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MINIMAL VALUE FOR THE TYPE OF AN ENTIRE FUNCTION OF ORDER $\rho \in (0, 1)$, WHOSE ZEROS LIE IN AN ANGLE AND HAVE A PRESCRIBED DENSITY

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Abstract. In the work we find the minimal value that can be taken by the type of an entire function of order $\rho \in (0, 1)$ with zeroes of prescribed upper and lower densities and located in an angle of a fixed opening less than π. The main theorem generalizes the previous result by the author (the zeroes lie on one ray) and by A.Yu. Popov (only the upper density of zeros was taken into consideration). We distinguish and study in detail the case when the an entire function has a measurable sequence of zeroes. We provide applications of the obtained results to the uniqueness theorems for entire functions and to the completeness of exponential systems in the space of analytic in a circle functions with the standard topology of uniform convergence on compact sets.

Keywords: type of an entire function, upper and lower density of zeroes, uniqueness theorem, completeness of exponential system.

Mathematics Subject Classification: 30D15

1. Introduce

Let $\rho \in (0,1)$, $\beta > 0$, $\alpha \in [0,\beta]$. Let $f(z)$ be an entire function, whose zeroes are located in an angle of a fixed opening $\leqslant \pi$ and they form a sequence $\Lambda = \Lambda_f = (\lambda_n)_{n=1}^{\infty}$ with an *upper* and *lower* ρ*-densities*

$$\overline{\Delta}_{\rho}(\Lambda) \equiv \varlimsup_{n\to\infty} \frac{n}{|\lambda_n|^{\rho}} = \beta, \qquad \underline{\Delta}_{\rho}(\Lambda) \equiv \varliminf_{n\to\infty} \frac{n}{|\lambda_n|^{\rho}} \geqslant \alpha, \tag{1}$$

respectively. As usually, the zeroes are taken counting multiplicity and in ascending order of the absolute values.

We need to find the smallest possible value for the *type* of function $f(z)$ at order ρ determined by the formula

$$\sigma_{\rho}(f) \equiv \varlimsup_{r\to+\infty} r^{-\rho} \ln \max_{|z|=r} |f(z)|\,. \tag{2}$$

Without loss of generality we assume that

$$\Lambda \subset \Gamma_{\theta} \equiv \{z \in \mathbb{C} : |\arg z| \leqslant \theta\}\,, \tag{3}$$

where $\theta \in [0, \pi/2]$, by reducing the problem of finding the extremum

$$s_{\theta}(\alpha,\beta;\rho) \equiv \inf\left\{\sigma_{\rho}(f) : \Lambda = \Lambda_f \subset \Gamma_{\theta},\ \overline{\Delta}_{\rho}(\Lambda) = \beta,\ \underline{\Delta}_{\rho}(\Lambda) \geqslant \alpha\right\}. \tag{4}$$

We mention that as $\theta = 0$, we obtain the problem for entire functions with zeroes at a ray solved earlier by A.Yu. Popov [1] (for $\alpha = 0$) and by the author [2] (for each $\alpha \in [0,\beta]$).

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In the present paper, quantity $s_\theta(\alpha,\beta;\rho)$ is calculated for all $\theta\in[0,\pi/2]$. Apart from the introduction, the work consists of three parts. In the first part we obtain the lower bound for the type of a function defined in (2). The second part is devoted to the proof of the sharpness for this estimate. The result is formulated as the following theorem.

Theorem 1. *Suppose that we are given numbers $\rho\in(0,1)$, $\beta>0$, $\alpha\in[0,\beta]$, $\theta\in[0,\pi/2]$. Then the formula*

$$s_\theta(\alpha,\beta;\rho)=\frac{\pi\alpha}{\sin\pi\rho}\cos\rho\,\theta+\max_{a>0}\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a}\left(\beta\,a^{-\rho}-\alpha\,x^{-\rho}\right)\frac{x+\cos\theta}{x^2+2x\cos\theta+1}\,dx$$

holds true. The infimum in (4) is attained at some function with a sequence of zeroes Λ_0 located at two rays $\arg z=\pm\,\theta$ *such that $\overline{\Delta}_\rho(\Lambda_0)=\beta$, $\underline{\Delta}_\rho(\Lambda_0)=\alpha$.*

In the third part of the work Theorem 1 is employed to specify one uniqueness theorem by B.N. Khabibullin. We provide some applications to entire functions of exponential type and issues on the completeness of exponentials systems.

The extremal problem on finding $s_\theta(\alpha,\beta;\rho)$ as $\alpha=0$ (i.e., without taking into consideration the lower ρ-density of zeroes) was posed and solved by A.Yu. Popov [3]. He found the quantity

$$s_\theta(0,\beta;\rho)=\frac{\beta}{2}\max_{a>0}a^{-\rho}\ln(1+2a\cos\theta+a^2).$$

For the functions with a *measurable* sequence of zeroes $\Lambda=\Lambda_f=(\lambda_n)_{n=1}^\infty$, that is, which has a *$\rho$-density*

$$\Delta_\rho(\Lambda)\equiv\lim_{n\to\infty}\frac{n}{|\lambda_n|^\rho}=\beta,$$

by Theorem 1 we obtain the relation

$$s_\theta(\beta,\beta;\rho)=\frac{\pi\beta}{\sin\pi\rho}\cos\rho\,\theta.$$

We note that extremal value $s_\theta(\beta,\beta;\rho)$ is attained if all the zeroes of a function are located at the rays $\arg z=\pm\,\theta$, and at each of them they form measurable sequences with the same ρ-densities ($=\beta/2$), and $s_\theta(\beta,\beta;\rho)$ is surely not attained if these ρ-densities are different.

The state-of-art of the theory of extremal problems for the type of entire functions with zeroes at a ray or in an angle is exposed in surveys [3], [4].

We proceed to the proof of Theorem 1.

2. Estimate for the type of an entire function

Let $f(z)$ be an entire function of order $\rho\in(0,1)$. We assume that the sequence of all its zeroes $\Lambda=\Lambda_f=(\lambda_n)_{n=1}^\infty$ lies in an angle Γ_θ with a fixed $\theta\in[0,\pi/2]$ and has ρ-densities $\overline{\Delta}_\rho(\Lambda)=\beta$, $\underline{\Delta}_\rho(\Lambda)\geqslant\alpha$. Hereafter $\alpha\in(0,\beta]$, since the case $\alpha=0$ was considered in [3]. Let us prove the estimate

$$\sigma_\rho(f)\geqslant\frac{\pi\alpha}{\sin\pi\rho}\cos\rho\,\theta+\max_{a>0}\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a}\left(\beta\,a^{-\rho}-\alpha\,x^{-\rho}\right)\frac{x+\cos\theta}{x^2+2x\cos\theta+1}\,dx.\tag{5}$$

We can assume that $f(0)=1$. Then by Hadamard theorem (see [5, Ch. I, Sect. 10]) function $f(z)$ can be represented as the canonical product

$$f(z)=\prod_{n=1}^\infty\left(1-\frac{z}{\lambda_n}\right).\tag{6}$$

Due to (3) we write $\lambda_n = r_n e^{i\varphi_n}$, $|\varphi_n| \leqslant \theta$, $n \in \mathbb{N}$. Then by (6) we obtain

$$M_f(r) \equiv \max_{|z|=r} |f(z)| \geqslant |f(-r)| = \prod_{n=1}^{\infty} \left|1 + \frac{r}{\lambda_n}\right| = \prod_{n=1}^{\infty} \left|1 + \frac{r}{r_n} e^{-i\varphi_n}\right|$$

$$= \prod_{n=1}^{\infty} \sqrt{1 + \frac{2r}{r_n}\cos\varphi_n + \left(\frac{r}{r_n}\right)^2} \geqslant \prod_{n=1}^{\infty} \sqrt{1 + \frac{2r}{r_n}\cos\theta + \left(\frac{r}{r_n}\right)^2}.$$

We denote by $n_\Lambda(\tau) = \sum_{|\lambda_n| \leqslant \tau} 1$ the counting function of sequence Λ, or, equivalently, of the sequence $|\Lambda| \equiv (|\lambda_n|)_{n=1}^{\infty} = (r_n)_{n=1}^{\infty}$. We also observe that formulae (1) can be written as

$$\overline{\Delta}_\rho(\Lambda) = \varlimsup_{t\to+\infty} \frac{n_\Lambda(t)}{t^\rho} = \beta, \quad \underline{\Delta}_\rho(\Lambda) = \varliminf_{t\to+\infty} \frac{n_\Lambda(t)}{t^\rho} \geqslant \alpha. \tag{7}$$

Standard employing of Stiltjes integral gives

$$\ln M_f(r) \geqslant \sum_{n=1}^{\infty} \ln \sqrt{1 + \frac{2r}{r_n}\cos\theta + \left(\frac{r}{r_n}\right)^2} = \frac{1}{2}\int_0^{+\infty} \ln\left(1 + \frac{2r}{\tau}\cos\theta + \left(\frac{r}{\tau}\right)^2\right) d\,n_\Lambda(\tau).$$

Integrating by parts and using the conditions

$$f(0) = 1, \qquad n_\Lambda(\tau) = O(\tau^\rho), \quad \tau \to +\infty,$$

eliminating the non-integral terms, we arrive at the relation

$$\frac{1}{2}\int_0^{\infty} \ln\left(1 + \frac{2r}{\tau}\cos\theta + \left(\frac{r}{\tau}\right)^2\right) d\,n_\Lambda(\tau) = \int_0^{+\infty} n_\Lambda(\tau)\, \frac{r\,(\tau\cos\theta + r)}{\tau\,(\tau^2 + 2r\tau\cos\theta + r^2)}\, d\tau.$$

By the change of variable $\tau = rt$ and the notations

$$\varphi_r(t) \equiv \frac{n_\Lambda(rt)}{(rt)^\rho}, \qquad K(t) \equiv \frac{t^{\rho-1}(t\cos\theta + 1)}{t^2 + 2t\cos\theta + 1}, \qquad t > 0, \tag{8}$$

we get the estimate

$$r^{-\rho} \ln M_f(r) \geqslant \int_0^{+\infty} \varphi_r(t)\, K(t)\, dt, \qquad r > 0. \tag{9}$$

In the above integral, function $\varphi_r(t)$ satisfies the conditions

$$\varlimsup_{t\to+\infty} \varphi_r(t) = \beta, \qquad \varliminf_{t\to+\infty} \varphi_r(t) \geqslant \alpha,$$

for fixed r, while kernel $K(t)$ is positive as $t > 0$ for arbitrary value of parameter $\theta \in [0, \pi/2]$ (see (7), (8)). This is why in further estimates we can employ the method developed in [2] for the case, when zeroes Λ are located at a single ray ($\theta = 0$). We fix an arbitrary number $a > 0$ and let $\eta = \eta(r) \equiv \varphi_r(1/a)$. We have

$$\varlimsup_{r\to+\infty} \eta(r) = \beta, \qquad \varliminf_{r\to+\infty} \eta(r) \geqslant \alpha.$$

Let $\alpha' \in (0, \alpha)$. As it was shown in [2], there exists a number $c > 0$ such that for all $r \geqslant ac$ and $t \geqslant c/r$ the inequality $\varphi_r(t) \geqslant \psi_r(t)$ holds true, where function $\psi_r(t)$ is defined for positive t by the formula

$$\psi_r(t) \equiv \begin{cases} \alpha', & t \notin \left[\dfrac{1}{a}, \left(\dfrac{\eta}{\alpha'}\right)^{1/\rho}\dfrac{1}{a}\right], \\[2ex] \dfrac{\eta}{(at)^\rho}, & t \in \left[\dfrac{1}{a}, \left(\dfrac{\eta}{\alpha'}\right)^{1/\rho}\dfrac{1}{a}\right]. \end{cases} \tag{10}$$

By (9) it follows that

$$r^{-\rho}\ln M_f(r) \geqslant \int\limits_{c/r}^{+\infty} \psi_r(t)\,K(t)\,dt, \qquad r \geqslant ac. \tag{11}$$

Substituting expressions for $K(t)$ in (8) and for $\psi_r(t)$ in (10) into (11) and extracting the known integral (see, for instance, [6, Prob. 4.174])

$$\int\limits_0^{+\infty} K(t)\,dt = \frac{\pi}{\sin\pi\rho}\,\cos\rho\,\theta, \tag{12}$$

we obtain the estimate

$$r^{-\rho}\ln M_f(r) \geqslant \frac{\pi\alpha'}{\sin\pi\rho}\,\cos\rho\,\theta + \int\limits_{1/a}^{(1/a)(\eta/\alpha')^{1/\rho}} \frac{(\eta a^{-\rho}-\alpha' t^{\rho})\,(t\cos\theta+1)}{t\,(t^2+2t\cos\theta+1)}\,dt - \alpha'\int\limits_0^{c/r} K(t)\,dt.$$

We pass to the limit over the sequence of r, on which $\eta=\eta(r)$ tends to β. In view of (2) we have

$$\sigma_\rho(f) \geqslant \frac{\pi\alpha'}{\sin\pi\rho}\,\cos\rho\,\theta + \int\limits_{1/a}^{(1/a)(\beta/\alpha')^{1/\rho}} \frac{(\beta a^{-\rho}-\alpha' t^{\rho})\,(t\cos\theta+1)}{t\,(t^2+2t\cos\theta+1)}\,dt.$$

In order to obtain estimate (5), it remains to make the change of variable $t=1/x$ in the integral and employ the freedom in the choice of numbers $\alpha'\in(0,\alpha)$ and $a>0$.

3. Proof of the sharpness of the estimate

Let us show that estimate (5) can be attained. In order to do it, we choose sequence Λ_0 on the rays $\arg z=\pm\theta$ so that

$$\overline{\Delta}_\rho(\Lambda_0)=\beta, \qquad \underline{\Delta}_\rho(\Lambda_0)=\alpha, \tag{13}$$

and the canonical product

$$f_0(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right), \qquad \lambda_n\in\Lambda_0, \tag{14}$$

has the type

$$\sigma_\rho(f_0)=\frac{\pi\alpha}{\sin\pi\rho}\,\cos\rho\,\theta + \max_{a>0}\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a} \left(\beta\,a^{-\rho}-\alpha\,x^{-\rho}\right)\frac{x+\cos\theta}{x^2+2x\cos\theta+1}\,dx. \tag{15}$$

For the parameters of the problem we assume that

$$\rho\in(0,1), \quad \beta>0, \quad \alpha\in(0,\beta], \quad \theta\in(0,\pi/2],$$

and it corresponds to the situation not studied before. The cases $\alpha\in(0,\beta)$ and $\alpha=\beta$ will be treated separately.

Suppose first that $\alpha\in(0,\beta)$. We first employ the construction of an extremal sequence proposed by the author in [2] for $\theta=0$. We choose an auxiliary positive sequence $(m_k)_{k=1}^{\infty}$ with the property

$$m_1>1, \quad m_{k+1}=m_k^4, \qquad k\in\mathbb{N},$$

and construct the sequence $(r_j)_{j=1}^{\infty}\subset\mathbb{R}_+$ following the rule: in the segments $[m_k,\,m_k^2-1]$ and $\Big[(\beta/\alpha)^{1/\rho}\,m_k^2,\,m_{k+1}\Big)$ points r_j^ρ form an arithmetical progression with the step $2/\alpha$; in the segments $(m_k^2-1,\, m_k^2]$ points r_j^ρ form an arithmetical progression with the step $\frac{2\rho}{(\beta-\alpha)m_k^2}$; the segments $\left(m_k^2,\, (\beta/\alpha)^{1/\rho}\, m_k^2\right)$ contain no points r_j^ρ. In accordance with [2], the upper and lower ρ-densities of the sequence $(r_j)_{j=1}^\infty$ are equal to $\beta/2$ and $\alpha/2$, respectively. Letting

$$\Lambda_0 \equiv \left(r_j\, e^{-i\theta}\right)_{j=1}^\infty \bigcup \left(r_j\, e^{i\theta}\right)_{j=1}^\infty,$$

we arrive immediately at (13). By sequence Λ_0 we defined canonical product (14). We observe that

$$f_0(z) = \prod_{j=1}^\infty \left(1 - \frac{z}{r_j}\, e^{i\theta}\right) \left(1 - \frac{z}{r_j}\, e^{-i\theta}\right) = \prod_{j=1}^\infty \left(1 - \frac{2z}{r_j} \cos\theta + \left(\frac{z}{r_j}\right)^2\right)$$

that implies

$$M_{f_0}(r) = \max_{|z|=r} |f_0(z)| = f_0(-r) = \prod_{j=1}^\infty \left(1 + \frac{2r}{r_j} \cos\theta + \left(\frac{r}{r_j}\right)^2\right).$$

Since counting function $n_{\Lambda_0}(\tau)$ of sequence Λ_0 is a doubled counting function of the sequence $(r_j)_{j=1}^\infty$, reproducing the appropriate arguments from Section 2, we arrive at the representation

$$r^{-\rho} \ln M_{f_0}(r) \;=\; \int\limits_0^{+\infty} \varphi_{0,r}(t)\, K(t)\, dt, \qquad r > 0, \tag{16}$$

where $\varphi_{0,r}(t) \equiv \frac{n_{\Lambda_0}(rt)}{(rt)^\rho}$ and $K(t)$ is defined in (8). Thus, function (14) provides the identity in (9).

Up to residual terms making no influence on type (2), function $\varphi_{0,r}(t)$ with a parameter $r > 0$ coincides with function $\Phi_r(t)$, which is defined for $t > 0$ by the formulae

$$\Phi_r(t) \equiv \alpha, \quad t \in \left(0,\, \frac{m_1}{r}\right], \quad \Phi_r(t)\,|_{\left[\frac{m_k}{r},\, \frac{m_{k+1}}{r}\right]} \equiv \begin{cases} \alpha, & t \notin \left[\dfrac{m_k^2}{r},\, \left(\dfrac{\beta}{\alpha}\right)^{1/\rho} \dfrac{m_k^2}{r}\right], \\ \beta \left(\dfrac{m_k^2}{rt}\right)^\rho, & t \in \left[\dfrac{m_k^2}{r},\, \left(\dfrac{\beta}{\alpha}\right)^{1/\rho} \dfrac{m_k^2}{r}\right], \end{cases}$$

where $k \in \mathbb{N}$ (for details see [2]). Hence, it follows from (16) that

$$\sigma_\rho(f_0) \;=\; \varlimsup_{r\to+\infty} \int\limits_0^{+\infty} \Phi_r(t)\, K(t)\, dt. \tag{17}$$

For the sake of brevity, we introduce some notations. Let

$$g(a) \equiv \int\limits_{a(\alpha/\beta)^{1/\rho}}^{a} \frac{(\beta\, a^{-\rho} - \alpha\, x^{-\rho})\, (x + \cos\theta)}{x^2 + 2x\cos\theta + 1}\, dx = \int\limits_{1/a}^{(\beta/\alpha)^{1/\rho}(1/a)} \left(\frac{\beta}{(at)^\rho} - \alpha\right) K(t)\, dt. \tag{18}$$

Since function $g(a)$ is continuous and positive as $a > 0$ and

$$\lim_{a\to+0} g(a) = \lim_{a\to+\infty} g(a) = 0, \tag{19}$$

there exists a point $a_0 > 0$ such that $g(a_0) = \max_{a>0} g(a)$. For $t > 0$ we let

$$\psi_0(t) \equiv \begin{cases} \alpha, & t \notin \left[\frac{1}{a_0}, \left(\frac{\beta}{\alpha}\right)^{1/\rho} \frac{1}{a_0}\right], \\ \frac{\beta}{(a_0 t)^\rho}, & t \in \left[\frac{1}{a_0}, \left(\frac{\beta}{\alpha}\right)^{1/\rho} \frac{1}{a_0}\right]. \end{cases} \tag{20}$$

In view of definitions (18), (20), estimate (5) can be rewritten as

$$\sigma_\rho(f) \geqslant \frac{\pi\alpha}{\sin\pi\rho} \cos\rho\,\theta \,+\, g(a_0) = \int_0^{+\infty} \psi_0(t)\, K(t)\, dt.$$

In particular,

$$\sigma_\rho(f_0) \geqslant \int_0^{+\infty} \psi_0(t)\, K(t)\, dt. \tag{21}$$

Relation (15) to be justified is equivalent to the formula

$$\sigma_\rho(f_0) = \int_0^{+\infty} \psi_0(t)\, K(t)\, dt. \tag{22}$$

Let us prove that

$$\varlimsup_{r\to+\infty} \int_0^{+\infty} \left(\Phi_r(t) - \psi_0(t)\right)\, K(t)\, dt \leqslant 0; \tag{23}$$

this will imply identity (22). Indeed, by (21), (17), (23) we have

$$\begin{aligned} \int_0^{+\infty} \psi_0(t)\, K(t)\, dt \leqslant \sigma_\rho(f_0) &= \varlimsup_{r\to+\infty} \int_0^{+\infty} \left(\Phi_r(t) - \psi_0(t)\right)\, K(t)\, dt + \int_0^{+\infty} \psi_0(t)\, K(t)\, dt \\ &\leqslant \int_0^{+\infty} \psi_0(t)\, K(t)\, dt, \end{aligned}$$

which implies (22).

Thus, it remains to check inequality (23) expressing a “closedness” of weighted counting function $\varphi_{0,r}(t)$ of sequence Λ_0 to “extremal” function $\psi_0(t)$ in (20). Let us first obtain the representation

$$\int_0^{+\infty} \left(\Phi_r(t) - \psi_0(t)\right)\, K(t)\, dt \,=\, \sum_{k=1}^{\infty} g\left(\frac{r}{m_k^2}\right) - g(a_0) \tag{24}$$

basing on (18), (20). In order to do it, we write

$$\psi_0(t) - \alpha \equiv 0, \qquad t \notin \left[\frac{1}{a_0}, \left(\frac{\beta}{\alpha}\right)^{1/\rho} \frac{1}{a_0}\right], \qquad \int_{1/a_0}^{(\beta/\alpha)^{1/\rho}(1/a_0)} \left(\psi_0(t) - \alpha\right)\, K(t)\, dt \,=\, g(a_0).$$

Moreover,

$$\Phi_r(t)-\alpha\equiv 0,\qquad t\notin\bigcup_{k=1}^{\infty}\left[\frac{m_k^2}{r},\ \left(\frac{\beta}{\alpha}\right)^{1/\rho}\frac{m_k^2}{r}\right]\equiv T_r,$$

$$\int\limits_{T_r}(\Phi_r(t)-\alpha)\ K(t)\,dt\ =\sum_{k=1}^{\infty}\ \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)}\left(\beta\left(\frac{m_k^2}{rt}\right)^{\rho}-\alpha\right)\ K(t)\,dt\ =\sum_{k=1}^{\infty}g\left(\frac{r}{m_k^2}\right).$$

Hence,

$$\begin{aligned}\int\limits_0^{+\infty}(\Phi_r(t)-\psi_0(t))\ K(t)\,dt&=\int\limits_0^{+\infty}(\Phi_r(t)-\alpha)\ K(t)\,dt-\int\limits_0^{+\infty}(\psi_0(t)-\alpha)\ K(t)\,dt\\&=\int\limits_{T_r}(\Phi_r(t)-\alpha)\ K(t)\,dt\ -\int\limits_{1/a_0}^{(\beta/\alpha)^{1/\rho}(1/a_0)}(\psi_0(t)-\alpha)\ K(t)\,dt\\&=\sum_{k=1}^{\infty}g\left(\frac{r}{m_k^2}\right)-g(a_0),\end{aligned}$$

and we obtain (24).

Let us estimate the sum in (24) for $r\in\left[m_s^2,\,m_{s+1}^2\right]$ and for a fixed $s\in\mathbb{N}$ by splitting it into three parts:

$$\sum_{k=1}^{\infty}g\left(\frac{r}{m_k^2}\right)-g(a_0)=\sum_{k=1}^{s-1}g\left(\frac{r}{m_k^2}\right)+\sum_{k=s+2}^{\infty}g\left(\frac{r}{m_k^2}\right)+\left(g\left(\frac{r}{m_s^2}\right)+g\left(\frac{r}{m_{s+1}^2}\right)-g(a_0)\right).$$

Employing the inequality $K(t)\leqslant t^{\rho-1}$, $t>0$, while estimating the first sum and neglecting the negative term under the integral, we have

$$\begin{aligned}0<\sum_{k=1}^{s-1}g\left(\frac{r}{m_k^2}\right)&=\sum_{k=1}^{s-1}\ \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)}\left(\beta\left(\frac{m_k^2}{rt}\right)^{\rho}-\alpha\right)K(t)\,dt\\&\leqslant\sum_{k=1}^{s-1}\ \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)}\left(\beta\left(\frac{m_k^2}{rt}\right)^{\rho}-\alpha\right)t^{\rho-1}\,dt\leqslant\beta\sum_{k=1}^{s-1}\left(\frac{m_k^2}{r}\right)^{\rho}\int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)}\frac{dt}{t}\\&=\frac{\beta}{\rho}\ln\frac{\beta}{\alpha}\sum_{k=1}^{s-1}\left(\frac{m_k^2}{r}\right)^{\rho}\leqslant\frac{\beta}{\rho}\ln\frac{\beta}{\alpha}\sum_{k=1}^{s-1}\left(\frac{m_k}{m_s}\right)^{2\rho}.\end{aligned}$$

By the choice of sequence $(m_k)_{k=1}^{\infty}$ we have $\lim\limits_{s\to\infty}\sum\limits_{k=1}^{s-1}\left(\frac{m_k}{m_s}\right)^{2\rho}=0$, since

$$\sum_{k=1}^{s-1}\left(\frac{m_k}{m_s}\right)^{2\rho}<s\left(\frac{m_{s-1}}{m_s}\right)^{2\rho}=\frac{s}{m_s^{3\rho/2}}.$$

It yields

$$\lim_{s\to\infty}\ \sup_{r\in\left[m_s^2,\,m_{s+1}^2\right]}\ \sum_{k=1}^{s-1}g\left(\frac{r}{m_k^2}\right)=0. \tag{25}$$

Employing another inequality $K(t) \leqslant t^{\rho-2}$, $t>0$, while estimating the second sum and again neglecting the negative term under the integral, we have

$$0 < \sum_{k=s+2}^{\infty} g\left(\frac{r}{m_k^2}\right) = \sum_{k=s+2}^{\infty} \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)} \left(\beta\left(\frac{m_k^2}{rt}\right)^{\rho} - \alpha\right) K(t)\,dt$$

$$\leqslant \sum_{k=s+2}^{\infty} \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)} \left(\beta\left(\frac{m_k^2}{rt}\right)^{\rho} - \alpha\right) t^{\rho-2}\,dt \leqslant \beta \sum_{k=s+2}^{\infty} \left(\frac{m_k^2}{r}\right)^{\rho} \int\limits_{m_k^2/r}^{(\beta/\alpha)^{1/\rho}\left(m_k^2/r\right)} \frac{dt}{t^2}$$

$$=\beta\left(1-\left(\frac{\alpha}{\beta}\right)^{1/\rho}\right) \sum_{k=s+2}^{\infty} \left(\frac{m_k^2}{r}\right)^{\rho-1} \leqslant \beta\left(1-\left(\frac{\alpha}{\beta}\right)^{1/\rho}\right) \sum_{k=s+2}^{\infty} \left(\frac{m_{s+1}}{m_k}\right)^{2(1-\rho)}.$$

The choice of sequence $(m_k)_{k=1}^{\infty}$ ensures the condition

$$\lim_{s\to\infty} \sum_{k=s+2}^{\infty} \left(\frac{m_{s+1}}{m_k}\right)^{2(1-\rho)} = 0,$$

since

$$\sum_{k=s+2}^{\infty} \left(\frac{m_{s+1}}{m_k}\right)^{2(1-\rho)} \leqslant \sum_{k=s+2}^{\infty} \left(\frac{m_{k-1}}{m_k}\right)^{2(1-\rho)} = \sum_{k=s+2}^{\infty} \frac{1}{m_k^{3(1-\rho)/2}}.$$

Hence,

$$\lim_{s\to\infty} \sup_{r\in\left[m_s^2,\, m_{s+1}^2\right]} \sum_{k=s+2}^{\infty} g\left(\frac{r}{m_k^2}\right) = 0. \tag{26}$$

Let us estimate the expression

$$g\left(\frac{r}{m_s^2}\right) + g\left(\frac{r}{m_{s+1}^2}\right) - g(a_0), \qquad r \in \left[m_s^2,\, m_{s+1}^2\right],$$

basing on the definition of point a_0 and property (19) of function $g(a)$. We consider two possible cases: $r \in [m_s^2,\, m_s m_{s+1}]$ and $r \in \left[m_s m_{s+1},\, m_{s+1}^2\right]$. In the first case we have $\frac{r}{m_{s+1}^2} \leqslant \frac{m_s}{m_{s+1}}$ and

$$g\left(\frac{r}{m_s^2}\right) - g(a_0) + g\left(\frac{r}{m_{s+1}^2}\right) \leqslant g\left(\frac{r}{m_{s+1}^2}\right) \to 0, \qquad s\to\infty.$$

In the second case we get $\dfrac{r}{m_s^2} \geqslant \dfrac{m_{s+1}}{m_s}$ and

$$g\left(\frac{r}{m_{s+1}^2}\right) - g(a_0) + g\left(\frac{r}{m_s^2}\right) \leqslant g\left(\frac{r}{m_s^2}\right) \to 0, \qquad s\to\infty.$$

Therefore, we can state that

$$\overline{\lim_{s\to\infty}} \sup_{r\in\left[m_s^2,\, m_{s+1}^2\right]} \left(g\left(\frac{r}{m_s^2}\right) + g\left(\frac{r}{m_{s+1}^2}\right) - g(a_0)\right) \leqslant 0. \tag{27}$$

Combining (24)–(27), we obtain (23).

Thus, in the case $\alpha \in (0,\beta)$, entire function $f_0(z)$ constructed by rule (14) satisfies (15) and is an extremal one in problem (4) as $\theta \in (0,\pi/2]$.

The case $\alpha = \beta$ is technically much simpler than the previous one, but it has certain features. In accordance with (5), the type of an entire function of order $\rho \in (0,1)$ satisfies the inequality

$$\sigma_\rho(f) \geqslant \frac{\pi\beta}{\sin\pi\rho} \cos\rho\,\theta, \tag{28}$$

if the sequence of its zeroes $\Lambda = \Lambda_f$ lies in the angle

$$\Gamma_\theta = \{z \in \mathbb{C} : |\arg z| \leqslant \theta\}$$

with $\theta \in (0, \pi/2]$ and has ρ-density

$$\overline{\Delta}_\rho(\Lambda) = \underline{\Delta}_\rho(\Lambda) = \Delta_\rho(\Lambda) = \lim_{n\to\infty} \frac{n}{|\lambda_n|^\rho} = \beta. \tag{29}$$

The excluded value $\theta = 0$ is not interesting in view of extremal problem (4) as $\alpha = \beta$, since, as it is known, the type of entire function of order $\rho \in (0, 1)$, whose zeroes are located at a single ray and are measurable with ρ-density β, can be always found by the simple formula

$$\sigma_\rho(f) = \frac{\pi\beta}{\sin \pi\rho}.$$

However, the situation becomes more complicated, when in restriction (3) for the location of zeroes the opening of the angle is positive.

Let us show that estimate (28) is sharp. In order to do it, we choose a measurable sequence $(r_j)_{j=1}^\infty \subset \mathbb{R}_+$ with ρ-density $\beta/2$ and we let

$$\Lambda_0 = \left(r_j\, e^{-i\theta}\right)_{j=1}^\infty \bigcup \left(r_j\, e^{i\theta}\right)_{j=1}^\infty, \quad f_0(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{\lambda_n}\right), \qquad \lambda_n \in \Lambda_0.$$

Sequence Λ_0 is located symmetrically on the sides of angle Γ_θ and has ρ-density $\Delta_\rho(\Lambda_0) = \beta$ obeying (29), while $f_0(z)$ satisfies representation (16). Arguing in the standard way, we fix $\varepsilon > 0$ and choose $t_0 = t_0(\varepsilon) > 0$ such that as $rt \geqslant t_0$, the relation

$$\varphi_{0,r}(t) = \frac{n_{\Lambda_0}(rt)}{(rt)^\rho} \leqslant \beta + \varepsilon$$

holds true. Then

$$r^{-\rho} \ln M_{f_0}(r) = \int_0^{+\infty} \varphi_{0,r}(t)\, K(t)\, dt =$$

$$= \int_{t_0/r}^{+\infty} \varphi_{0,r}(t)\, K(t)\, dt + \int_0^{t_0/r} \varphi_{0,r}(t)\, K(t)\, dt \leqslant (\beta + \varepsilon) \int_{t_0/r}^{+\infty} K(t)\, dt + o(1), \quad r \to +\infty.$$

Since $\varepsilon > 0$ is arbitrary, in view of formulae (2), (12) we obtain

$$\sigma_\rho(f_0) \leqslant \frac{\pi\beta}{\sin \pi\rho} \cos \rho\, \theta.$$

Thus, constructed function $f_0(z)$ makes (28) the identity. The proof of Theorem 1 is complete.

Let us discuss some nuances useful for understanding the matter. We first observe that function $f_0(z)$ provided in the concluding part of the proof of Theorem 1 has a rather regular growth. Let us describe a natural generalization of this example.

At the ray $\arg z = -\theta$ we choose an arbitrary measurable sequence Λ_1 with ρ-density $\beta/2$, while on ray $\arg z = \theta$ we choose an arbitrary measurable sequece Λ_2 with ρ-density $\beta/2$. Then sequence $\Lambda_0 \equiv \Lambda_1 \bigcup \Lambda_2$ possesses property (29). Let us check that function (14) constructed by such sequence Λ_0 has the type

$$\sigma_\rho(f_0) = \frac{\pi\beta}{\sin \pi\rho} \cos \rho\, \theta. \tag{30}$$

Function $f_0(z)$ is that of completely regular growth but now, generally speaking, its zeroes are not located symmetrically w.r.t. the real axis. In accordance with [5, Ch. II, Sect. 2], the *indicator* of $f_0(z)$ is calculated by the formula

$$h_\rho(f_0,\varphi) \equiv \overline{\lim_{r\to+\infty}}\, r^{-\rho}\ln|f_0(re^{i\varphi})| = \frac{\pi\beta}{2\sin\pi\rho}\,\left(h_\rho(\varphi+\theta)+h_\rho(\varphi-\theta)\right), \qquad 0\leqslant\varphi\leqslant 2\pi,$$

where $h_\rho(\varphi)$ stands for a 2π-periodic continuation of function $\cos\rho(\varphi-\pi)$ from $[0,2\pi]$ on $\mathbb{R}$. By straightforward calculations we get

$$h_\rho(f_0,\varphi) = \frac{\pi\beta}{\sin\pi\rho}\cdot\begin{cases}\cos\rho(\pi-\theta)\cdot\cos\rho\varphi, & 0\leqslant\varphi\leqslant\theta,\\ \cos\rho\theta\cdot\cos\rho(\varphi-\pi), & \theta\leqslant\varphi\leqslant 2\pi-\theta,\\ \cos\rho(\pi-\theta)\cdot\cos\rho(2\pi-\varphi), & 2\pi-\theta\leqslant\varphi\leqslant 2\pi.\end{cases}$$

Then by the inequality $\cos\rho\,\theta\geqslant\cos\rho(\pi-\theta)$ we obtain

$$\sigma_\rho(f_0) \;=\; \max_{0\leqslant\varphi<2\pi} h_\rho(f_0,\varphi) \;=\; h_\rho(f_0,\pi) \;=\; \frac{\pi\beta}{\sin\pi\rho}\cos\rho\,\theta,$$

that justifies (30).

On the other hand, even for functions of completely regular growth, the inequality in (28) can be strict. Indeed, let Λ consist of two measurable sequence, one of which has ρ-density $\beta_1\geqslant 0$ and is located at the ray $\arg z = -\theta$, while the other has ρ-density $\beta_2\geqslant 0$, $\beta_2\neq\beta_1$ and is located at the ray $\arg z=\theta$, and $\beta_1+\beta_2=\beta$. Identities (29) again hold true. By such sequence Λ we define canonical product (6). Excluding by the restriction $\beta_2\neq\beta_1$ the case of a “well” function $f_0(z)$, we still deal with function $f(z)$ of completely regular growth. In terms of notations in the formula for $h_\rho(f_0,\varphi)$, indicator$h_\rho(f,\varphi)$ is of the form [5, Ch. II, Sect. 2]

$$h_\rho(f,\varphi) = \frac{\pi}{\sin\pi\rho}\,\left(\beta_1\,h_\rho(\varphi+\theta)+\beta_2\,h_\rho(\varphi-\theta)\right), \qquad 0\leqslant\varphi\leqslant 2\pi.$$

By simple transformations we arrive at the extended writing:

$$h_\rho(f,\varphi) = \frac{\pi}{\sin\pi\rho}\cdot\begin{cases}A_{\pi-\theta}\,\cos\left(\rho\varphi-\varphi_{\pi-\theta}\right), & 0\leqslant\varphi\leqslant\theta,\\ A_\theta\,\cos\left(\rho(\varphi-\pi)-\varphi_\theta\right), & \theta\leqslant\varphi\leqslant 2\pi-\theta,\\ A_{\pi-\theta}\,\cos\left(\rho(2\pi-\varphi)-\varphi_{\pi-\theta}\right), & 2\pi-\theta\leqslant\varphi\leqslant 2\pi,\end{cases}$$

where for simplicity we denote

$$A_\theta \equiv \sqrt{\beta^2\cos^2\rho\,\theta \,+\, (\beta_2-\beta_1)^2\sin^2\rho\,\theta}, \qquad \varphi_\theta \equiv \arctan\left(\frac{\beta_2-\beta_1}{\beta}\tan\rho\,\theta\right).$$

Due to the restrictions for the parameters, the inequalities

$$A_\theta > \beta\cos\rho\,\theta, \qquad |\varphi_\theta|\leqslant\rho\,\theta$$

hold true. We take $\varphi^*\equiv\pi+\varphi_\theta/\rho$. Then $\theta\leqslant\pi-\theta\leqslant\varphi^*\leqslant\pi+\theta\leqslant 2\pi-\theta$. Substituting φ^* into the expression of the indicator, we obtain

$$\sigma_\rho(f)\geqslant h_\rho(f,\varphi^*) = \frac{\pi}{\sin\pi\rho}\,A_\theta > \frac{\pi\beta}{\sin\pi\rho}\cos\rho\,\theta = \sigma_\rho(f_0).$$

Thus, in contrast to $f_0(z)$, function $f(z)$ is not extremal.

4. Uniqueness theorems and the completeness of exponentials systems

The main result of the paper allows us to obtain new uniqueness theorems for entire functions and theorems on completeness of exponentials systems. Such applications of Theorem 1 in the case $\theta = 0$ are given in work [2]; a detailed study of the general situation $\theta \in [0, \pi/2]$ requires a separate publication. We briefly dwell on some applications. For instance, a natural extension of the result by B.N. Khabibullin [7, Thm. 4] is the following statement.

Theorem 2. *Let $\rho \in (0,1)$ and let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers with a finite upper ρ-density $\beta > 0$ and with a lower ρ-density $\geqslant \alpha \in [0, \beta]$ located in some angle of opening $2\theta \leqslant \pi$. If the type at order ρ of an entire function f vanishing at Λ is less than*

$$\frac{2^{\rho}\sqrt{\pi}\,\Gamma(1-\rho/2)}{\Gamma((1-\rho)/2)}\, s_{\theta}(\alpha,\, \beta;\, \rho) \;=\; \frac{\sin \pi\rho}{\pi}\,\Gamma(\rho)\,\Gamma^2(1-\rho/2)\; s_{\theta}(\alpha,\, \beta;\, \rho),$$

where $s_{\theta}(\alpha,\, \beta;\, \rho)$ is introduced in Theorem 1, then $f \equiv 0$ on $\mathbb{C}$.

The formulation of Theorem 2 involves Euler's Γ-function. The proof can be obtained by direct combination of Theorem 4 in [7] and by our Theorem 1.

Now we provide a corollary of Theorem 1 concerning even entire functions of exponential type, which play an important role in various branches of complex analysis, for instance, in the theory of Dirichlet series (see [8]).

Theorem 3. *Let $\beta > 0$, $\alpha \in [0, \beta]$, $\theta \in [0, \pi/4]$, and let*

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right), \qquad |\arg \lambda_n| \leqslant \theta,$$

and

$$\overline{\lim_{n\to\infty}}\, \frac{n}{|\lambda_n|} = \beta, \qquad \underline{\lim_{n\to\infty}}\, \frac{n}{|\lambda_n|} \geqslant \alpha.$$

Then the exponential type

$$\sigma(F) \equiv \overline{\lim_{r\to+\infty}}\, r^{-1} \ln \max_{|z|=r} |F(z)|$$

of function $F(z)$ satisfies the exact inequality

$$\sigma(F) \geqslant s_{2\theta}(\alpha, \beta; 1/2), \tag{31}$$

where the quantity in the right hand side

$$s_{2\theta}(\alpha, \beta; 1/2) = \pi\alpha\, \cos\theta \,+\, \max_{a>0} \int\limits_{a(\alpha/\beta)^2}^{a} \left(\frac{\beta}{\sqrt{a}} - \frac{\alpha}{\sqrt{x}}\right) \frac{x + \cos 2\theta}{x^2 + 2x\cos 2\theta + 1}\, dx \tag{32}$$

comes from Theorem 1.

In order to prove, it is sufficient to consider the entire function

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\mu_n}\right), \qquad \mu_n = \lambda_n^2,$$

of order $\rho = 1/2$ with zeroes

$$(\mu_n)_{n=1}^{\infty} \subset \Gamma_{2\theta} = \{z \in \mathbb{C} : |\arg z| \leqslant 2\theta\}\,, \qquad 2\theta \in [0, \pi/2],$$

to take into consideration that

$$\overline{\lim_{n\to\infty}}\, \frac{n}{|\mu_n|^{1/2}} = \beta, \qquad \underline{\lim_{n\to\infty}}\, \frac{n}{|\mu_n|^{1/2}} \geqslant \alpha, \qquad \sigma_{1/2}(f) = \sigma(F),$$

and to apply Theorem 1.

Without using the lower density of zeroes ($\alpha = 0$), estimate (31) becomes

$$\sigma(F) \geqslant \frac{\beta}{2} \max_{a>0} \frac{1}{\sqrt{a}} \ln\left(a^2 + 2a\cos 2\theta + 1\right).$$

If the sequence of zeroes of $F(z)$ has the density ($\alpha = \beta$), then (31) becomes the estimate

$$\sigma(F) \geqslant \pi\beta \cos\theta.$$

All the estimates are sharp. The integral (32) can be calculated in terms of elementary functions also in the case $\alpha \in (0, \beta)$, but the final expression is so bulky that it is not reasonable to provide it here.

Theorems 2, 3 imply immediately the following statement.

Theorem 4. *Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers with a finite upper density $\beta > 0$ and with lower density $\geqslant \alpha \in [0, \beta]$ such that $|\arg \lambda_n| \leqslant \theta$, where $\theta \in [0, \pi/4]$. Let entire function F vanishes at set $\pm\Lambda$ and its exponential type less than*

$$\frac{\Gamma^2(3/4)}{\sqrt{\pi}}\, s_{2\theta}(\alpha, \beta; 1/2),$$

where $s_{2\theta}(\alpha, \beta; 1/2)$ is given by formula (32), and the scalar coefficient $\Gamma^2(3/4)/\sqrt{\pi}$ is equal to $0.8472\ldots$. Then $F \equiv 0$ on $\mathbb{C}$.

In order to show the possibility of application of Theorem 4 to the exponential approximation in a complex domain, we recall some definitions. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ and $\Lambda(\lambda)$ denotes the number of appearance of point λ in sequence Λ. We say that the system of (multiple) exponentials

$$E_\Lambda \equiv \left\{z^{n-1}e^{\lambda z} : \lambda \in \Lambda,\ n = 1, 2, \ldots, \Lambda(\lambda)\right\}, \qquad z \in \mathbb{C},$$

is complete in a circle

$$K_R \equiv \{z \in \mathbb{C} : |z| < R\}, \qquad R > 0,$$

if it is complete in space $A(K_R)$ of functions analytic in this circle equipped with the topology of uniform convergence on compact sets in K_R. The symbol $R(\Lambda)$ denotes the *completeness circle radius* of sequence Λ, i.e., the supremum of all radii of circles K_R, in which system E_Λ is dense. We denote by $\sigma_{inf}(\Lambda)$ the infimum of values $\sigma > 0$, for which there exists an entire function $F \not\equiv 0$ of exponential type $\leqslant \sigma$ such that F vanishes at Λ (counting multiplicities): $F(\Lambda) = 0$. In accordance with the well-known criteria of the completeness of system E_Λ in space $A(K_R)$ (see, for instance, [9, Sect. 3.3.1]) the identity

$$\sigma_{inf}(\Lambda) = R(\Lambda)$$

holds true.

For fixed $\beta > 0$, $0 \leqslant \alpha \leqslant \beta$, $0 \leqslant \theta \leqslant \pi/4$ we introduce the class $P_\theta(\alpha, \beta)$, consisting of all possible sequences $\Lambda = (\lambda_n)_{n=1}^{\infty}$ of complex numbers of upper density $\beta > 0$ and lower density $\geqslant \alpha \in [0, \beta]$ such that $|\arg \lambda_n| \leqslant \theta$. We let

$$R_\theta(\alpha, \beta) \equiv \inf_{\Lambda \in P_\theta(\alpha, \beta)} R(\pm\Lambda). \tag{33}$$

Our aim is to estimate characteristics $R_\theta(\alpha, \beta)$ as exact as possible. Simply saying, we need to find with a good sharpness the radius of the maximal circle in which each system of exponentials with exponents generated by a sequence $\pm\Lambda$ in class $P_\theta(\alpha, \beta)$ is complete.

At present, the best known estimates for $R_\theta(\alpha, \beta)$ can be obtained by combining Theorem 4 with classical inequality (see. [10, Sect. 2.5])

$$\sigma(F) \geqslant \beta \exp\left\{\frac{\alpha}{\beta} - 1\right\}.$$

This inequality is valid for the exponential type of each entire function F with the sequence of zeroes having upper density β and lower density $\geqslant \alpha$.

Theorem 5. *Suppose that we are given numbers $\beta > 0$, $0 \leqslant \alpha \leqslant \beta$, $0 \leqslant \theta \leqslant \pi/4$, and quantity $s_{2\theta}(\alpha, \beta; 1/2)$ is calculated by rule (32). Then for the extremal completeness radius $R_\theta(\alpha,\, \beta)$ defined by formula (33) the two-sided estimate*

$$\max\left\{\frac{\Gamma^2(3/4)}{\sqrt{\pi}}\, s_{2\theta}(\alpha, \beta; 1/2);\;\; 2\beta\, e^{\alpha/\beta - 1}\right\} \leqslant R_\theta(\alpha,\, \beta) \leqslant s_{2\theta}(\alpha, \beta; 1/2)$$

holds true. For instance, for the system of exponentials with measurable exponents we have

$$\beta \max\left\{\Gamma^2(3/4)\sqrt{\pi}\cos\theta;\;\; 2\right\} \leqslant R_\theta(\beta,\, \beta) \leqslant \pi\beta\cos\theta,$$

where scalar coefficient $\Gamma^2(3/4)\sqrt{\pi} = 2.6614\ldots$. In particular,

$$2.6614\ldots\,\beta \leqslant R_0(\beta,\, \beta) \leqslant \pi\beta.$$

In conclusion we observe that Theorem 3 can be extended for the functions invariant w.r.t. the rotation by the angle $2\pi/s$, where $s = 3, 4, \ldots$, following work [7]. The same remark is true for other results of Section 4 of the present paper.

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