lemma 4. *A D-invariant subspace $W(S, \langle c; d\rangle)$ admits a weak spectral synthesis, its annulating submodule is $\mathcal{J}(\varphi, \langle c; d\rangle)$, where $\varphi = \mathcal{F}(S)$.*

Proof. By the arguments given after the **duality principle** we see that the first statement of the lemma is implied by the first one. We denote by $\mathcal{J}_1$ the annulating submodule of the subspace $W(S, \langle c; d\rangle)$. In accordance with the **duality principle** we have the inclusion

$$\mathcal{J}_1 \subset \mathcal{J}(\varphi, \langle c; d\rangle).$$

Since the zero set $\Lambda_{\mathcal{J}_1}$ of the submodule $\mathcal{J}_1$ coincides with the zero set Λ_φ of the function φ and the indicator segment of this submodule is equal to $[c; d]$, it follows from (3.3) that the quantity $\rho(\Lambda_{\mathcal{J}_1})$ is less than the half of the length of the segment $[c; d]$. Statement 3) of Theorem 2 in work [3] states that in this case the submodule $\mathcal{J}_1$ is weakly localizable only if it is stable.

As we mentioned earlier [4, Introduction], the module $\mathcal{P}(a; b)$ is a a bornological and b-stable space (the latter notion was introduced in the work [5]). This is why it belongs to the class of the topological modules, for which it was proved in the work [7] (Proposition 4.2 and Remark 1 in the end of Subsection 1 of Section 4) that the stability of the submodule $\mathcal{J} \subset \mathcal{P}(a; b)$ at each point $\lambda \in \mathbb{C}$ is implied by its stability at some single point. Thus, in order to prove the identity

$$\mathcal{J}_1 = \mathcal{J}(\varphi, \langle c; d\rangle)$$

(which is equivalent to the weak localizable property of the submodule $\mathcal{J}_1$), it is sufficient to check the stability of the submodule $\mathcal{J}_1$ at some point $\mu \notin \Lambda_\varphi$. Without loss of generality we can assume that $\mu = 0$, $\varphi(0) = 1$.

Let $\psi \in \mathcal{J}_1$, $\psi(0) = 0$. The function ψ is the limit of a generalized sequence of the form $(a_1 e^{\mathrm{i}h_1 z} + \cdots + a_m e^{\mathrm{i}h_m z})\varphi$ in the topology of the space $\mathcal{P}(a; b)$, where $h_j \in \langle c; d\rangle \div [c_\varphi; d_\varphi]$, $j = 1, \ldots, m$; $\mathrm{i}[c_\varphi; d_\varphi]$ is the indicator diagram of the function φ (coinciding with $\operatorname{ch}\operatorname{supp} S$ by Paley-Wiener theorem). Since it is obvious that $e^{\mathrm{i}h' z}\varphi \to e^{\mathrm{i}hz}\varphi$ as $h' \to h$ in the topology of $\mathcal{P}(a; b)$, we can assume that

$$h_j \in (c; d) \div [c_\varphi; d_\varphi], \quad j = 1, \ldots, m. \tag{3.4}$$

By the definition of the topology in $\mathcal{P}(a; b)$ it is easy to obtain that the generalized sequence

$$\left(a_1 \frac{e^{\mathrm{i}h_1 z} - 1}{z} + \cdots + a_m \frac{e^{\mathrm{i}h_m z} - 1}{z}\right)\varphi \tag{3.5}$$

converges to the function $\frac{\psi}{z}$.

By belongings (3.4), each element of generalized sequence (3.5) belongs to a *localizable* submodule $\mathcal{J}(\varphi, (c; d))$ of the module $\mathcal{P}(c; d)$. By the duality principle and a well-known result on a spectral synthesis in the kernel of the convolution operator (see, for instance, [20, Thm. 16.4.1]), this submodule coincides with the annulating submodule $\mathcal{J}_2 \subset \mathcal{P}(c; d)$ of a D-invariant subspace $W(S, (c; d)) \subset \mathcal{E}(c; d)$, where

$$W(S, (c; d)) = \{f \in \mathcal{E}(c; d) : \ (S * f)(h) = (S_h, f) = 0, \ \forall h \in (c; d) \ \div \operatorname{ch}\operatorname{supp} S\}.$$

In view of the above, we conclude that each function in generalized sequence (3.5) belongs to the submodule $\mathcal{J}_2 = \mathcal{J}(\varphi, (c; d))$, which, in its turn, is contained in $\mathcal{J}_1$. And therefore, the limiting function $\frac{\psi}{z}$ of generalized sequence (3.5) satisfies also the belonging $\frac{\psi}{z} \in \mathcal{J}_1$, that is, the submodule $\mathcal{J}_1$ is stable at point 0. This fact implies both statements of the lemma. □

Remark 1. *We note that the proven lemma is true if we replace condition (3.3) by a weaker condition: the length of the segment $\operatorname{ch}\operatorname{supp} S$ is less than $(d - c)$ (the latter can be equal to $+\infty$).*

Theorem 4. *Each D-invariant subspace with a discrete spectrum σ_W satisfying the condition $2\rho(\sigma_W) < |I_W|$, where $|I_W| \leqslant +\infty$ is the length of the segment I_W, can be represented as the set of the solutions to two (probably, coinciding) homogeneous convolution equations:*

$$f \in W \Longleftrightarrow \begin{cases} (S_1 * f)(h) = 0, & h \in I_W \div ch\,supp\, S_1, \\ (S_2 * f)(h) = 0, & h \in I_W \div ch\,supp\, S_2. \end{cases} \tag{3.6}$$

Proof. Corollary 2 in the work [3] states that a D-invariant subspace W satisfying the assumptions of the proven theorem admits a weak spectral synthesis and its annulating submodule $\mathcal{J}$ is weakly localizable. It is easy to see that this submodule contains a function φ_1 in $\mathcal{P}_0(a;b)$ with an indicator diagram compactly embedded in the segment $\mathrm{i}I_W$. And thus, in accordance with Statement 1 of Theorem 1 and the duality principle

$$\mathcal{J} = \overline{\mathcal{J}\,(\varphi_1, I_W) + \mathcal{J}\,(\varphi_2, I_W)},$$

where function $\varphi_2 \in \mathcal{J} \bigcap \mathcal{P}_0(a;b)$ has the same indicator diagram as the function φ_1. Applying Lemma 4, in view of the reflexivity of the space $\mathcal{E}(a;b)$ we obtain relation (3.6) with $S_1 = \mathcal{F}^{-1}(\varphi_1)$, $S_2 = \mathcal{F}^{-1}(\varphi_2)$. □

Theorem 5. *If a D-invariant subspace W admits a weak spectral synthesis and $\overline{I}_W \subset (a;b)$, then there exist distributions S_1, $S_2 \in W^0$ (probably, $S_1 = S_2$) such that*

$$f \in W \Longleftrightarrow \begin{cases} (S_1, D^j f) = 0, & j = 0, 1, 2, \dots, \\ (S_2, D^j f) = 0, & j = 0, 1, 2, \dots \end{cases} \tag{3.7}$$

Proof. The annulating submodule $\mathcal{J} = \mathcal{F}(W^0)$ is weakly localizable and satisfies the assumptions of Theorem 3. Therefore, either $\mathcal{J} = \mathcal{J}_\varphi$ or $\mathcal{J} = \mathcal{J}_{\varphi_1,\varphi_2}$. In view of the duality principle and the reflexivity of the space $\mathcal{E}(a;b)$, it follows that (3.7) holds true with $S_1 = \mathcal{F}^{-1}(\varphi_1)$, $S_2 = \mathcal{F}^{-1}(\varphi_2)$ (at that $S_1 = S_2$, if $\mathcal{J}$ is the principle submodule). □

BIBLIOGRAPHY

1. J. Sebastian-e-Silva. "On some classes of locally convex spaces important in applications" // Matematika. Sbornik Perevodov. **1**, 60–77 (1957). (in Russian).
2. B.Y. Levin (in collaboration with Yu. Lyubarskii, M. Sodin, V. Tkachenko). *Lectures on entire functions.* (Rev. Edition). AMS, Providence, Rhode Island (1996).
3. N.F. Abuzyarova. *Spectral synthesis in the Schwartz space of infinitely differentiable functions* // Dokl. Akad. Nauk. **457**:5, 510-513 (2014). [Dokl. Math. **90**:1, 479-482 (2014).]
4. N.F. Abuzyarova. *Closed submodules in the module of entire functions of exponential type and polynomial growth on the real axis* // Ufimskij Matem. Zhurn. **6**:4, 3–18 (2014). [Ufa Math. J. **6**:4, 3–17 (2014).]
5. I.F. Krasičkov-Ternovskiĭ. *Local description of closed ideals and submodules of analytic functions of one variable. I* // Izvestia AN SSSR. Ser. Matem. **43**:1, 44–66 (1979). [Math. USSR-Izvestiya. **14**:1, 41-60 (1980).]
6. I.F. Krasičkov-Ternovskiĭ. *Invariant subspaces of analytic functions. I. Spectral analysis on convex regions* // Matem. Sbornik. **87(129)**:4, 459–489 (1972). [Math. USSR-Sbornik. **16**:4, 471–500 (1972).]
7. I.F. Krasičkov-Ternovskiĭ. *Local description of closed ideals and submodules of analytic functions of one variable. II* // Izvestia AN SSSR. Ser. Matem. **43**:2, 309–341 (1979). [Math. USSR-Izvestiya. **14**:2, 289–316 (1980).]
8. A. Aleman, A. Baranov, Yu. Belov. *Subspaces of C^∞ invariant under the differentiation* // J. Funct. Anal. **268**:8, 2421–2439 (2015).

9. N.F. Abuzyarova. *Some properties of principal submodules in the module of entire functions of exponential type and polynomial growth on the real axis* // Ufimskij Matem. Zhurn. **8**:1, 3–14 (2016). [Ufa Math. J. **8**:1, 1–12 (2016).]
10. N.F. Abuzyarova. *A property of subspaces admitting spectral synthesis* // Matem. Sbornik. **190**:4, 3–22 (1999). [Sbornik: Math. **190**:4, 481–499 (1999).]
11. N.F. Abuzyarova. *Finitely generated submodules in the module of entire functions determined by restrictions on the indicator function* // Matem. Zametki. **71**:1, 3–17 (2002). [Math. Notes. **71**:1, 3–16 (2002).]
12. B.N. Khabibullin. *Closed submodules of holomorphic functions with two generators* // Funkts. Anal. Pril. **38**:1, 65–80 (2004). [Funct. Anal. Appl. **38**:1, 52–64 (2004).]
13. A.M. Sedletskii. *Analytic Fourier transforms and exponential approximations. I* // Sovr. Matem. Fundament. Napravlenia. **5**, 3–152 (2003). [J. Math. Sci. **129**:6, 4251–4408 (2005).]
14. R.S. Yulmukhametov. *Approximation of subharmonic functions* // Anal. Math. **11**:3, 257–282 (1985). (in Russian).
15. R.S. Yulmukhametov. *Solution of the Ehrenpreis factorization problem* // Matem. Sborn. **190**:4, 123–157 (19990. [Sb. Math. **190**:4, 567–629 (1999).]
16. A. Aleman, B. Korenblum. *Derivation-invariant subspaces of* C^∞ // Comp. Meth. Funct. Theory. **8**:2, 493–512 (2008).
17. L. Hörmander. *The analysis of linear partial differential operators I: distribution theory and Fourier analysis.* Springer, Berlin, (1980).
18. A. Beurling, P. Malliavin. *On the closure of characters and the zeros of entire functions* // Acta Math. **118**:1-4, 79–93 (1967).
19. A.V. Abanin. *Ultra-differentiable functions and ultra-distributions.* Nauka, Moscow (2007). (in Russian).
20. L. Hörmander. *The analysis of linear partial differential operators I: differential operators with constant coefficients.* Springer, Berlin, (1983).
21. P. Koosis. *The logarithmic integral I.* Cambridge Univ. Press, Cambridge (1998).
22. R.P. Boas, Jr. *Entire functions.* Acad. Press, New-York (1954).
23. L.I. Ronkin. *Introduction to the theory of entire functions of several variables.* Nauka, Moscow (1971). [Transl. Math. Monog. **44**. Amer. Math. Soc., Providence, R.I. (1974).]
24. S.J. Favorov. *On the addition of the indicators of entire and subharmonic functions of several variables* // Matem. Sborn. **105(147)**:1, 128–140 (1978). [Math. USSR. Sb. **34**:1, 119–130 (1978).]

Natalia Fairbakhovna Abuzyarova,
Bashkir State University,
Zaki Validi str. 32,
450074, Ufa, Russia
E-mail: abnatf@gmail.com