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ON FRECHÈT DIFFERENTIABILITY OF COST FUNCTIONAL IN OPTIMAL CONTROL OF COEFFICIENTS OF ELLIPTIC EQUATIONS

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Abstract. In the work we consider nonlinear optimal control problems for semi-linear elliptic equations with discontinuous data and solutions (states) with controls in the boundary conditions of conjugation of heterogeneous media and in the right hand side of the state equation. We prove the differentiability and Lipshitz continuity for the grid analogue of the cost functional for extremum problems.

Keywords: optimal control problem, semi-linear elliptic equations, cost functional, differentiability, Lipschitz continuity.

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1. Introduction

In the present work we consider the problem on optimal control of processes described by elliptic equations in heterogeneous anisotropic media with discontinuous coefficients and solutions (states) subject to the boundary interface conditions of non-ideal contact type. The problems for equations of mathematical physics (EMP) with non-ideal contact conditions often arise in modeling various processes in the mechanics of continua, the theory of elasticity, heat-conducting, diffusion. The discontinuity of the coefficients and solution occurs when the media is heterogeneous and consists in several parts with different properties or when the domain contains thin layers S with physical properties sharply different from the main media (see [1]-[3]). Assuming such layers S very thin and weakly penetrable, their influence on the studied physical process, i.e., the contact conditions, can be described by the relations (see, for instance, [1]):

$$p(x) = \left(\frac{\partial u}{\partial N_S}\right)^- = \left(\frac{\partial u}{\partial N_S}\right)^+ = \theta(x)[u], \quad x \in S,$$

$$\left(\frac{\partial u}{\partial N_S}\right)^\pm = \left(\sum_{\alpha=1}^{2} k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\cos(n, x_\alpha)\right)^\pm,$$

where $[u] = u^+(x) - u^-(x)$ is the jump of function $u(x)$ on S, $p(x)$ is an unknown flow of the matter (heat) through an elementary area, $\theta(x) \geqslant \theta_0 > 0$ is a given function, $S = \overline{\Omega}^- \cap \overline{\Omega}^+$ is interface between the media, $\Omega^- \cap \Omega^+ = \emptyset$, Ω^- and Ω^+ are some domains such that $\Omega = \Omega^- \cup \Omega^+ \cup S$ is a bounded domain.

In the most cases, mathematical optimizations of the processes can not be solved analytically and they require application of numerical methods and their computer realizations. The

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numerical solving of optimal control problems (NSOCP) is in a wide sense related with studying the following issues:

1. The formulation of optimization problem ensuring the existence of solution on the set of admissible controls being a set of some infinite-dimensional vector space;
2. The reduction of optimal control problem to a sequence of finite-dimensional problem guaranteeing the convergence in some sense of solutions to the finite-dimensional problems to the solutions of original optimal control problems;
3. Numerical solving of the finite-dimensional problems.

The problems for EMF with discontinuous coefficients and solution are not well studied (see the survey in [4]). Important results for optimal control problems described by nonlinear EMP with discontinuous coefficients and solutions were obtained in works [4]-[6], where there were developed new methods for studying optimal control problems described by nonlinear EMP with discontinuous coefficients and solutions; the methods were based on constructing and studying difference approximations of extremal problems, establishing the estimates for the approximation accuracy w.r.t. the state and functional, and on the regularization of approximations.

The present work is a natural continuation of [4]-[6]. Here we study nonlinear optimal control problems described by semi-linear elliptic coefficients with discontinuous coefficients and solutions (states) with interface boundary condition of non-ideal contact type. As the control, the coefficients in the boundary condition of the interface between heterogeneous media and the right hand side of the state equation serve. The work is aimed on solving the third step of NSOCP, namely, on developing effective numerical methods of solving constructed finite-dimensional of grip optimal control problems. We note that such issues were not considered before. For the numerical realization of finite dimensional optimal control we prove the differentiability and Lipschitz continuity for the grid functional of the approximating grid problems. We obtain effective procedures of calculating the gradient of minimized grid functionals employing solutions of direct problems and associated auxiliary adjoint problems.

In thermal terms, the formulated problems can be interpreted as optimal control problems of controlling the coefficient in the boundary condition of the interface between two different heat-conducting media $\theta(x)$ and coefficients $f_1(x)$ and $f_2(x)$ characterizing the presence in media Ω_1 and Ω_2 of internal heat sources, respectively, by which the heat can be released or absorbed inside the media. At that, the coefficient in the interface boundary condition characterizes the thermal resistance of the non-ideal contact of heterogeneous media [1], [3].

2. Formulation of problem

Let

$$\Omega = \{r = (r_1, r_2) \in \mathbb{R}^2 : 0 \leqslant r_\alpha \leqslant l_\alpha, \alpha = 1, 2\}$$

be a rectangle in $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma$. Suppose that domain Ω is splitted by an "internal contact boundary" $\overline{S} = \{r_1 = \xi,\ 0 \leqslant r_2 \leqslant l_2\}$, where $0 < \xi < l_1$, into subdomains $\Omega_1 \equiv \Omega^- = \{0 < r_1 < \xi, \quad 0 < r_2 < l_2\}$ and $\Omega_2 \equiv \Omega^+ = \{\xi < r_1 < l_1, \quad 0 < r_2 < l_2\}$ with boundaries $\partial\Omega_1 \equiv \partial\Omega^-$ and $\partial\Omega_2 \equiv \partial\Omega^+$. Thus, $\Omega = \Omega_1 \cup \Omega_2 \cup \overline{S}$ and $\partial\Omega$ is the internal boundary of domain Ω. By $\overline{\Gamma}_k$ we denote the boundaries of domains Ω_k without S, $k = 1, 2$. Thus, $\partial\Omega_k = \overline{\Gamma}_k \cup S$, where parts Γ_k, $k = 1, 2$ are open non-empty sets in $\partial\Omega_k$, $k = 1, 2$; $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega = \Gamma$. By n_α, $\alpha = 1, 2$ we denote the outward normal to boundary $\partial\Omega_\alpha$ of domain Ω_α, $\alpha = 1, 2$. Let $n = n(x)$ be the unit normal to S at some point $x \in S$ oriented, for instance, so that normal n is the outward one for S w.r.t. domain Ω_1, i.e., normal n is directed inside domain Ω_2. In what follows, while formulating boundary value problems for the states of control processes, S is a straight line along which the coefficients and solutions of boundary value problems are discontinuous, while in domains Ω_1 and Ω_2 they possess certain smoothness.

We consider the following Dirichlet problem for the semi-linear elliptic equation with discontinuous coefficients and solutions:

Find function $u(x)$ defined on $\overline{\Omega}$ of the form $u(x) = u_1(x)$, $x \in \overline{\Omega}_1$, $u(x) = u_2(x)$, $x \in \overline{\Omega}_2$, where components u_k, $k = 1, 2$, satisfy the conditions:

1) functions $u_k(x)$, $k = 1, 2$, defined on $\overline{\Omega}_k = \Omega_k \cup \partial\Omega_k$, $k = 1, 2$, satisfy the equations

$$-\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) + d(x) q(u) = f(x), \quad x \in \Omega_1 \cup \Omega_2, \tag{1a}$$

in Ω_k, $k = 1, 2$, while on boundaries $\partial\Omega_k \setminus S = \overline{\Gamma}_k$ the conditions

$$u(x) = 0, \quad x \in \partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2, \tag{1b}$$

are imposed;

2) Functions $u_k(x)$, $k = 1, 2$, satisfy additional coefficients on the discontinuity line S for the coefficients and solution so that these conditions allow us to "glue" solutions $u_1(x)$ and $u_2(x)$ along interface boundary S of domains Ω_1 and Ω_2:

$$k_1^{(1)}(x) \frac{\partial u_1}{\partial x_1} = k_1^{(2)}(x) \frac{\partial u_2}{\partial x_1} = \theta(x_2) \left(u_2(x) - u_1(x) \right), \quad x \in S, \tag{1c}$$

where

$$u(x) = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2, \end{cases} \qquad q(\xi) = \begin{cases} q_1(\xi_1), & \xi_1 \in \mathbb{R}, \\ q_2(\xi_2), & \xi_2 \in \mathbb{R}, \end{cases}$$

$$k_\alpha(x), d(x), f(x) = \begin{cases} k_\alpha^{(1)}(x), d_1(x), f_1(x), & x \in \Omega_1, \\ k_\alpha^{(2)}(x), d_2(x), f_2(x), & x \in \Omega_2, \quad \alpha = 1, 2. \end{cases}$$

Here $\left[u\right] = u_2(x) - u_1(x)$ is the jump of function $u(x)$ at S, $k_\alpha(x)$, $\alpha = 1, 2$, $d(x)$ are given functions defined independently in Ω_1 and Ω_2 and having a first kind jump at S, $q_\alpha(\xi_\alpha)$, $\alpha = 1, 2$, are given functions defined for $\xi_\alpha \in \mathbb{R}$, $\alpha = 1, 2$, $g(x) = (f_1(x), f_2(x), \theta(x))$ is a control. For the given functions we assume that $k_\alpha(x) \in W_\infty^1(\Omega_1) \times W_\infty^1(\Omega_2)$, $\alpha = 1, 2$, $d(x) \in L_\infty(\Omega_1) \times L_\infty(\Omega_2)$, $0 < \nu \leqslant k_\alpha(x) \leqslant \overline{\nu}$, $\alpha = 1, 2$, $0 \leqslant d_0 \leqslant d(x) \leqslant \overline{d}_0$, $x \in \Omega_1 \cup \Omega_2$, $\nu, \overline{\nu}, d_0, \overline{d}_0$ are given constants, functions $q_\alpha(\xi_\alpha)$ defined on $\mathbb{R}$ with values on $\mathbb{R}$ satisfy the conditions:

$$q_\alpha(0) = 0, \quad 0 < q_0 \leqslant \big(q_\alpha(\xi_\alpha) - q_\alpha(\overline{\xi}_\alpha) \big) / \big(\xi_\alpha - \overline{\xi}_\alpha \big) \leqslant L_q < \infty,$$

for all ξ_α, $\overline{\xi}_\alpha \in \mathbb{R}$, $\xi_\alpha \neq \overline{\xi}_\alpha$, $\alpha = 1, 2$, $L_q = \mathrm{Const}$.

We introduce the set of admissible controls:

$$\begin{aligned} &U = \prod_{\alpha=1}^{3} U_\alpha \subset H = L_2(\Omega_1) \times L_2(\Omega_2) \times L_2(S), \\ &U_\alpha = \big\{ g_\alpha(x) = f_\alpha(x) \in L_2(\Omega_\alpha) : \ \varrho_\alpha \leqslant f_\alpha(x) \leqslant \overline{\varrho}_\alpha \text{ a.e. on } \Omega_\alpha \big\}, \\ &\alpha = 1, 2; \quad U_3 = \big\{ g_3(x) = \theta(x) \in L_2(S) : \ 0 < \varrho_3 \leqslant \theta(x) \leqslant \overline{\varrho}_3 \text{ a.e. on } S \big\}, \end{aligned} \tag{2}$$

where ϱ_α, $\overline{\varrho}_\alpha$, $\alpha = \overline{1, 3}$, are given numbers.

We introduce the cost functional $J : U \to \mathbb{R}^1$ as

$$g \to J(g) = \int_{\Omega_1} \left| u(r_1, r_2; g) - u_0^{(1)}(r) \right|^2 d\Omega_1 = I(u(r; g)), \tag{3}$$

where $u_0^{(1)} \in W_2^1(\Omega_1)$ is a given function.

The problem of optimal control is to find a control $g_* \in U$ that minimizes the functional $g \to J(g)$ on set $U \subset H$, more precisely, we need to minimize functional (3) on solutions $u(r) = u(r; g)$ to problem (1) associated with all admissible controls $g = \big(f_1, f_2, \theta \big) \in U$.

In what follows we shall need some spaces introduced in work [6]. We provide their definitions for the completeness of the exposition. In particular, we consider the space $V(\Omega^{(1,2)})$, $\Omega^{(1,2)} = \Omega_1 \cup \Omega_2$, of pairs of functions $u = (u_1, u_2)$: $V(\Omega^{(1,2)}) = \big\{ u = (u_1, u_2) \in W_2^1(\Omega_1) \times W_2^1(\Omega_2) \big\}$, where $W_2^1(\Omega_k)$, $k=1,2$, are Sobolev spaces of functions defined in subdomains Ω_k, with boundaries $\partial\Omega_k$, $k=1,2$, respectively, and with the norms [7]-[9]:

$$\|u_k\|^2_{W_2^1(\Omega_k)} = \int\limits_{\Omega_k}\left[\sum_{\alpha=1}^{2}\left(\frac{\partial u_k}{\partial x_\alpha}\right)^2 + u_k^2\right]d\Omega_k, \quad k=1,2.$$

Space $V = V(\Omega^{(1,2)})$ equipped with scalar product and norm $(u,\vartheta)_V = \sum\limits_{k=1}^{2}(u_k,\vartheta_k)_{W_2^1(\Omega_k)}$, $\|u\|_V^2 = \sum\limits_{k=1}^{2}\|u_k\|^2_{W_2^1(\Omega_k)}$, is a Hilbert space.

In Hilbert space $V(\Omega^{(1,2)})$ we can introduce an equivalent norm

$$\|u\|_*^2 = \sum_{k=1}^{2}\int\limits_{\Omega_k}\sum_{\alpha=1}^{2}\left(\frac{\partial u_k}{\partial x_\alpha}\right)^2 d\Omega_k + \sum_{k=1}^{2}\int\limits_{\Gamma_k} u_k^2\,d\Gamma_k + \int\limits_{S}[u]^2\,dS,$$

where $\big[u\big] = u_2(x)-u_1(x) = u^+(x)-u^-(x)$ is a jump of function $u(x)$ on S. Here $u_2(x)=u^+(x)$, $x\in S$, and $u_1(x)=u^-(x)$, $x\in S$, are the traces of function $u(x)$ on S while approaching S in $\Omega_2=\Omega^+$ and $\Omega_1=\Omega^-$, respectively. We note that the condition $u(x)\in V(\Omega^{(1,2)})$ implies that $[u(x)]\in L_2(S)$, since in this case the theorem on traces [7]-[9] is valid for each side S^+, S^- of the boundary of contact S (the restriction operator is continuous as that from $W_2^1(\Omega^{\pm})$ into $L_2(S)$). Theorems on traces to Ω_1 and Ω_2 allow us to define two traces for each function $u(x)\in V(\Omega^{(1,2)})$ by the operators of restriction on $S^{\pm}$, i.e., from opposite sides (while approaching in Ω_1 and in Ω_2), which in general case are different.

Let $\mathring{\Gamma}_k$ be a part of $\partial\Omega_k$. By $W_2^1\big(\Omega_k;\mathring{\Gamma}_k\big)$ we denote a closed subspace of space $W_2^1(\Omega_k)$, in which a dense set is that of functions in $C^1(\overline{\Omega}_k)$ vanishing in the vicinity of $\mathring{\Gamma}_k\subset\partial\Omega_k$, $k=1,2$, which is a part $\mathring{\Gamma}_k$ of boundary $\partial\Omega_k$, $k=1,2$.

We introduce the space $\mathring{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$ of pairs of functions $u=(u_1,u_2)$:

$$\mathring{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)}) = \big\{u=(u_1,u_2)\in W_2^1(\Omega_1;\Gamma_1)\times W_2^1(\Omega_2;\Gamma_2)\big\}$$

with the norm (see [4]):

$$\|u\|^2_{\mathring{V}_{\Gamma_1,\Gamma_2}} = \sum_{k=1}^{2}\int\limits_{\Omega_k}\sum_{\alpha=1}^{2}\left(\frac{\partial u_k}{\partial x_\alpha}\right)^2 d\Omega_k + \int\limits_{S}[u]^2\,dS.$$

For each fixed control $g=\big(f_1,f_2,\theta\big)\in U$, a solution to direct problem (1) is a function $u(g)\in\mathring{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$ satisfying the identity

$$\begin{aligned} Q(u,\vartheta) = \int\limits_{\Omega_1\cup\Omega_2}\left[\sum_{\alpha=1}^{2}k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\frac{\partial\vartheta}{\partial x_\alpha} + d(x)\,q(u)\,\vartheta\right]d\Omega_0 + \int\limits_{S}\theta(x)[u][\vartheta]dS \\ = \int\limits_{\Omega_1\cup\Omega_2} f(x)\,\vartheta d\Omega_0 = l(\vartheta), \quad \text{for all } \vartheta\in\mathring{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)}). \end{aligned} \tag{4}$$

Remark 1. *In what follows we make the following assumption on the smoothness of the direct problem (which is similar to the assumption made in* [5] *in studying difference scheme for the problems with the same interface conditions): a solution to boundary value problem* (1) *belongs to* $W_2^2(\Omega_1)\times W_2^2(\Omega_2)$*, more precisely, it belongs to space*

$$\mathring{\hat{V}}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)}) = \mathring{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})\cap\big\{u=(u_1,u_2)\in W_2^2(\Omega_1)\times W_2^2(\Omega_2)\big\},$$

and for each fixed control $g \in U$ the estimate

$$\sum_{k=1}^{2} \|u_k(x,g)\|_{W_2^2(\Omega_k)} \leqslant M \sum_{k=1}^{2} \|f_k(x)\|_{L_2(\Omega_k)}, \quad \forall g \in U, \quad \text{where } M = Const > 0.$$

hold true.

Remark 2. *Hereinafter, by M, $\widetilde{M}$, M_0, C, C_0, $\widetilde{C}_0$, C_k, $k = \overline{1,3}$ we denote various positive constants independent of solution $u(r;g)$ and control $g \in U$ (grid solution $y(x;\Phi_h)$, grid control $\Phi_h \in U_h$).*

3. Difference approximation of optimization problems

For numerical solving of optimal control problems we consider the problem on approximations of infinite-dimensional optimization problems (1)-(3) by a sequence of finite-dimensional optimal control problems. In what follows we shall construct the approximations of the problems by the grid method (see [1]). In order to approximate problems (1)-(3), we shall need some grids on $[0, l_\alpha]$, $\alpha = 1, 2$, and in $\overline{\Omega}$. We note we can always construct a grid on $[0, l_1]$ so that the point $x_1 = \xi$ is its node. While solving practical problems, it is reasonable to find uniform steps $h_1^{(1)}$ and $h_1^{(2)}$ in domains $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively, and subject to the location of the point $x_1 = \xi$, the number of nodes should be found by the assumption $h_1^{(1)} \approx h_1^{(2)}$. We let $x_1^{(i_1)} - x_1^{(i_1-1)} = h_1$, $i_1 = \overline{1, N_1}$ and $x_2^{(i_2)} - x_2^{(i_2-1)} = h_2$, $i_2 = \overline{1, N_2}$. The value of x_1 at the point $x_1 = \xi$ is denoted by x_ξ and the corresponding index of the node is denoted by $N_{1\xi}$, $1 < N_{1\xi} < N_1 - 1$. We introduce the grids:

$$\begin{aligned}
&\overline{\omega}_1^{(1)} = \big\{x_1^{(i_1)} = i_1 h_1 \in [0,\xi] : \ i_1 = \overline{0, N_{1\xi}}, \ N_{1\xi} h_1 = \xi\big\},\\
&\overline{\omega}_1^{(2)} = \big\{x_1^{(i_1)} = i_1 h_1 \in [\xi, l_1] : \ i_1 = \overline{N_{1\xi}, N_1}, \ N_{1\xi} h_1 = l_1\big\},\\
&\omega_1^{(1)} = \overline{\omega}_1^{(1)} \setminus \{x_1 = 0, x_1 = \xi\}, \quad \omega_1^{(2)} = \overline{\omega}_1^{(2)} \setminus \{x_1 = \xi, x_1 = l_1\};\\
&\overline{\omega}_2 = \{x_2^{(i_2)} = i_2 h_2 \in [0, l_2] : \ i_2 = \overline{0, N_2}, N_2 h_2 = l_2\}, \quad \omega_2 = \overline{\omega}_2 \setminus \{x_2 = 0, x_2 = l_2\};\\
&\overline{\omega}_1 = \overline{\omega}_1^{(1)} \cup \overline{\omega}_1^{(2)}; \quad \omega_1 = \omega_1^{(1)} \cup \omega_1^{(2)}; \quad \overline{\omega}^{(1)} = \overline{\omega}_1^{(1)} \times \overline{\omega}_2; \quad \overline{\omega}^{(2)} = \overline{\omega}_1^{(2)} \times \overline{\omega}_2;\\
&\omega^{(1)} = \omega_1^{(1)} \times \omega_2; \quad \omega^{(2)} = \omega_1^{(2)} \times \omega_2;\\
&\overline{\omega} \equiv \overline{\omega}^{(1,2)} = \overline{\omega}^{(1)} \cup \overline{\omega}^{(2)} = \big(\overline{\omega}_1^{(1)} \cup \overline{\omega}_1^{(2)}\big) \times \overline{\omega}_2\\
&\quad = \big\{x_1^{(i_1)} = i_1 h_1, \ i_1 = \overline{0, N_1}, \ N_{1\xi} h_1 = \xi, \ (N_1 - N_{1\xi}) h_1 = l_1 - \xi, \ 1 < N_{1\xi} < N_1 - 1\big\} \times \overline{\omega}_2,\\
&\omega \equiv \omega^{(1,2)} = \omega^{(1)} \times \omega^{(2)}; \quad \omega_1^{(1)+} = \overline{\omega}_1^{(1)} \cap (0, \xi], \quad \omega_1^{(1)-} = \overline{\omega}_1^{(1)} \cap [0, \xi),\\
&\omega_1^{(2)-} = \overline{\omega}_1^{(2)} \cap [\xi, l_1), \quad \omega^{(1)(+1)} = \omega_1^{(1)+} \times \overline{\omega}_2;\\
&\gamma_S = \big\{x_1 = \xi, \ x_2 = h_2, 2h_2, \ldots, (N_2 - 1) h_2\big\} = \big\{x_1 = \xi, \ x_2^{(i_2)} = i_2 h_2, \quad i_2 = \overline{1, N_2 - 1}\big\};\\
&\gamma^{(k)} = \partial\omega^{(k)} \setminus \gamma_S; \quad \omega_1^{(1)+} \times \omega_2 = \omega^{(1)} \cup \gamma_S = \overline{\omega}^{(1)} \setminus \gamma^{(1)};
\end{aligned}$$

$\partial\omega^{(k)} = \overline{\omega}^{(k)} \setminus \omega^{(k)}$ is the set of boundary nodes of grid $\overline{\omega}^{(k)}$, $k = 1, 2$.

Let us introduce scalar products, norms and semi-norms of grid functions, which will be used in what follows (for a more detailed description see [4]). The set of grid functions $y_1(x)$ defined on the grid $\overline{\omega}^{(1)} = \overline{\omega}_1^{(1)} \times \overline{\omega}_2 \subset \overline{\Omega}_1 \equiv \overline{\Omega}^-$ is denoted by $H_h^{(1)}(\overline{\omega}^{(1)})$, while the set of grid functions $y_2(x)$ defined on the grid $\overline{\omega}^{(2)} = \overline{\omega}_1^{(2)} \times \overline{\omega}_2 \subset \overline{\Omega}_2 \equiv \overline{\Omega}^+$ is denoted by $H_h^{(2)}(\overline{\omega}^{(2)})$. The set $H_h^{(k)}(\overline{\omega}^{(k)})$, $k = 1, 2$, equipped with the scalar product and the norm

$$(y_k, \nu_k)_{L_2(\overline{\omega}^{(k)})} = \sum_{\overline{\omega}^{(k)}} y_k(x)\, \nu_k(x)\, \hbar_1 \hbar_2, \quad \|y_k\|_{L_2(\overline{\omega}^{(k)})} = (y_k, y_k)^{1/2}_{L_2(\overline{\omega}^{(k)})},$$

is denoted by $L_2(\overline{\omega}^{(k)})$, $k = 1, 2$. Here $\hbar_1 = \hbar_1(x_1)$ is the mean step of grids $\overline{\omega}_1^{(1)}$ and $\overline{\omega}_1^{(2)}$, while $\hbar_2 = \hbar_2(x_2)$ is the mean step of the grid $\overline{\omega}_2$, [1]. By $W_2^1(\overline{\omega}^{(1)})$ and $W_2^1(\overline{\omega}^{(2)})$ we denote the spaces of grid functions defined on grids $\overline{\omega}^{(1)}$ and $\overline{\omega}^{(2)}$, respectively, with the scalar products and the norm:

$$(y_k, \nu_k)_{W_2^1(\overline{\omega}^{(k)})} = \sum_{\omega_1^{(k)+} \times \overline{\omega}_2} y_{k\overline{x}_1} \nu_{k\overline{x}_1} h_1 \hbar_2 + \sum_{\overline{\omega}_1^{(k)} \times \omega_2^+} y_{k\overline{x}_2} \nu_{k\overline{x}_2} \hbar_1 h_2 + (y_k, \nu_k)_{L_2(\overline{\omega}^{(k)})},$$

$$\|y_k\|^2_{W_2^1(\overline{\omega}^{(k)})} = \|\nabla y_k\|^2 + \|y_k\|^2_{L_2(\overline{\omega}^{(k)})}, \quad k = 1, 2,$$

where

$$\|\nabla y_k\|^2 = \sum_{\omega_1^{(k)+} \times \overline{\omega}_2} y^2_{k\overline{x}_1} h_1 \hbar_2 + \sum_{\overline{\omega}_1^{(k)} \times \omega_2^+} y^2_{k\overline{x}_2} \hbar_1 h_2, \quad k = 1, 2.$$

We have introduced space $V(\overline{\omega}^{(1,2)})$ of the pairs of grid functions $y = (y_1, y_2)$ determined by the relation $V(\overline{\omega}^{(1,2)}) = \{y = (y_1, y_2) \in W_2^1(\overline{\omega}^{(1)}) \times W_2^1(\overline{\omega}^{(2)})\}$. Being equipped with the scalar product and the norm

$$(y, \nu)_{V(\overline{\omega}^{(1,2)})} = \sum_{k=1}^{2} (y_k, \nu_k)_{W_2^1(\overline{\omega}^{(k)})}, \quad \|y\|^2_{V(\overline{\omega}^{(1,2)})} = \sum_{k=1}^{2} \|y_k\|^2_{W_2^1(\overline{\omega}^{(k)})},$$

space $V(\overline{\omega}^{(1,2)})$ is a Hilbert one. Assume that $\gamma^{(k)} = \partial\omega^{(k)} \setminus \gamma_S$ is the subset of boundary nodes $\partial\omega^{(k)}$ of grid $\overline{\omega}^{(k)} \subset \overline{\Omega}_k$, $k = 1, 2$. By $L_2(\overline{\omega}^{(k)}; \gamma^{(k)})$ we denote the normed subspace of spaceo of grid functions $L_2(\overline{\omega}^{(k)})$ vanishing at $\gamma^{(k)}$, $k = 1, 2$, with the norms

$$\|y_k\|^2_{L_2(\overline{\omega}^{(k)}; \gamma^{(k)})} = \sum_{x \in \omega^{(k)}} y_k^2(x) h_1 h_2 + \frac{1}{2} \sum_{x \in \gamma_S} y_k^2(x) h_1 h_2$$
$$= \sum_{x \in \omega^{(k)}} y_k^2(x) h_1 h_2 + \frac{1}{2} \sum_{x_2 \in \omega_2} y_k^2(\xi, x_2) h_1 h_2, \quad k = 1, 2,$$

induced by the scalar products

$$(y_k, v_k)_{L_2(\overline{\omega}^{(k)}; \gamma^{(k)})} = \sum_{x \in \omega^{(k)}} y_k(x) v_k(x) h_1 h_2 + \frac{1}{2} \sum_{x \in \gamma_S} y_k(x) v_k(x) h_1 h_2, \quad k = 1, 2.$$

By $W_2^1(\overline{\omega}^{(k)}; \gamma^{(k)})$ we denote the subspace of space of grid functions $W_2^1(\overline{\omega}^{(k)})$ vanishing at $\gamma^{(k)}$, $k = 1, 2$. We introduce the spaces $\overset{\circ}{H}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$ and $\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$ of the pairs of grid functions $y = (y_1, y_2)$:

$$\overset{\circ}{H}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)}) = \left\{y = (y_1, y_2) \in L_2(\overline{\omega}^{(1)}; \gamma^{(1)}) \times L_2(\overline{\omega}^{(2)}; \gamma^{(2)})\right\},$$
$$\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)}) = \left\{y = (y_1, y_2) \in W_2^1(\overline{\omega}^{(1)}; \gamma^{(1)}) \times W_2^1(\overline{\omega}^{(2)}; \gamma^{(2)})\right\},$$

with the norms

$$\|y\|^2_{\overset{\circ}{H}_{\gamma^{(1)}\gamma^{(2)}}} = \sum_{k=1}^{2} \|y_k\|^2_{L_2(\overline{\omega}^{(k)}; \gamma^{(k)})}, \quad \|y\|^2_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}} = \|\nabla y_k\|^2 + \|[y]\|^2_{L_2(\gamma_S)},$$

where

$$\|y_k\|^2_{L_2(\gamma_S)} = \left(y_k, y_k\right)_{L_2(\gamma_S)}, \quad \left(y_k, \nu_k\right)_{L_2(\gamma_S)} = \sum_{x \in \gamma_S} h_2 y_k(x)\, \nu_k(x), \quad k = 1, 2.$$

By $H_h^{(1)}(\omega^{(1)} \cup \gamma_S) \equiv L_2(\omega^{(1)} \cup \gamma_S)$ we denote the space of grid functions $v_{1h}(x)$, $x \in \omega^{(1)} \cup \gamma_S$ defined on grid $\omega^{(1)} \cup \gamma_S$ with the scalar product and the norm

$$(v_{1h}, \tilde{v}_{1h})_{H_h^{(1)}(\omega^{(1)} \cup \gamma_S)} = \sum_{x \in \omega^{(k)}} v_{1h}(x) \tilde{v}_{1h}(x) h_1 h_2 + \frac{1}{2} \sum_{x \in \gamma_S} v_{1h}(x) \tilde{v}_{1h}(x) h_1 h_2,$$

$$\|v_{1h}(x)\|^2_{H_h^{(1)}(\omega^{(1)}\cup\gamma_S)} = (v_{1h}, v_{1h})_{H_h^{(1)}(\omega^{(1)}\cup\gamma_S)}.$$

In the same we introduce the space of grid functions $H_h^{(2)}(\omega^{(2)}\cup\gamma_S) \equiv L_2(\omega^{(2)}\cup\gamma_S)$ (see [4]).

We associate the following difference approximations with the optimal control problems (1)-(3): minimize grid functional

$$J_h(\Phi_h) = \sum_{x\in\overline{\omega}^{(1)}} \left|y(x,\Phi_h) - u_{0h}^{(1)}(x)\right|^2 \hbar_1\hbar_2 = \|y(\Phi_h) - u_{0h}^{(1)}\|^2_{L_2(\overline{\omega}^{(1)})}, \tag{5}$$

under the conditions that the grid function $y(\Phi_h) = (y_1(\Phi_h), y_2(\Phi_h)) \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$ called a solution to difference boundary value problem (difference scheme) for problem (1) satisfies the summatory identity

$$\begin{aligned}
Q_h(y,v) = &\Bigg\{ \sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)} y_{1\overline{x}_1} v_{1\overline{x}_1} h_1h_2 \\
&+ \Bigg(\sum_{\omega_1^{(1)}}\sum_{\omega_2^{+}} a_{2h}^{(1)} y_{1\overline{x}_2} v_{1\overline{x}_2} h_1h_2 + \frac{1}{2}\sum_{\omega_2^{+}} a_{2h}^{(1)}(\xi,x_2) y_{1\overline{x}_2}(\xi,x_2) v_{1\overline{x}_2}(\xi,x_2) h_1h_2 \Bigg)\Bigg\} \\
&+ \Bigg\{ \sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)} y_{2\overline{x}_1} v_{2\overline{x}_1} h_1h_2 \\
&+ \Bigg(\sum_{\omega_1^{(2)}}\sum_{\omega_2^{+}} a_{2h}^{(2)} y_{2\overline{x}_2} v_{2\overline{x}_2} h_1h_2 + \frac{1}{2}\sum_{\omega_2^{+}} a_{2h}^{(2)}(\xi,x_2) y_{2\overline{x}_2}(\xi,x_2) v_{2\overline{x}_2}(\xi,x_2) h_1h_2 \Bigg)\Bigg\} \\
&+ \sum_{\omega_2} \Phi_{3h}(x_2)\big[y(\xi,x_2)\big]\big[v(\xi,x_2)\big] h_2 \\
&+ \Bigg\{ \Bigg(\sum_{\omega^{(1)}} d_{1h}(x) q_1(y_1(x)) v_1(x) h_1h_2 + \frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2) q_1(y_1(\xi,x_2)) v_1(\xi,x_2) h_1h_2 \Bigg) \\
&+ \Bigg(\sum_{\omega^{(2)}} d_{2h}(x) q_2(y_2(x)) v_2(x) h_1h_2 + \frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2) q_2(y_2(\xi,x_2)) v_2(\xi,x_2) h_1h_2 \Bigg)\Bigg\} \\
= &\Bigg\{ \Bigg(\sum_{\omega^{(1)}} \Phi_{1h}(x) v_1(x) h_1h_2 + \frac{1}{2}\sum_{\omega_2} \Phi_{1h}(\xi,x_2) v_1(\xi,x_2) h_1h_2 \Bigg) \\
&+ \Bigg(\sum_{\omega^{(2)}} \Phi_{2h}(x) v_2(x) h_1h_2 + \frac{1}{2}\sum_{\omega_2} \Phi_{2h}(\xi,x_2) v_2(\xi,x_2) h_1h_2 \Bigg)\Bigg\} = l_h(v),
\end{aligned} \tag{6}$$

for each grid function $v(\Phi_h) = (v_1(\Phi_h), v_2(\Phi_h)) \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$, while grid controls $\Phi_h = \big(\Phi_{1h}, \Phi_{2h}, \Phi_{3h}\big)$ are such that

$$\begin{aligned}
&\Phi_h(x) \in U_h = \prod_{k=1}^{3} U_{kh} \subset H_h = L_2(\omega^{(1)}\cup\gamma_S) \times L_2(\omega^{(2)}\cup\gamma_S) \times L_2(\omega_2), \\
&U_{\alpha h} = \big\{\Phi_{\alpha h} \in L_2(\omega^{(\alpha)}\cup\gamma_S) : 0 < \varrho_\alpha \leqslant \Phi_{\alpha h}(x) \leqslant \overline{\varrho}_\alpha, \text{ a.e. in } \omega^{(\alpha)}\cup\gamma_S\big\}, \\
&\alpha = 1,2; \quad U_3 = \big\{\Phi_{3h}(x_2) \in L_2(\omega_2) : 0 < \varrho_3 \leqslant \Phi_{3h}(x) \leqslant \overline{\varrho}_3, \text{ a.e. in } \omega_2\big\},
\end{aligned} \tag{7}$$

where ϱ_k, $\overline{\varrho}_k$, $k = \overline{1,3}$ are given numbers.

Here $a_{\alpha h}^{(1)}(x)$, $a_{\alpha h}^{(2)}(x)$, $d_{\alpha h}(x)$, $\alpha = 1,2$, $u_{0h}^{(1)}(x)$ are the grid approximations of functions $k_\alpha^{(1)}(r)$, $k_\alpha^{(2)}(r)$, $d_\alpha(r)$, $\alpha = 1,2$, $u_0^{(1)}(r)$ defined via Steklov averages (see [6]).

Remark 3. *The proof of the well-posedness for optimal control problems* (1)-(3)*, the well-posedness of their difference approximations by grid optimal control problems (5)-(7), the* *convergence of the approximations by state, functional, control, the corresponding approximation estimate and regularizations of the approximations can be made by the methods in* [4]–[6].

Let us write explicitly difference scheme (6) at the nodes of the grid $\overline{\omega} = \overline{\omega}_1 \cup \overline{\omega}_2 = \overline{\omega}^{(1,2)}$: Find function $y = (y_1, y_2)$ defined on $\overline{\omega} = \overline{\omega}_1 \cup \overline{\omega}_2 = \overline{\omega}^{(1,2)}$, $y(x) = y_1(x)$ for $x \in \overline{\omega}^{(1)}$, $y(x) = y_2(x)$ for $x \in \overline{\omega}^{(2)}$, where components $y_1(x)$ and $y_2(x)$ satisfy the following conditions:

1) Grid function y_1 satisfies the equation

$$-\Big(a^{(1)}_{1h}(x) y_{1\overline{x}_1}\Big)_{x_1} - \Big(a^{(1)}_{2h}(x) y_{1\overline{x}_2}\Big)_{x_2} + d_{1h}(x) q_1(y_1) = \Phi_{1h}(x),\ x \in \omega^{(1)},$$

in $\omega^{(1)}$ and the condition $y_1(x) = 0$, $x \in \gamma^{(1)}$, on the boundary $\gamma^{(1)} = \partial\omega^{(1)} \setminus \gamma_S$.

2) Grid function y_2 satisfies the equation

$$-\Big(a^{(2)}_{1h}(x) y_{2\overline{x}_1}\Big)_{x_1} - \Big(a^{(2)}_{2h}(x) y_{2\overline{x}_2}\Big)_{x_2} + d_{2h}(x) q_2(y_2) = \Phi_{2h}(x),\ x \in \omega^{(2)},$$

in $\omega^{(2)}$ and the condition $y_2(x) = 0$, $x \in \gamma^{(2)}$, on the boundary $\gamma^{(2)} = \partial\omega^{(2)} \setminus \gamma_S$.

3) Functions y_1 and y_2 are related by additional conditions on $\gamma_S = \{x_1 = \xi, x_2 \in \omega_2\}$:

$$\begin{aligned}
&\frac{2}{h_1}\Big[a^{(1)}_{1h}(\xi, x_2) y_{1\overline{x}_1}(\xi, x_2) + \Phi_{3h}(x_2) y_1(\xi, x_2)\Big] + d_{1h}(\xi, x_2) q_1(y_1(\xi, x_2)) \\
&- \Big(a^{(1)}_{2h}(\xi, x_2) y_{1\overline{x}_2}(\xi, x_2)\Big)_{x_2} = \Phi_{1h}(\xi, x_2) + \frac{2}{h_1}\Phi_{3h}(x_2) y_2(\xi, x_2),\ x \in \gamma_S, \\
&- \frac{2}{h_1}\Big[a^{(2)}_{1h}(\xi + h_1, x_2) y_{2x_1}(\xi, x_2) - \Phi_{3h}(x_2) y_2(\xi, x_2)\Big] + d_{2h}(\xi, x_2) q_2(y_2(\xi, x_2)) \\
&- \Big(a^{(2)}_{2h}(\xi, x_2) y_{2\overline{x}_2}(\xi, x_2)\Big)_{x_2} = \Phi_{2h}(\xi, x_2) + \frac{2}{h_1}\Phi_{3h}(x_2) y_1(\xi, x_2),\ x \in \gamma_S.
\end{aligned}$$

4. Differentiability of grid functional $J_h(\Phi_h)$

For numerical realization [10] of finite-dimensional optimal control we first need to prove the differentiability and Lipschitz continuity of grid functional for approximating grid problems (5)-(7).

Let us show that functional $J_h(\Phi_h)$ is differentiable w.r.t. $\Phi_h = \big(\Phi_{1h}, \Phi_{2h}, \Phi_{3h}\big)$ on $U_{\alpha h}$, $\alpha = 1, 2, 3$, in space $\widetilde{B}_h = L_2(\omega^{(1)} \cup \gamma_S) \times L_2(\omega^{(2)} \cup \gamma_S) \times L_\infty(\omega_2)$. In order to do it, we take arbitrary controls Φ_h, $\Phi_h + \Delta\Phi_h \in U_h$. Let $y(\Phi_h)$ and $y(\Phi_h + \Delta\Phi_h)$ be the solutions to problem (6) associated with controls Φ_h and $\Phi_h + \Delta\Phi_h$, while $J_h(\Phi_h)$ and $J_h(\Phi_h + \Delta\Phi_h)$ be the associated values of functional J_h. We denote

$$\Delta y(x) = y(x; \Phi_h + \Delta\Phi_h) - y(x; \Phi_h), \quad \Delta J_h(\Phi_h) = J_h(\Phi_h + \Delta\Phi_h) - J_h(\Phi_h).$$

Let us obtain the problem for increment $\Delta y = \Delta y(x)$. In order to do it, we rewrite summatory identity for the solution (6) associated with the control $\Phi_h + \Delta\Phi_h$:

$$\begin{aligned}
&\sum_{\omega^{(1)+}_1}\sum_{\omega_2} a^{(1)}_{1h} y_{1\overline{x}_1}(\Phi_h + \Delta\Phi_h) v_{1\overline{x}_1} h_1 h_2 + \Bigg(\sum_{\omega^{(1)}_1}\sum_{\omega^+_2} a^{(1)}_{2h} y_{1\overline{x}_2}(\Phi_h + \Delta\Phi_h) v_{1\overline{x}_2} h_1 h_2 \\
&+ \frac{1}{2}\sum_{\omega^+_2} a^{(1)}_{2h}(\xi, x_2) y_{1\overline{x}_2}(\xi, x_2; \Phi_h + \Delta\Phi_h)\, v_{1\overline{x}_2}(\xi, x_2) h_1 h_2\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)} y_{2\overline{x}_1}(\Phi_h+\Delta\Phi_h) v_{2\overline{x}_1} h_1 h_2 + \Bigg(\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)} y_{2\overline{x}_2}(\Phi_h+\Delta\Phi_h)\, v_{2\overline{x}_2} h_1 h_2 \\
&+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2) y_{2\overline{x}_2}(\xi,x_2;\Phi_h+\Delta\Phi_h)\, v_{2\overline{x}_2}(\xi,x_2) h_1 h_2\Bigg) \\
&+\sum_{\omega_2}\big(\Phi_{3h}(x_2)+\Delta\Phi_{3h}(x_2)\big)\big[y(\xi,x_2;\Phi_h+\Delta\Phi_h)\big]\big[v(\xi,x_2)\big] h_2 \\
&+\Bigg(\sum_{\omega^{(1)}} d_{1h}(x) q_1(y_1(x;\Phi_h+\Delta\Phi_h))\, v_1(x) h_1 h_2 \\
&+\frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2) q_1(y_1(\xi,x_2;\Phi_h+\Delta\Phi_h))\, v_1(\xi,x_2) h_1 h_2\Bigg) \qquad (8)\\
&+\Bigg(\sum_{\omega^{(2)}} d_{2h}(x) q_2(y_2(x;\Phi_h+\Delta\Phi_h)) v_2(x) h_1 h_2 \\
&+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2) q_2(y_2(\xi,x_2))\, v_2(\xi,x_2;\Phi_h+\Delta\Phi_h) h_1 h_2\Bigg) \\
&=\Bigg(\sum_{\omega^{(1)}}(\Phi_{1h}(x)+\Delta\Phi_{1h})\, v_1(x) h_1 h_2 + \frac{1}{2}\sum_{\omega_2}(\Phi_{1h}+\Delta\Phi_{1h})(\xi,x_2)\, v_1(\xi,x_2) h_1 h_2\Bigg) \\
&+\Bigg(\sum_{\omega^{(2)}}(\Phi_{2h}(x)+\Delta\Phi_{2h})\ v_2(x) h_1 h_2 + \frac{1}{2}\sum_{\omega_2}(\Phi_{2h}+\Delta\Phi_{2h})(\xi,x_2)\, v_2(\xi,x_2) h_1 h_2\Bigg).
\end{aligned}
$$

Deducting identity (6) from (8), we obtain

$$
\begin{aligned}
&\sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)}\big(y_{1\overline{x}_1}(x;\Phi_h+\Delta\Phi_h)-y_{1\overline{x}_1}(x;\Phi_h)\big) v_{1\overline{x}_1} h_1 h_2 \\
&+\Bigg(\sum_{\omega_1^{(1)}}\sum_{\omega_2^+} a_{2h}^{(1)}\big(y_{1\overline{x}_2}(x;\Phi_h+\Delta\Phi_h)-y_{1\overline{x}_2}(x;\Phi_h)\big) v_{1\overline{x}_2} h_1 h_2 \\
&+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(1)}(\xi,x_2)\big(y_{1\overline{x}_2}(\xi,x_2;\Phi_h+\Delta\Phi_h)-y_{1\overline{x}_2}(\xi,x_2;\Phi_h)\big) v_{1\overline{x}_2}(\xi,x_2) h_1 h_2\Bigg) \\
&+\sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)}\big(y_{2\overline{x}_1}(x;\Phi_h+\Delta\Phi_h)-y_{2\overline{x}_1}(x;\Phi_h)\big) v_{2\overline{x}_1} h_1 h_2 \\
&+\Bigg(\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)}\big(y_{2\overline{x}_2}(x;\Phi_h+\Delta\Phi_h)-y_{2\overline{x}_2}(x;\Phi_h)\big) v_{2\overline{x}_2} h_1 h_2 \\
&+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2)\big(y_{2\overline{x}_2}(\xi,x_2;\Phi_h+\Delta\Phi_h)-y_{2\overline{x}_2}(\xi,x_2;\Phi_h)\big) v_{2\overline{x}_2}(\xi,x_2) h_1 h_2\Bigg) \\
&+\sum_{\omega_2}\Big\{\big(\Phi_{3h}(x_2)+\Delta\Phi_{3h}(x_2)\big)\big[y(\Phi_h+\Delta\Phi_h)\big]-\Phi_{3h}(x_2)\big[y(\Phi_h)\big]\Big\}\big[v(\xi,x_2)\big] h_2 \\
&+\Bigg(\sum_{\omega^{(1)}} d_{1h}(x)\big(q_1(y_1(x;\Phi_h+\Delta\Phi_h))-q_1(y_1(x;\Phi_h))\big) v_1(x) h_1 h_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)\big(q_1(y_1(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_1(y_1(\xi,x_2;\Phi_h))\big)v_1(\xi,x_2)h_1h_2\Big)\\
&+\Big(\sum_{\omega^{(2)}} d_{2h}(x)\big(q_2(y_2(x;\Phi_h+\Delta\Phi_h))-q_2(y_2(x;\Phi_h))\big)v_2(x)h_1h_2\\
&+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)\big(q_2(y_2(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_2(y_2(\xi,x_2;\Phi_h))\big)v_2(\xi,x_2)h_1h_2\Big)\\
&=\Big(\sum_{\omega^{(1)}}\Delta\Phi_{1h}(x)\,v_1(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{1h}(\xi,x_2)\,v_1(\xi,x_2)h_1h_2\Big)\\
&+\Big(\sum_{\omega^{(2)}}\Delta\Phi_{2h}(x)\,v_2(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{2h}(\xi,x_2)\,v_2(\xi,x_2)h_1h_2\Big)
\end{aligned}
$$

for all $v(x)\in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$.

Taking into consideration that $y(x;\Phi_h+\Delta\Phi_h)=y(x;\Phi_h)+\Delta y(x)$, we obtain the following problem for increment Δy:

$$
\begin{aligned}
&\sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)}(\Delta y_1)_{\overline{x}_1}\,v_{1\overline{x}_1}h_1h_2+\Big(\sum_{\omega_1^{(1)}}\sum_{\omega_2^+} a_{2h}^{(1)}(\Delta y_1)_{\overline{x}_2}\,v_{1\overline{x}_2}h_1h_2\\
&+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(1)}(\xi,x_2)(\Delta y_1)_{\overline{x}_2}(\xi,x_2)\,v_{1\overline{x}_2}(\xi,x_2)h_1h_2\Big)\\
&+\sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)}(\Delta y_2)_{\overline{x}_1}\,v_{2\overline{x}_1}h_1h_2+\Big(\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)}(\Delta y_2)_{\overline{x}_2}\,v_{2\overline{x}_2}h_1h_2\\
&+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2)(\Delta y_2)_{\overline{x}_2}(\xi,x_2)\,v_{2\overline{x}_2}(\xi,x_2)h_1h_2\Big)+\sum_{\omega_2}\Big\{\Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big]\\
&+\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2,\Phi_h)\big]+\Delta\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2,\Phi_h)\big]\Big\}\big[v(\xi,x_2)\big]h_2\\
&+\sum_{\omega^{(1)}} d_{1h}(x)\big(q_1(y_1(x;\Phi_h+\Delta\Phi_h))-q_1(y_1(x;\Phi_h))\big)v_1(x)h_1h_2\\
&+\frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)\big(q_1(y_1(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_1(y_1(\xi,x_2;\Phi_h))\big)v_1(\xi,x_2)h_1h_2\\
&+\sum_{\omega^{(2)}} d_{2h}(x)\big(q_2(y_2(x;\Phi_h+\Delta\Phi_h))-q_2(y_2(x;\Phi_h))\big)v_2(x)h_1h_2\\
&+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)\big(q_2(y_2(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_2(y_2(\xi,x_2;\Phi_h))\big)v_2(\xi,x_2)h_1h_2\\
&=\sum_{\omega^{(1)}}\Delta\Phi_{1h}(x)\,v_1(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{1h}(\xi,x_2)\,v_1(\xi,x_2)h_1h_2\\
&+\sum_{\omega^{(2)}}\Delta\Phi_{2h}(x)\,v_2(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{2h}(\xi,x_2)\,v_2(\xi,x_2)h_1h_2,
\end{aligned}
\tag{9}
$$

for each grid function $v=\big(v_1(\Phi_h),v_2(\Phi_h)\big)\in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$.

The increment of functional $J_h(\Phi_h)$ can be represented as

$$\begin{aligned} \Delta J_h(\Phi_h) &= J_h(\Phi_h + \Delta\Phi_h) - J_h(\Phi_h) \\ &= \sum_{x\in\overline{\omega}^{(1)}} \big|y(x;\Phi_h) + \Delta y - u_{0h}^{(1)}(x)\big|^2 \hbar_1\hbar_2 - \sum_{x\in\overline{\omega}^{(1)}} \big|y(x;\Phi_h) - u_{0h}^{(1)}(x)\big|^2 \hbar_1\hbar_2 \\ &= 2\sum_{\overline{\omega}^{(1)}} \big(y(x;\Phi_h) - u_{0h}^{(1)}(x)\big)\, \Delta y\, \hbar_1\hbar_2 + \sum_{\overline{\omega}^{(1)}} \big(\Delta y\big)^2 \hbar_1\hbar_2. \end{aligned} \tag{10}$$

For further transformations of formula (10) for the increment of the functional, we introduce function $\psi \equiv \psi(x;\Phi_h)$ as a solution to an auxiliary boundary value problem (adjoint problem):

$$\begin{aligned} &-\Big(a_{1h}^{(1)}(x)\psi_{1\overline{x}_1}\Big)_{x_1} - \Big(a_{2h}^{(1)}(x)\psi_{1\overline{x}_2}\Big)_{x_2} + d_{1h}(x)\, q_{1y_1}\psi_1(x) = -2\Big(y(x) - u_{0h}^{(1)}(x)\Big), \quad x\in\omega^{(1)}, \\ &\psi_1(x) = 0, \quad \gamma^{(1)} = \partial\omega^{(1)}\setminus\gamma_S; \\ &-\Big(a_{1h}^{(2)}(x)\psi_{2\overline{x}_1}\Big)_{x_1} - \Big(a_{2h}^{(2)}(x)\psi_{2\overline{x}_2}\Big)_{x_2} + d_{2h}(x)\, q_{2y_2}\psi_2(x) = 0, \quad x\in\omega^{(2)}, \\ &\psi_2(x) = 0, \quad x\in\gamma^{(2)} = \partial\omega^{(2)}\setminus\gamma_S; \\ &\frac{2}{h_1}\Big[a_{1h}^{(1)}(\xi,x_2)\psi_{1\overline{x}_1}(\xi,x_2) + \Phi_{3h}(x_2)\psi_1(\xi,x_2)\Big] + d_{1h}(\xi,x_2)\, q_{1y_1}\psi(\xi,x_2)) \\ &\quad - \Big(a_{2h}^{(1)}(\xi,x_2)\psi_{1\overline{x}_2}(\xi,x_2)\Big)_{x_2} = -2\Big(y(\xi,x_2) - u_{0h}^{(1)}(\xi,x_2)\Big) \\ &\qquad\qquad + \frac{2}{h_1}\Phi_{3h}(x_2)\psi_2(\xi,x_2), \quad x\in\gamma_S = \{x_1=\xi, x_2\in\omega_2\}, \\ &\quad -\frac{2}{h_1}\Big[a_{1h}^{(2)}(\xi+h_1,x_2)\psi_{2x_1}(\xi,x_2) - \Phi_{3h}(x_2)\psi_2(\xi,x_2)\Big] + d_{2h}(\xi,x_2)\, q_{2y_2}(\xi,x_2)) \\ &\quad - \Big(a_{2h}^{(2)}(\xi,x_2)\psi_{2\overline{x}_2}(\xi,x_2)\Big)_{x_2} = \frac{2}{h_1}\Phi_{3h}(x_2)\psi_1(\xi,x_2), \quad x\in\gamma_S = \{x_1=\xi, x_2\in\omega_2\}. \end{aligned} \tag{11}$$

A solution to adjoint problem (11) is a function $\psi(\Phi_h) \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$ satisfying the summatory identity

$$\begin{aligned} &\sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)}\psi_{1\overline{x}_1}v_{1\overline{x}_1}h_1h_2 + \Bigg(\sum_{\omega_1^{(1)}}\sum_{\omega_2^+} a_{2h}^{(1)}\psi_{1\overline{x}_2}v_{1\overline{x}_2}h_1h_2 \\ &+ \frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(1)}(\xi,x_2)\psi_{1\overline{x}_2}(\xi,x_2)\, v_{1\overline{x}_2}(\xi,x_2)h_1h_2\Bigg) + \sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)}\psi_{2\overline{x}_1}v_{2\overline{x}_1}h_1h_2 \\ &+ \Bigg(\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)}\psi_{2\overline{x}_2}v_{2\overline{x}_2}h_1h_2 + \frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2)\psi_{2\overline{x}_2}(\xi,x_2)\, v_{2\overline{x}_2}(\xi,x_2)h_1h_2\Bigg) \\ &+ \sum_{\omega_2}\Phi_{3h}(x_2)\big[\psi(\xi,x_2)\big]\big[v(\xi,x_2)\big]h_2 + \sum_{\omega^{(1)}} d_{1h}(x)q_{1y_1}\psi_1(x)\, v_1(x)h_1h_2 \\ &+ \frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)q_{1y_1}(\xi,x_2)\psi_1(\xi,x_2)\, v_1(\xi,x_2)h_1h_2 \\ &+ \sum_{\omega^{(2)}} d_{2h}(x)q_{2y_2}\psi_2(x)\, v_2(x)h_1h_2 + \frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)q_{2y_2}\psi_2(\xi,x_2)\, v_2(\xi,x_2)h_1h_2 \\ &= -2\sum_{\omega^{(1)}}\Big(y(x) - u_{0h}^{(1)}(x)\Big)v_1(x)h_1h_2 - \sum_{\omega_2}\Big(y(\xi,x_2) - u_{0h}^{(1)}(\xi,x_2)\Big)v_1(\xi,x_2)h_1h_2, \end{aligned} \tag{12}$$

for all $v \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$.

Let us show that the increment of the functional satisfies the representation:

$$\begin{aligned} \Delta J_h(\Phi_h) =& J_h(\Phi_h + \Delta\Phi_h) - J_h(\Phi_h) \\ =& - \sum_{\omega^{(1)}\cup\gamma_S} \Delta\Phi_{1h}(x)\,\psi_1(x)\,\hbar_1 h_2 - \sum_{\omega^{(2)}\cup\gamma_S} \Delta\Phi_{2h}(x)\,\psi_2(x)\,\hbar_1 h_2 \\ &+ \sum_{\omega_2} \Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big]\big[\psi(\xi,x_2)\big]h_2 + R_h, \end{aligned} \tag{13}$$

where

$$\begin{aligned} R_h &= \sum_{k=1}^{6} R_{hk}, \quad R_{h1} = \sum_{\omega_1^{(1)+}\times\omega_2} \big(\Delta y_1\big)^2 \hbar_1 h_2, \\ R_{h2} &= \sum_{\omega_1^{(1)}\times\omega_2} d_{1h}(x)\psi_1(x)\big(q_1(y_1+\Delta y_1) - q_1(y_1) - q_{1y_1}\Delta y_1\big)h_1h_2, \\ R_{h3} &= \sum_{\omega_1^{(2)}\times\omega_2} d_{2h}(x)\psi_2(x)\big(q_2(y_2+\Delta y_2) - q_2(y_2) - q_{2y_2}\Delta y_2\big)h_1h_2, \\ R_{h4} &= \frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)\psi_1(\xi,x_2)\Big(q_1(y_1+\Delta y_1) - q_1(y_1) - q_{1y_1}\Delta y_1(\xi,x_2)\Big)h_1h_2, \\ R_{h5} &= \frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)\psi_2(\xi,x_2)\Big(q_2(y_2+\Delta y_2) - q_2(y_2) - q_{2y_2}\Delta y_2(\xi,x_2)\Big)h_1h_2, \\ R_{h6} &= \sum_{\omega_2} \Delta\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2)\big]\big[\psi(\xi,x_2)\big]h_2. \end{aligned} \tag{14}$$

Indeed, letting $v = \psi$ in (9), we obtain

$$\begin{aligned} &\sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)}(\Delta y_1)_{\overline{x}_1}\psi_{1\overline{x}_1}h_1h_2 \\ &+ \Big(\sum_{\omega_1^{(1)}}\sum_{\omega_2^+} a_{2h}^{(1)}(\Delta y_1)_{\overline{x}_2}\psi_{1\overline{x}_2}h_1h_2 + \frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(1)}(\xi,x_2)(\Delta y_1)_{\overline{x}_2}(\xi,x_2)\psi_{1\overline{x}_2}(\xi,x_2)h_1h_2\Big) \\ &+ \sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)}(\Delta y_2)_{\overline{x}_1}\psi_{2\overline{x}_1}h_1h_2 \\ &+ \Big(\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)}(\Delta y_2)_{\overline{x}_2}\psi_{2\overline{x}_2}h_1h_2 + \frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2)(\Delta y_2)_{\overline{x}_2}(\xi,x_2)\psi_{2\overline{x}_2}(\xi,x_2)h_1h_2\Big) \\ &+ \sum_{\omega_2}\Big\{\Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big] + \Phi_{3h}(x_2)\big[\Delta y(\xi,x_2,\Phi_h)\big] \\ &\qquad + \Delta\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2,\Phi_h)\big]\Big\}\big[\psi(\xi,x_2)\big]h_2 \\ &+ \Big(\sum_{\omega^{(1)}} d_{1h}(x)\big(q_1(y_1(x;\Phi_h+\Delta\Phi_h)) - q_1(y_1(x;\Phi_h))\big)\psi_1(x)h_1h_2 \\ &+ \frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)\big(q_1(y_1(\xi,x_2;\Phi_h+\Delta\Phi_h)) - q_1(y_1(\xi,x_2;\Phi_h))\big)\psi_1(\xi,x_2)h_1h_2\Big) \end{aligned} \tag{15}$$

$$+\Big(\sum_{\omega^{(2)}} d_{2h}(x)\big(q_2(y_2(x;\Phi_h+\Delta\Phi_h))-q_2(y_2(x;\Phi_h))\big)\psi_2(x)h_1h_2$$
$$+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)\big(q_2(y_2(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_2(y_2(\xi,x_2;\Phi_h))\big)\psi_2(\xi,x_2)h_1h_2\Big)$$
$$=\sum_{\omega^{(1)}}\Delta\Phi_{1h}(x)\,\psi_1(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{1h}(\xi,x_2)\,\psi_1(\xi,x_2)h_1h_2$$
$$+\sum_{\omega^{(2)}}\Delta\Phi_{2h}(x)\,\psi_2(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{2h}(\xi,x_2)\,\psi_2(\xi,x_2)h_1h_2.$$

Letting $v=\Delta y$ in (12), we obtain

$$\sum_{\omega_1^{(1)+}}\sum_{\omega_2} a_{1h}^{(1)}\psi_{1\overline{x}_1}\Delta y_{1\overline{x}_1}h_1h_2+\sum_{\omega_1^{(1)}}\sum_{\omega_2^+} a_{2h}^{(1)}\psi_{1\overline{x}_2}\Delta y_{1\overline{x}_2}h_1h_2$$
$$+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(1)}(\xi,x_2)\psi_{1\overline{x}_2}(\xi,x_2)\Delta y_{1\overline{x}_2}(\xi,x_2)h_1h_2+\sum_{\omega_1^{(2)+}}\sum_{\omega_2} a_{1h}^{(2)}\psi_{2\overline{x}_1}\Delta y_{2\overline{x}_1}h_1h_2$$
$$+\sum_{\omega_1^{(2)}}\sum_{\omega_2^+} a_{2h}^{(2)}\psi_{2\overline{x}_2}\Delta y_{2\overline{x}_2}h_1h_2+\frac{1}{2}\sum_{\omega_2^+} a_{2h}^{(2)}(\xi,x_2)\psi_{2\overline{x}_2}(\xi,x_2)\Delta y_{2\overline{x}_2}(\xi,x_2)h_1h_2$$
$$+\sum_{\omega_2}\Phi_{3h}(x_2)\big[\psi(\xi,x_2)\big]\big[\Delta y(\xi,x_2)\big]h_2+\sum_{\omega^{(1)}} d_{1h}(x)q_{1y_1}\psi_1(x)\Delta y_1(x)h_1h_2 \tag{16}$$
$$+\frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)q_{1y_1}(\xi,x_2)\psi_1(\xi,x_2)\Delta y_1(\xi,x_2)h_1h_2$$
$$+\sum_{\omega^{(2)}} d_{2h}(x)q_{2y_2}\psi_2(x)\Delta y_2(x)h_1h_2+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)q_{2y_2}\psi_2(\xi,x_2)\Delta y_2(\xi,x_2)h_1h_2$$
$$=-2\sum_{\omega^{(1)}}\Big(y(x)-u_{0h}^{(1)}(x)\Big)\Delta y_1(x)h_1h_2-\sum_{\omega_2}\Big(y(\xi,x_2)-u_{0h}^{(1)}(\xi,x_2)\Big)\Delta y_1(\xi,x_2)h_1h_2.$$

We deduct identity (16) from (15):

$$2\Big\{\sum_{\omega^{(1)}}\big(y(x)-u_{0h}^{(1)}\big)\,\Delta y_1\,h_1h_2+\frac{1}{2}\sum_{\omega_2}\big(y(\xi,x_2)-u_{0h}^{(1)}(\xi,x_2)\big)\,\Delta y_1(\xi,x_2)\,h_1h_2\Big\}$$
$$=\sum_{\omega_2}\Big\{\Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big]+\Delta\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2,\Phi_h)\big]\Big\}\big[\psi(\xi,x_2)\big]h_2$$
$$-\sum_{\omega^{(1)}}\Delta\Phi_{1h}(x)\psi_1(x)h_1h_2-\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{1h}(\xi,x_2)\,\psi_1(\xi,x_2)h_1h_2$$
$$-\sum_{\omega^{(2)}}\Delta\Phi_{2h}(x)\psi_2(x)h_1h_2-\frac{1}{2}\sum_{\omega_2}\Delta\Phi_{2h}(\xi,x_2)\,\psi_2(\xi,x_2)h_1h_2$$
$$+\sum_{\omega^{(1)}} d_{1h}(x)\psi_1(x)\Big(q_1(y_1(x;\Phi_h+\Delta\Phi_h))-q_1(y_1(x;\Phi_h))-q_{1y_1}\Delta y_1\Big)h_1h_2 \tag{17}$$
$$+\sum_{\omega^{(2)}} d_{2h}(x)\psi_2(x)\Big(q_2(y_2(x;\Phi_h+\Delta\Phi_h))-q_2(y_2(x;\Phi_h))-q_{2y_2}\Delta y_2\Big)h_1h_2$$

$$+\frac{1}{2}\sum_{\omega_2} d_{1h}(\xi,x_2)\,\psi_1(\xi,x_2)\Big(q_1(y_1(\Phi_h+\Delta\Phi_h))-q_1(y_1(\Phi_h))-q_{1y_1}\Delta y_1\Big)h_1h_2$$
$$+\frac{1}{2}\sum_{\omega_2} d_{2h}(\xi,x_2)\,\psi_2(\xi,x_2)\Big(q_2(y_2(\Phi_h+\Delta\Phi_h))-q_2(y_2(\Phi_h))-q_{2y_2}\Delta y_2\Big)h_1h_2.$$

Substituting (17) into (10), we arrive at representation (13)–(14) for the increment of functional $J_h(\Phi_h)$.

Let us estimate increment Δy. Letting $v=\Delta y$ in identity (9) for the increment and taking into consideration that $\Phi_h=\big(\Phi_{1h},\Phi_{2h}\Phi_{3h}\big)\in U_h$, $\Phi_h+\Delta\Phi_h\in U_h$, we find

$$\begin{aligned}C\big\|\Delta y\big\|^2_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant&\Big|\sum_{\omega^{(1)}}\Delta\Phi_{1h}\,\Delta y_1\,h_1h_2\Big|+\frac{1}{2}\Big|\sum_{\omega_2}\Delta\Phi_{1h}\,\Delta y_1\,h_1h_2\Big|\\&+\Big|\sum_{\omega^{(2)}}\Delta\Phi_{2h}\,\Delta y_2\,h_1h_2\Big|+\frac{1}{2}\Big|\sum_{\omega_2}\Delta\Phi_{2h}\,\Delta y_2\,h_1h_2\Big|\\&+\Big|\sum_{\omega_2}\Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big]\big[\Delta y(\xi,x_2,\Phi_h)\big]h_2\Big|.\end{aligned}\tag{18}$$

Let us estimate the right hand side of (18). We have

$$\begin{aligned}\Big|\sum_{\omega_1^{(1)}\times\omega_2}\Delta\Phi_{1h}\,\Delta y_1\,h_1h_2\Big|&\leqslant\big\|\Delta\Phi_{1h}\big\|_{L_2(\omega_1^{(1)}\times\omega_2)}\|\Delta y_1\|_{L_2(\omega_1^{(1)}\times\omega_2)}\\&\leqslant C\big\|\Delta\Phi_{1h}\big\|_{L_2(\omega_1^{(1)}\times\omega_2)}\|\Delta y_1\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})},\end{aligned}$$
$$\frac{1}{2}\Big|\sum_{\omega_2}\Delta\Phi_{1h}(\xi,x_2)\,\Delta y_1(\xi,x_2)\,h_1h_2\Big|\leqslant C\,\frac{1}{2}\big\|\Delta\Phi_{1h}\big\|_{L_2(\gamma_S)}\|\Delta y_1\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}.$$

In the same way:

$$\begin{aligned}\Big|\sum_{\omega_1^{(2)}\times\omega_2}\Delta\Phi_{2h}\,\Delta y_2\,h_1h_2\Big|&\leqslant C\big\|\Delta\Phi_{2h}\big\|_{L_2(\omega_1^{(2)}\times\omega_2)}\|\Delta y_2\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})},\\\frac{1}{2}\Big|\sum_{\omega_2}\Delta\Phi_{2h}(\xi,x_2)\,\Delta y_2(\xi,x_2)\,h_1h_2\Big|&\leqslant C\,\frac{1}{2}\big\|\Delta\Phi_{2h}\big\|_{L_2(\gamma_S)}\|\Delta y_2\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}.\end{aligned}\tag{19}$$

Employing the identity from [4], we obtain

$$\begin{aligned}&\Big|\sum_{\omega_2}\Delta\Phi_{3h}(x_2)\big[y(\xi,x_2,\Phi_h)\big]\big[\Delta y(\xi,x_2,\Phi_h)\big]h_2\Big|\\&\qquad\leqslant\big\|\Delta\Phi_{3h}\big\|_{L_\infty(\omega_2)}\big\|[y]\big\|_{L_2(\gamma_S)}\big\|[\Delta y]\big\|_{L_2(\gamma_S)}\\&\qquad\leqslant C\big\|\Delta\Phi_{3h}\big\|_{L_\infty(\omega_2)}\|y\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\|\Delta y\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}.\end{aligned}\tag{20}$$

Taking into consideration estimates (19), (20), by inequality (18) we find the desired estimate

$$\big\|\Delta y\big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant C_0\Big\{\sum_{\alpha=1}^{2}\big\|\Delta\Phi_{\alpha h}\big\|_{L_2(\omega^{(\alpha)}\cup\gamma_S)}+\big\|\Delta\Phi_{3h}\big\|_{L_\infty(\omega_2)}\Big\}.\tag{21}$$

We proceed to estimating the solution of adjoint problem (12). Letting $v=\psi$ in identity (12) and estimate the left hand side in (12), we obtain

$$C\big\|\psi\big\|^2_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant 2\Big|\sum_{\omega_1^{(1)+}\times\omega_2}\big(y(x;\Phi_h)-u_0^h(x)\big)\,\psi_1(x)\,h_1h_2\Big|.\tag{22}$$

It is easy to estimate the right hand side of (22):

$$2\Big|\sum_{\omega_1^{(1)+}\times\omega_2}\big(y(x;\Phi_h)-u_0^h(x)\big)\,\psi_1\,h_1h_2\Big|\leqslant M_0\|y-u_0^h\|_{L_2(\omega_1^{(1)+}\times\omega_2)}\|\psi\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}.$$

This yields

$$\big\|\psi\big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant\overline{M}_0\|y-u_0^h\|_{L_2(\omega_1^{(1)+}\times\omega_2)}. \tag{23}$$

To proceed in estimate the right hand side of inequality (23), we employ the statement from [6]:

$$\|y(\Phi_h)\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant M\sum_{\alpha=1}^{2}\|\Phi_{\alpha h}\|_{L_2(\omega^{(\alpha)}\cup\gamma_S)},\ \forall\Phi_h\in U_h. \tag{24}$$

Then by (24),

$$\sup_{\Phi_h\in U_h}\big\|y\big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant M=\text{Const},$$

and by (23) we obtain

$$\big\|\psi\big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\leqslant\widetilde{M}=\text{Const},\qquad\forall\Phi_h\in U_h.$$

We proceed to estimating quantity R_h in (13)–(14). We have

$$|R_{h2}|\leqslant\sum_{\omega_1^{(1)}\times\omega_2}\Big|d_{1h}(x)\psi_1(x)\big[q_1(y_1+\Delta y_1)-q_1(y_1)-q_{1y_1}\Delta y_1\Big|h_1h_2.$$

We impose an additional restriction for function $q(y)$:

$$\big|q_s'(s_1)-q_s'(s_2)\big|\leqslant\overline{L}_q|s_1-s_2|\text{ for all }s_1,s_2\in\mathbb{R},\quad\overline{L}_q=\text{Const}>0.$$

It implies easily the following inequality

$$\Big|q_i(y_i+\Delta y_i)-q_i(y_i)-q_i'(y_i)\Delta y_i\Big|\leqslant\frac{\overline{L}_q}{2}\big|\Delta y_i\big|^2,\quad i=1,2.$$

Then

$$\|R_{h2}|\leqslant\frac{\overline{L}_q}{2}\,\overline{d}_0\sum_{\omega_1^{(1)}\times\omega_2}\big|\Delta y_1\big|^2\big|\psi_1\big|h_1h_2\leqslant C\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})}\big\|\psi\big\|_{W_2^1(\overline{\omega}^{(1)})},$$

$$|R_{h3}|\leqslant C\big\|\Delta y_2\big\|^2_{W_2^1(\overline{\omega}^{(2)})}\big\|\Delta\psi_2\big\|_{W_2^1(\overline{\omega}^{(2)})},$$

$$\begin{aligned}\big|R_{h4}\big|&=\Big|\frac12\sum_{\omega_2}d_{1h}(\xi,x_2)\,\psi_1(\xi,x_2)\Big(q_1(y_1+\Delta y_1)-q_1(y_1)-q_{1y_1}\Delta y_1\Big)h_1h_2\Big|\\&\leqslant C\big\|\Delta\psi_1\big\|_{W_2^1(\overline{\omega}^{(1)})}\Big\{\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})}+\big\|\Delta y_1\big\|_{W_2^1(\overline{\omega}^{(1)})}\Big\},\end{aligned}$$

$$\big|R_{h5}\big|\leqslant C\big\|\Delta\psi_2\big\|_{W_2^1(\overline{\omega}^{(2)})}\Big\{\big\|\Delta y_2\big\|^2_{W_2^1(\overline{\omega}^{(2)})}+\big\|\Delta y_2\big\|_{W_2^1(\overline{\omega}^{(2)})}\Big\},$$

$$\big|R_{h1}\big|\leqslant\sum_{\omega_1^{+}\times\omega_2}\big|\Delta y_1\big|^2\hbar_1h_2\leqslant C\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})},$$

$$\begin{aligned}\big|R_{h6}\Big|&\leqslant\sum_{\omega_2}\big|\Delta\Phi_{3h}(x_2)\big[\Delta y(\xi,x_2)\big]\big[\psi(\xi,x_2)\big]\Big|h_2\\&\leqslant\big\|\Delta\Phi_{3h}\big\|_{L_\infty(\omega_2)}\sum_{\omega_2}\Big|\big[\Delta y(\xi,x_2)\big]\big[\psi(\xi,x_2)\big]\Big|h_2\end{aligned}$$

$$\leqslant C\big\|\Delta\Phi_{3h}\big\|_{L_\infty(\omega_2)}\big\|\Delta y\big\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\big\|\psi\big\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}.$$

Thus, for the increment of functional $J_h(\Phi_h)$ we have obtained the representation:

$$\Delta J_h(\Phi_h) = -\sum_{\omega^{(1)}\cup\gamma_S}\Delta\Phi_{1h}\,\psi_1\,\hbar_1 h_2 - \sum_{\omega^{(2)}\cup\gamma_S}\Delta\Phi_{2h}\,\psi_2\,\hbar_1 h_2 + \sum_{\omega_2}\Delta\Phi_{3h}\big[y(\xi,x_2)\big]\big[\psi(\xi,x_2)\big]h_2 + o\big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\big), \tag{25}$$

where $\widetilde{B}_h = L_2(\omega^{(1)}\cup\gamma_S)\times L_2(\omega^{(2)}\cup\gamma_S)\times L_\infty(\omega_2)$.

It is easy to see that the increment of functional $J_h(\Phi_h)$ can be written as

$$\Delta J_h(\Phi_h) = \left(\frac{\partial J_h}{\partial\Phi_{1h}},\Delta\Phi_{1h}\right)_{L_2(\omega^{(1)}\cup\gamma_S)} + \left(\frac{\partial J_h}{\partial\Phi_{2h}},\Delta\Phi_{2h}\right)_{L_2(\omega^{(2)}\cup\gamma_S)} + \left(\frac{\partial J_h}{\partial\Phi_{3h}},\Delta\Phi_{3h}\right)_{L_2(\omega_2)} + o\big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\big), \tag{26}$$

where

$$\begin{aligned}
&\frac{\partial J_h}{\partial\Phi_h} = \left(\frac{\partial J_h}{\partial\Phi_{1h}},\frac{\partial J_h}{\partial\Phi_{2h}},\frac{\partial J_h}{\partial\Phi_{3h}}\right),\\
&\frac{\partial J_h}{\partial\Phi_{1h}} = -\psi_1(x),\quad x\in\omega^{(1)}\cup\gamma_S,\qquad \frac{\partial J_h}{\partial\Phi_{2h}} = -\psi_2(x),\quad x\in\omega^{(2)}\cup\gamma_S,\\
&\frac{\partial J_h}{\partial\Phi_{3h}} = \big[y(\xi,x_2)\big]\big[\psi(\xi,x_2)\big],\quad x_2\in\omega_2.
\end{aligned} \tag{27}$$

Now we can rewrite the formula for the increment of functional $J_h(\Phi_h)$ as

$$\Delta J_h(\Phi_h) = < J_h'(\Phi_h),\Delta\Phi_h > + o\big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\big), \tag{28}$$

where

$$< J_h'(\Phi_h),\Delta\Phi_h > = \left(\frac{\partial J_h}{\partial\Phi_{1h}},\Delta\Phi_{1h}\right)_{L_2(\omega^{(1)}\cup\gamma_S)} + \left(\frac{\partial J_h}{\partial\Phi_{2h}},\Delta\Phi_{2h}\right)_{L_2(\omega^{(2)}\cup\gamma_S)} + \left(\frac{\partial J_h}{\partial\Phi_{3h}},\Delta\Phi_{3h}\right)_{L_2(\omega_2)}. \tag{29}$$

Thus, in formula (28) for the increment of the functional, the first term is a linear bounded functional on $\widetilde{B}_h = L_2(\omega^{(1)}\cup\gamma_S)\times L_2(\omega^{(2)}\cup\gamma_S)\times L_\infty(\omega_2)$ w.r.t. $\Phi_h = \big(\Phi_{1h},\Phi_{2h},\Phi_{3h}\big)$, while the second term is of order $o\big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\big)$. It means that functional $J_h(\Phi_h)$ is Frechèt differentiable on set U_h in space $\widetilde{B}_h$. At that, the gradient of functional $J_h(\Phi_h)$ at point $\Phi_h\in U_h$ is given by (27) and the first component in (27) is an analogue of the partial derivative of functional $J_h(\Phi_h) = J_h(\Phi_{1h},\Phi_{2h},\Phi_{3h})$ w.r.t. variable Φ_{1h}, the second and the third components are analogues of the derivatives w.r.t. variables Φ_{2h} and Φ_{3h}, respectively.

Thus, we have proved the following theorem.

Theorem 4.1. *Suppose that function $q(s)$ is defined on $\mathbb{R}$ with values in $\mathbb{R}$ and satisfies the conditions: $q(0)=0$, $q(s)$ is differentiable w.r.t. s, the first derivative $q_s'(s)$ satisfies the restrictions*

$$\begin{aligned}
&0 < q_0 \leqslant q_s'(s) < L_q < \infty,\\
&\big|q_s'(s_1) - q_s'(s_2)\big| \leqslant \overline{L}_q|s_1 - s_2| \text{ for all } s_1, s_2\in\mathbb{R},\quad L_q,\overline{L}_q = Const > 0.
\end{aligned}$$

Let $k_\alpha(x)\in W^1_\infty(\Omega_1)\times W^1_\infty(\Omega_2)$, $\alpha=1,2$, $d(x)\in L_\infty(\Omega_1)\times L_\infty(\Omega_2)$. Then grid functional $J_h(\Phi_h)$ is Frechèt differentiable w.r.t. Φ_h on U_h in space $\widetilde{B}_h = L_2(\omega^{(1)}\cup\gamma_S)\times L_2(\omega^{(2)}\cup\gamma_S)\times L_\infty(\omega_2)$, and gradient $J_h'(\Phi_h)$ at point $(\Phi_h) = (\Phi_{1h},\Phi_{2h},\Phi_{3h})$ is given by (29), (27).

It can be shown that grid functional $J_h(\Phi_h)$ belongs to class $C^{1,1}(\widetilde{B}_h)$, where $\widetilde{B}_h = L_2(\omega^{(1)} \cup \gamma_S) \times L_2(\omega^{(2)} \cup \gamma_S) \times L_\infty(\omega_2)$, that is,

$$\big\| J_h'(\Phi_h + \Delta\Phi_h) - J_h'(\Phi_h) \big\| \leqslant C \big\| \Delta\Phi_h \big\|_{\widetilde{B}_h}. \tag{30}$$

Indeed, employing Lemmata 2.1-2.3 in [4], for each $\eta = \big(\eta_1, \eta_2, \eta_3\big) \in \widetilde{B}_h$ we have

$$\begin{aligned}
\Big| < J_h'(\Phi_h + \Delta\Phi_h) - J_h'(\Phi_h), \eta > \Big| &= \Big| \sum_{\omega^{(1)} \cup \gamma_S} \left(\frac{\partial J_h(\Phi_h + \Delta\Phi_h; x)}{\partial \Phi_{1h}} - \frac{\partial J_h(\Phi_h; x)}{\partial \Phi_{1h}} \right) \eta_1(x) \hbar_1 h_2 \\
&+ \sum_{\omega^{(2)} \cup \gamma_S} \left(\frac{\partial J_h(\Phi_h + \Delta\Phi_h; x)}{\partial \Phi_{2h}} - \frac{\partial J_h(\Phi_h; x)}{\partial \Phi_{2h}} \right) \eta_2(x) \hbar_1 h_2 \Big| \\
&+ \sum_{\omega_2} \left(\frac{\partial J_h(\Phi_h + \Delta\Phi_h; x)}{\partial \Phi_{3h}} - \frac{\partial J_h(\Phi_h; x)}{\partial \Phi_{3h}} \right) \eta_3(x) h_2 \Big| \\
&\leqslant C_1 \big\| \Delta\psi_1(\Phi_h) \big\|_{W_2^1(\overline{\omega}^{(1)})} \big\| \eta_1 \big\|_{L_2(\omega^{(1)} \cup \gamma_S)} + C_2 \big\| \Delta\psi_2(\Phi_h) \big\|_{W_2^1(\overline{\omega}^{(2)})} \big\| \eta_2 \big\|_{L_2(\omega^{(2)} \cup \gamma_S)} \\
&+ C_3 \big\| \eta_3 \big\|_{L_\infty(\omega_2)} \Big\{ \big\| y \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} \big\| \Delta\psi \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} \\
&+ \big\| \Delta y \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} \big\| \psi \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} + \big\| \Delta y \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} \big\| \Delta\psi \big\|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}} \Big\}.
\end{aligned}$$

Let us estimate increment $\Delta\psi$. In order to do it, by the approach used in obtaining problem (9), let us find a problem for increment $\Delta\psi = \psi(\Phi_h + \Delta\Phi_h) - \psi(\Phi_h)$:

$$\begin{aligned}
&\sum_{\omega_1^{(1)+}} \sum_{\omega_2} a_{1h}^{(1)} (\Delta\psi_1)_{\overline{x}_1} v_{1\overline{x}_1} h_1 h_2 + \sum_{\omega_1^{(1)}} \sum_{\omega_2^+} a_{2h}^{(1)} (\Delta\psi_1)_{\overline{x}_2} v_{1\overline{x}_2} h_1 h_2 \\
&+ \frac{1}{2} \sum_{\omega_2^+} a_{2h}^{(1)}(\xi, x_2) (\Delta\psi_1)_{\overline{x}_2}(\xi, x_2)\, v_{1\overline{x}_2}(\xi, x_2) h_1 h_2 \\
&+ \sum_{\omega_1^{(2)+}} \sum_{\omega_2} a_{1h}^{(2)} (\Delta\psi_2)_{\overline{x}_1} v_{2\overline{x}_1} h_1 h_2 + \sum_{\omega_1^{(2)}} \sum_{\omega_2^+} a_{2h}^{(2)} (\Delta\psi_2)_{\overline{x}_2} v_{2\overline{x}_2} h_1 h_2 \\
&+ \frac{1}{2} \sum_{\omega_2^+} a_{2h}^{(2)}(\xi, x_2) (\Delta\psi_2)_{\overline{x}_2}(\xi, x_2)\, v_{2\overline{x}_2}(\xi, x_2) h_1 h_2 \\
&+ \sum_{\omega_2} \Phi_{3h}(x_2) \big[\Delta\psi\big] \big[v\big] h_2 + \sum_{\omega^{(1)}} d_{1h}(x) q_{1y_1} \Delta\psi_1(x)\, v_1(x) h_1 h_2 \\
&+ \frac{1}{2} \sum_{\omega_2} d_{1h}(\xi, x_2) q_{1y_1} \Delta\psi_1(\xi, x_2)\, v_1(\xi, x_2) h_1 h_2 \\
&+ \sum_{\omega^{(2)}} d_{2h}(x) q_{1y_1} \Delta\psi_1(x)\, v_2(x) h_1 h_2 + \frac{1}{2} \sum_{\omega_2} d_{2h}(\xi, x_2) q_{2y_2} \Delta\psi_2(\xi, x_2)\, v_2(\xi, x_2) h_1 h_2 \\
&= -2 \sum_{\omega_1^{(1)+} \times \omega_2} \Delta y_1(x)\, v_1(\xi, x_2) \hbar_1 h_2, \quad \forall v = (v_1, v_2) \in \mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)}).
\end{aligned} \tag{31}$$

Letting $v = \Delta\psi$ in identity (31), we get

$$\begin{aligned}
C \big\| \Delta\psi \big\|^2_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})} &\leqslant 2 \Big| \sum_{\omega_1^{(1)+} \times \omega_2} \Delta y_1(x)\, \Delta\psi_1(x) \hbar_1 h_2 \Big| \\
&\leqslant \widetilde{C}_0 \| \Delta y_1 \|_{W_2^1(\omega_1^{(1)+} \times \omega_2)} \| \Delta\psi \|_{\mathring{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})},
\end{aligned}$$

that is,

$$\big\|\Delta\psi\big\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})} \leqslant \widetilde{C}_0\left(\sum_{\alpha=1}^{2}\|\Delta\Phi_{\alpha h}\|_{L_2(\omega^{(\alpha)}\cup\gamma_S)} + \|\Delta\Phi_{3h}\|_{L_\infty(\omega_2)}\right) = \widetilde{C}_0\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}.$$

It yields

$$\big|< J_h'(\Phi_h+\Delta\Phi_h) - J_h'(\Phi_h), \eta >\big|$$
$$\leqslant C\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\left(\big\|\eta_1\big\|_{L_2(\omega^{(1)}\cup\gamma_S)} + \big\|\eta_2\big\|_{L_2(\omega^{(2)}\cup\gamma_S)} + \big\|\eta_3\big\|_{L_\infty(\omega_2)}\right) = C\big\|\eta\big\|_{\widetilde{B}_h}\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}.$$

Thus,

$$\big\|J_h'(\Phi_h+\Delta\Phi_h) - J_h'(\Phi_h)\big\| = \sup_{\eta\neq 0}\frac{\big|< J_h'(\Phi_h+\Delta\Phi_h) - J_h'(\Phi_h), \eta >\big|}{\big\|\eta\big\|_{\widetilde{B}_h}} \leqslant C\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}.$$

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold true. Then grid functional* $J_h(\Phi_h)$ *belongs to class* $C^{1,1}(\widetilde{B}_h)$*, where* $\widetilde{B}_h = L_2(\omega^{(1)}\cup\gamma_S)\times L_2(\omega^{(2)}\cup\gamma_S)\times L_\infty(\omega_2)$*, i.e., estimate (30) holds true.*

BIBLIOGRAPHY

1. A.A. Samarskii, V.B. Andreev. *Difference methods for elliptic equations.* Nauka, Moscow (1976). (in Russian).
2. A.A. Samarskii. *Theory of difference schemes.* Nauka, Moscow (1989). (in Russian).
3. E.M. Kartashov. *Analytic methods in heat theory for solid bodies.* Vysshaya shkola, Moscow (1985). (in Russian).
4. F.V. Lubyshev, A.R. Manapova, M.E. Fairuzov. *Approximations of optimal control problems for semilinear elliptic equations with discontinuous coefficients and solutions and with control in matching boundary conditions* // Zhurn. Vychisl. Matem. Matem. Fiz. **54**:11, 1767–1792 (2014). [Comp. Math. Math. Phys. **54**:11, 1700–1724 (2014).]
5. F.V. Lubyshev. *Finite difference approximations of optimal control problems for semilinear elliptic equations with discontinuous coefficients and solutions* // Zhurn. Vychisl. Matem. Matem. Fiz. **52**:8, 1378–1399 (2012). [Comp. Math. Math. Phys. **52**:8, 1094–1114 (2012).]
6. A.R. Manapova, F.V. Lubyshev. *Accuracy estimate with respect to state of finite-dimensional approximations for optimization problems for semi-linear elliptic equations with discontinuous coefficients and solutions* // Ufimskij Matem. Zhurn. **6**:3, 72–87 (2014). [Ufa Math. J. **6**:3, 69–84 (2014).]
7. S.L. Sobolev. *Some applications of functional analysis in mathematical physics.* Siberian Branch of AS USSR, Novosibirsk (1962). (in Russian).
8. O.A. Ladyzhenskaya. *The boundary value problems of mathematical physics.* Nauka, Moscow (1973). [Applied Mathematical Sciences, V. 49. Springer, NY (1985).]
9. H. Gaewski, K. Gröger, K. Zacharius. *Nichtlineare operatorgleichungen und operatordifferentialgleichungen.* Akademie-Verlag, Berlin (1974).
10. F.P. Vasil'ev. *Optimization methods.* Factorial Press, Moscow (2002). (in Russian).

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