ON CONVERGENCE OF POLYNOMIAL SOLUTIONS OF MINIMAL SURFACE

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Abstract. In this paper we consider the polynomial approximate solutions of the Dirichlet problem for minimal surface equation. It is shown that under certain conditions on the geometric structure of the domain the absolute values of the gradients of the solutions are bounded as the degree of these polynomials increases. The obtained properties imply the uniform convergence of approximate solutions to the exact solution of the minimal surface equation.

Keywords: minimal surface equation, uniform convergence, approximate solution.

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1. INTRODUCTION

In numerical solving of boundary value problems for equations and systems of partial differential equations, a very important issue is the convergence of approximate solutions. The study of this issue is especially important for nonlinear equations since in this case there is a series of difficulties related with the impossibility of employing traditional methods and approaches used for linear equations. At present, a quite topical problem is to determine the conditions ensuring the uniform convergence of approximate solutions obtained by various methods for nonlinear equations and system of equations of variational kind. In this case, it is natural to employ variational methods of solving boundary value problems. And this is an issue on the justification of these methods arises, which is reduced to studying general properties of approximate solutions (see, for instance, [1], [2]).

2. FORMULATION OF THE PROBLEM

We consider the issue on convergence of approximate solutions for the minimal surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{f_{x_i}}{\sqrt{1+|\nabla f|^2}}\right) = 0 \qquad (1)$$

in domain Ω subject to the boundary condition

$$f|_{\partial\Omega} = \varphi|_{\partial\Omega}, \qquad (2)$$

where $\varphi \in C(\overline{\Omega})$. It should be noted that this Dirichlet problem is not solvable for an arbitrary domain (even with a smooth boundary). For planar domains the necessary and sufficient condition for the solvability of the Dirichlet problem for an arbitrary continuous boundary function $\varphi(x)$ is the convexity of this domain. In the space of dimension greater than two, such condition is the non-negativity of the mean curvature of the boundary w.r.t. the outward normal. Precise formulation and the proof of these results can be found in works [3]–[10]. In the present work we impose no conditions for domain Ω, but we assume that for a given boundary

function $\varphi(x)$ problem (1)-(2) is solvable. It is clear that for an arbitrary domain Ω, such functions $\varphi(x)$ exist.

We study the issue on uniform convergence of polynomial approximate solutions to the minimal surface equation constructed by means of algebraic polynomials. In work [11] a similar convergence problem for piece-wise linear approximate solution to boundary value problem (1)-(2) was solved, while in work [12] there was given a description of numerical realization of finite elements methods based on piece-wise linear functions. Let us provide required definitions.

Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain such that some polynomial $\psi(x, y)$ of degree at most N_0 in each variable satisfies $\psi(x, y) = 0$ as $(x, y) \in \partial\Omega$ and $\psi(x, y) > 0$ as $(x, y) \in \Omega$. For a natural N we denote by L_N the set of all polynomials of the form

$$v_N(x, y) = \psi(x, y) \sum_{n,m=1}^{N} c_{nm} x^n y^m.$$

It is clear that $v_N(x, y) = 0$ for $(x, y) \in \partial\Omega$. Assume that $\varphi \in C^1(\Omega)$. Consider the problem on finding a polynomial v_N^* on which the area integral

$$\sigma(\varphi + v_N) = \iint_{\Omega} \sqrt{1 + |\nabla\varphi + \nabla v_N|^2}\, dxdy \to \min, \quad v_N \in L_N. \tag{3}$$

attains its minimum. It is easy to show that solution v_N^* to problem (3) satisfies the identity

$$\iint_{\Omega} \frac{\langle \nabla\varphi + \nabla v_N^*, \nabla v_N \rangle}{\sqrt{1 + |\nabla\varphi + \nabla v_N^*|^2}}\, dxdy = 0 \quad \forall\, v_N \in L_N. \tag{4}$$

Theorem 1. *Problem* (3) *is uniquely solvable.*

Proof. It is clear the area $\sigma(\varphi + v_N)$ is a function $\sigma(c_{11}, c_{12}, \ldots, c_{NN})$ of finitely many variables c_{nm}, $n, m = 1, \ldots, N$. This function is obviously continuous. At that,

$$\lim_{|c|\to\infty} \sigma(c_{nm}) = +\infty,$$

where $|c| = \max_{1 \leqslant n,m \leqslant N} |c_{nm}|$. Therefore, there exists a set of numbers c_{nm}^*, at which function $\sigma(c_{11}, c_{12}, \ldots, c_{NN})$ attains its minimal value. We denote by v_N^* the polynomial associated with this set of coefficients. It satisfies condition (4).

Let us show the uniqueness. Assume that there exists one more function $v_N^1 \in L_N$ being a solution to problem (3). It also satisfies identity (4). Deducting one identity from the other for $v_N = v_N^* - v_N^1$, we obtain

$$\iint_{\Omega} \left(\frac{\langle \nabla f^*, \nabla f^* - \nabla f^1 \rangle}{\sqrt{1 + |\nabla f^*|^2}} - \frac{\langle \nabla f^1, \nabla f^* - \nabla f^1 \rangle}{\sqrt{1 + |\nabla f^1|^2}} \right) dxdy = 0, \tag{5}$$

where $f^* = \varphi + v_N^*$, $f^1 = \varphi + v_N^1$.

We shall make use of the identity

$$\left\langle \frac{\xi}{\sqrt{1 + |\xi|^2}} - \frac{\eta}{\sqrt{1 + |\eta|^2}}, \xi - \eta \right\rangle \geqslant \frac{|\xi - \eta|^2}{\sqrt{1 + |\xi|^2}(\sqrt{1 + |\xi|^2}\sqrt{1 + |\eta|^2} + |\xi||\eta| + 1)}, \tag{6}$$

which holds for all vectors $\xi, \eta \in \mathbb{R}^2$ and it can be obtained as follows. We first observe that

$$\sqrt{1 + |\xi|^2} \geqslant \sqrt{1 + |\eta|^2} + \frac{\langle \eta, \xi - \eta \rangle}{\sqrt{1 + |\eta|^2}}.$$

Then

$$\left\langle \frac{\xi}{\sqrt{1+|\xi|^2}} - \frac{\eta}{\sqrt{1+|\eta|^2}}, \xi - \eta \right\rangle = -\frac{\langle \xi, \eta - \xi \rangle}{\sqrt{1+|\xi|^2}} - \frac{\langle \eta, \xi - \eta \rangle}{\sqrt{1+|\eta|^2}}$$

$$\geqslant \sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta - \xi \rangle}{\sqrt{1+|\xi|^2}}$$

$$= \frac{\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2} - \langle \xi, \eta \rangle - 1}{\sqrt{1+|\xi|^2}}$$

$$\geqslant \frac{|\xi - \eta|^2}{\sqrt{1+|\xi|^2}(\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2} + |\xi||\eta| + 1)}.$$

Thus, it follows from inequalities (6) and identity (5) that

$$\iint\limits_{\Omega} \frac{|\nabla f^* - \nabla f^1|^2}{\sqrt{1+|\nabla f^*|^2}(\sqrt{1+|\nabla f^*|^2}\sqrt{1+|\nabla f^1|^2} + |\nabla f^*||\nabla f^1| + 1)} dxdy \leqslant 0.$$

Hence, $\nabla f^* \equiv \nabla f^1$. This yields that $\nabla v_N^* \equiv \nabla v_N^1$. Since polynomials v_N^* and v_N^1 vanish on boundary $\partial\Omega$, we have $v_N^*(x,y) = v_N^1(x,y)$ for all $(x,y) \in \Omega$. The uniqueness is proven. $\square$

Definition. *Function* $f^* = \varphi + v_N^*$, $v_N^* \in L_N$ *is called a polynomial solution to boundary value problem* (1)–(2) *if identity* (4) *holds true for each polynomial.*

In what follows we shall be interested in the issue on uniform convergence of a sequence of polynomial solutions $\varphi + v_N^*$ as $N \to \infty$. First of all we shall show that under certain conditions, the gradients of these functions are bounded by a constant independent of N. This property will allow us to obtain an estimate for the rate of uniform convergence to the exact solution.

3. Estimate for the gradient of a polynomial solution

Let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to problem (1)-(2). We introduce quantity $\delta(\xi, \eta)$ for arbitrary vectors $\xi, \eta \in \mathbb{R}^2$

$$\delta(\xi, \eta) = \sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta - \xi \rangle}{\sqrt{1+|\xi|^2}}.$$

It is easy to see that $\delta(\xi, \eta) > 0$ for all $\xi \neq \eta$. We shall also make use of the following polynomial characteristics of a domain:

$$\lambda_N = \inf_{v \in L_N} \frac{\left(\iint\limits_{\Omega} |\nabla v|^2 \, dxdy\right)^{1/2}}{\sqrt{|\Omega|} \sup\limits_{\Omega} |\nabla v|} > 0,$$

where $|\Omega|$ is the area of domain Ω. It is clear that $0 < \lambda_N \leqslant 1$. We shall estimate the rate of convergence of λ_N to zero as $N \to \infty$ in the fifth section of the present paper.

Letting $\xi = \nabla f$, $\eta = \nabla\varphi + \nabla v_N^*$ and employing equation (1), we obtain

$$\iint\limits_{\Omega} \delta(\nabla f, \nabla\varphi + \nabla v_N^*) \, dxdy = \sigma(\varphi + v_N^*) - \sigma(f).$$

Employing the inequality (see (6))

$$\sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta - \xi \rangle}{\sqrt{1+|\xi|^2}} \geqslant \frac{|\xi - \eta|^2}{\sqrt{1+|\xi|^2}(\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2} + |\xi||\eta| + 1)},$$

we conclude that

$$\iint_{\Omega} \frac{|\nabla f - \nabla \varphi - \nabla v_N^*|^2 \, dxdy}{\sqrt{1+|\nabla f|^2}(\sqrt{1+|\nabla f|^2}\sqrt{1+|\nabla \varphi + \nabla v_N^*|^2}+|\nabla f||\nabla \varphi + \nabla v_N^*|+1)}$$
$$\leqslant \sigma(\varphi + v_N^*) - \sigma(f).$$

By the previous inequality we obtain

$$\iint_{\Omega} \frac{|\nabla f - \nabla \varphi - \nabla v_N^*|^2 \, dxdy}{\sqrt{1+|\nabla \varphi + \nabla v_N^*|^2}} \leqslant 3(1+P_0^2)(\sigma(\varphi + v_N^*) - \sigma(f)).$$

We let $K = \max\{\sup_{\Omega} |\nabla \varphi|, 1\}$ and $A_N = \sup_{\Omega} |\nabla v_N^*|$. Then

$$\iint_{\Omega} |\nabla f - \nabla \varphi - \nabla v_N^*|^2 \, dxdy \leqslant 3\sqrt{1+2K^2+2A_N^2}(1+P_0^2)(\sigma(\varphi + v_N^*) - \sigma(f)).$$

We introduce the notation $g = f - \varphi$. It is clear that $g(x,y) = 0$ as $(x,y) \in \partial\Omega$. Given an arbitrary function $h \in C(\overline{\Omega})$, by h_N we denote some approximation of function h by the functions in space L_N. The way how to do this approximation is not important and it will be specified in the next section. Employing the above notation, we have

$$\left(\iint_{\Omega} |\nabla f - \nabla \varphi - \nabla v_N^*|^2 \, dxdy\right)^{1/2} \geqslant \left(\iint_{\Omega} |\nabla g_N - \nabla v_N^*|^2 \, dxdy\right)^{1/2}$$
$$- \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy\right)^{1/2}.$$

Hence, since $g_N - v_N^* \in L_N$, we get

$$\sup_{\Omega} |\nabla g_N - \nabla v_N^*| \leqslant \frac{1}{\sqrt{|\Omega|}\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy\right)^{1/2}$$
$$+ \frac{1}{\sqrt{|\Omega|}\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla v_N^*|^2 \, dxdy\right)^{1/2}$$
$$\leqslant \frac{1}{\sqrt{|\Omega|}\,\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy\right)^{1/2} +$$
$$+ \frac{\left(3(1+P_0^2)\sqrt{1+2K^2+2A_N^2}\right)^{1/2}}{\sqrt{|\Omega|}\,\lambda_N}\sqrt{\sigma(\varphi + v_N^*) - \sigma(f)}.$$

We employ the following inequality

$$\frac{x}{\sqrt[4]{a+x^2}} \geqslant \sqrt[4]{a+x^2} - \frac{\sqrt{a}}{\sqrt[4]{a+x^2}},$$

which holds for $x \geqslant 0$ and $a > 0$. Letting $x = \sqrt{2}A_N$, $a = 1 + 2K^2$, by this inequality we obtain

$$A_N \leqslant \sup_{\Omega} |\nabla g_N| + \frac{1}{\sqrt{|\Omega|}\,\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy \right)^{1/2}$$

$$+ \left(\frac{3(1 + P_0^2)}{|\Omega|\,\lambda_N^2} (\sigma(\varphi + v_N^*) - \sigma(f)) \right)^{1/2} + 1. \tag{7}$$

Since function v_N^* is a solution to problem (3), we arrive at the estimate

$$\sup_{\Omega} |\nabla v_N^*| \leqslant \sup_{\Omega} |\nabla g_N| + \frac{1}{\sqrt{|\Omega|}\,\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy \right)^{1/2}$$

$$+ \left(\frac{3(1 + P_0^2)}{|\Omega|\,\lambda_N^2} (\sigma(\varphi + g_N) - \sigma(f)) \right)^{1/2} + 1. \tag{8}$$

In order to obtain the final inequality, we note that function $\varphi + g = f$ solves equation (1) and hence, $a'(0) = 0$, where $a(t) = \sigma(f + t(\varphi + g_N - f))$. Letting $f^t = f + t(\varphi + g^N - f)$, we have

$$\sigma(\varphi + g_N) - \sigma(f) = \int_0^1 ds \int_0^s a''(t)dt$$

$$= \int_0^1 ds \int_0^s dt \iint_{\Omega} \frac{(1 + |\nabla f^t|^2)|\nabla f - \nabla(\varphi + g_N)|^2 - \langle \nabla f^t, \nabla f - \nabla(\varphi + g_N) \rangle^2}{(1 + |\nabla f^t|^2)^{3/2}} dxdy$$

$$\leqslant \iint_{\Omega} |\nabla f - \nabla(\varphi + g_N)|^2 dxdy = \iint_{\Omega} |\nabla g - \nabla g_N|^2 dxdy.$$

Then by inequality (8) we arrive at the estimate

$$\sup_{\Omega} |\nabla v_N^*| \leqslant 1 + \sup_{\Omega} |\nabla g_N| + \frac{1 + \sqrt{3(1 + P_0^2)}}{\sqrt{|\Omega|}\,\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy \right)^{1/2}. \tag{9}$$

Thus, we have proved the following theorem.

Theorem 2. *Let f be a solution to equation (1) satisfying boundary condition (2). If $v_N^*(x,y) \in L_N$ is a solution to problem (3), its gradient satisfies estimate (9).*

Remark. *It follows from Theorem 2 that as $N \to \infty$, the gradients of approximated solutions $\varphi + v_N^*$ are uniformly bounded if the same is true for*

$$\frac{1}{\lambda_N} \left(\iint_{\Omega} |\nabla g - \nabla g_N|^2 \, dxdy \right)^{1/2}.$$

In this case the gradients of functions $g_N(x,y)$ are also bounded. To clarify these issues we need to estimate the rate of approximation of function g by polynomials g_N and to find out the behavior of sequence λ_N as $N \to \infty$.

4. APPROXIMATION OF SMOOTH FUNCTIONS BY POLYNOMIALS

Let Ω be a bounded domain with a boundary Γ, k be a natural number and function $\psi(x,y)$ satisfy the conditions:

1) function ψ is k times differentiable and its derivatives of kth order satisfy the Lipschitz condition;

2) $\psi(x,y) = 0$ as $(x,y) \in \Gamma$ and $\psi(x,y) \neq 0$ as $(x,y) \in \mathbb{R}^2 \setminus \Gamma$;

3) $|\nabla\psi(x,y)| > 0$ as $(x,y) \in \Gamma$.

Then, as it was shown in paper [13], given a function $u(x,y)$, which is k times continuously differentiable in $\overline{\Omega}$ and vanishes at Γ, we can find a sequence of polynomials $P_N(x,y)$ of degree $\leqslant N$ in each variable x, y such that

$$||u - \psi P_N||_{C^r(\Omega)} \leqslant \frac{\delta_N(u)}{N^{k-r}}, r = 0, 1, \ldots, k, \tag{10}$$

where $\delta_N(u) \to 0$ as $N \to \infty$.

Hereafter we assume that $g = f - \varphi \in C^k(\Omega)$. We specify the way of choosing function g_N: we let $g_N = \psi P_N$, where approximating polynomial is chosen for function $g = f - \varphi$. Applying for $r = 1$ estimates (10) for $u = f - \varphi$ in inequality (9), we obtain

$$\sup_{\Omega} |\nabla v_N^*| \leqslant 1 + K + P_0 + \frac{2 + \sqrt{3(1 + P_0^2)}}{\lambda_N} \frac{\delta_N(f - \varphi)}{N^{k-1}}. \tag{11}$$

By this inequality we see that the gradients of approximate solutions are bounded by a constant independent of N as $N \to \infty$ if λ_N tends to zero not faster than $O(1/N^{k-1})$. In the next section we shall study this issue.

5. ESTIMATE FOR λ_N

Let us provide an example of a lower bound for λ_N. It is clear that

$$\lambda_N = \inf_{v \in L_N} \frac{\left(\iint_{\Omega} |\nabla v|^2\, dxdy\right)^{1/2}}{\sqrt{|\Omega|}\, \sup_{\Omega} |\nabla v|} \geqslant \inf_{P} \frac{\left(\iint_{\Omega} |\nabla P|^2\, dxdy\right)^{1/2}}{\sqrt{|\Omega|}\, \sup_{\Omega} |\nabla P|},$$

where the infimum is taken over all polynomials of degree at most $N' = N + N_0$ in each variable.

We shall make use of the following inequality by A.A. Markov (see, for instance [14, Sect. 6]) for a polynomial $P(x)$ of one variable of degree N on a segment $[a,b]$

$$|P'(x)| \leqslant \frac{2N^2}{b - a} \max_{[a,b]} |P(x)|.$$

It implies easily a similar inequality for a rectangle in the case of polynomial of two variables. Let $P(x,y)$ be a polynomial, whose degree in each variable is at most N. We let

$$M = \max_{[a,b]\times[c,d]} |P(x,y)|.$$

Then by A.A. Markov inequality in each variable we have

$$\left|\frac{\partial P}{\partial x}\right| \leqslant \frac{2N^2}{b - a} \max_{[a,b]} |P(x,y)| \leqslant M\frac{2N^2}{b - a}, \quad \left|\frac{\partial P}{\partial y}\right| \leqslant M\frac{2N^2}{d - c}.$$

Therefore,

$$|\nabla P|^2 \leqslant 4N^4 M^2 \frac{(d - c)^2 + (b - a)^2}{(b - a)^2(d - c)^2}$$

or

$$|\nabla P| \leqslant 2N^2 M \frac{\sqrt{(d - c)^2 + (b - a)^2}}{(b - a)(d - c)} = 2N^2 M \frac{d}{S},$$

where d is the diagonal of the rectangle, and S is its area. We consider this inequality for a square with side $a > 0$. Then $d = a\sqrt{2}$, $S = a^2$. This is why

$$|\nabla P_N| \leqslant N^2 M \frac{2\sqrt{2}}{a}.$$

It is clear that this inequality is true for each square not necessarily having sides parallel to the axes. We let

$$\inf_P \frac{\left(\iint\limits_\Omega |P|^2\,dxdy\right)^{1/2}}{\sqrt{\Omega}\,\sup\limits_\Omega |P|} = \widetilde{\lambda}_N,$$

where the infimum is taken over all polynomials of degree at most N in each variable. We first find the estimate of $\widetilde{\lambda}_N$ for a square K with side $a > 0$. Let $z = (x,y) \in K$ and $z_0 \in K$ is such that $P(z_0) = \max\limits_K |P| = M$. Then

$$P(z_0) - P(z) \leqslant |z - z_0|\max_K |\nabla P| \leqslant N^2 \frac{2\sqrt{2}}{a} M|z - z_0|.$$

If $|z - z_0| < a/(4\sqrt{2}N^2)$, then $M - P(z) \leqslant \frac{M}{2}$. Thus, $P(x,y) \geqslant \frac{M}{2}$ for $(x,y) \in K \cap B_{\frac{a}{4\sqrt{2}N^2}}(z_0)$. Then

$$\iint\limits_K P^2(x,y)dxdy \geqslant \iint\limits_{K\cap B_{\frac{a}{4\sqrt{2}N^2}}(z_0)} P^2(z)dxdy \geqslant \frac{M^2}{4}\iint\limits_{K\cap B_{\frac{a}{4\sqrt{2}N^2}}(z_0)} dxdy \geqslant \frac{M^2}{16}\pi\frac{a^2}{32N^4}.$$

This is why

$$\left(\iint\limits_K P^2(x,y)dxdy\right)^{\frac{1}{2}} \geqslant \frac{Ma}{16\sqrt{2}N^2}\sqrt{\pi}.$$

Since P is an arbitrary admissible polynomial, then

$$\widetilde{\lambda}_N \geqslant \frac{1}{16}\sqrt{\frac{\pi}{2}}\frac{1}{N^2}.$$

We apply the proven inequality for partial derivative of polynomial P, which are also polynomials. Therefore,

$$\left(\iint\limits_K P_x^2(x,y)dxdy\right)^{\frac{1}{2}} \geqslant \widetilde{\lambda}_N \max_K |P_x|\sqrt{|K|}, \quad \left(\iint\limits_K P_y^2(x,y)dxdy\right)^{\frac{1}{2}} \geqslant \widetilde{\lambda}_N \max_K |P_y|\sqrt{|K|}.$$

It yields

$$|\nabla P|^2 \geqslant \widetilde{\lambda}_N^2 |K|((\max_K |P_x|)^2 + (\max_K |P_y|)^2) \geqslant \widetilde{\lambda}_N^2 |K|(\max_K |P_x|^2 + |P_y|^2) = |K|((\max_K |\nabla P|)^2.$$

Hence,

$$\frac{\left(\iint\limits_K |\nabla P|^2 dxdy\right)^{\frac{1}{2}}}{|K|\max\limits_K |\nabla P|} \geqslant \widetilde{\lambda}_N \geqslant \frac{1}{16N^2}\sqrt{\frac{\pi}{2}}.$$

Therefore,

$$\lambda_N \geqslant \frac{1}{16N^2}\sqrt{\frac{\pi}{2}}. \tag{12}$$

Thus, we have obtained the lower bound for λ_N in the case when Ω is a square with side $a > 0$. We observe that this estimate is independent of the size of square. Employing inequality (12), we obtain the estimate for an arbitrary planar domain Ω.

For each $z_0 \in \Omega$ there exists a maximal square $K \subset \overline{\Omega}$ with sides not necessarily parallel to the axes such that $z_0 \in K$. Let the side of the square be $a(z_0) > 0$. We denote

$$\Delta(\Omega) = \inf_{z_0 \in \Omega} a(z_0).$$

We shall assume that $\Delta(\Omega) > 0$. For instance, if Ω is a ball of radius $R > 0$, it is easy to see that for each point z_0 in this ball we have $a(z_0) = R\sqrt{2}$. This is why for the ball we have $\Delta(\Omega) = R\sqrt{2}$. As another example, we provide a domain inside the ellipse

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \ a > b.$$

At the point with the coordinates $(a, 0)$, the ellipse has the minimal curvature radius, which is equal to b^2/a. It is easy to obtain that each point $z_0 \in \Omega$ lies inside some ball of radius b^2/a lying inside the ellipse. This is why for the ellipse we have $\Delta(\Omega) \geqslant \sqrt{2}b^2/a$.

It is easy to estimate λ_N via $\Delta(\Omega)$. Indeed, let $z_0 \in \Omega$ be such that $\max\limits_{\Omega} |\nabla P| = |\nabla P(z_0)|$ and square $K \subset \overline{\Omega}$ contain this point. Then

$$\sqrt{\iint\limits_{\Omega} |\nabla P(x,y)|^2 dx dy} \geqslant \sqrt{\iint\limits_{K} |\nabla P(x,y)|^2 dx dy} \geqslant \sqrt{\frac{\pi}{2}} \frac{1}{16N^2} \sqrt{|K|} \max\limits_{K} |\nabla P|.$$

It implies easily the inequality

$$\lambda_N \geqslant \frac{\Delta(\Omega)}{16N^2 \sqrt{|\Omega|}} \sqrt{\frac{\pi}{2}}. \tag{13}$$

Hence, if domain Ω is such that each its point can be put inside a square $K \subset \overline{\Omega}$ with side $\Delta(\Omega) > 0$, then estimate (13) holds true. It is clear that if the domain has a smooth boundary, then $\Delta(\Omega) > 0$. For more general domains we first obtain the estimate of λ_N for a rhombus.

Assume that in plane with Cartesian coordinates (x, y) we are given a rhombus R with vertices at the points

$$(0, 0), \quad (a, 0), \quad (a\cos\alpha, a\sin\alpha), \quad (a(1 + \cos\alpha), a\sin\alpha),$$

where $\alpha \in (0, \pi/2]$, $a > 0$. By means of the linear transform

$$u = x\sin\alpha - y\cos\alpha, \quad v = y$$

on the plane with coordinates (u, v) we obtain square K with the vertices

$$(0, 0), \quad (a\sin\alpha, 0), \quad (0, a\sin\alpha), \quad (a\sin\alpha, a\sin\alpha),$$

Now let $P = P(x, y)$ be an arbitrary polynomial, whose degree in each variable does not exceed N. It is easy to see that

$$P_x^2 + P_y^2 = (P_u \sin\alpha)^2 + (-P_u \cos\alpha + P_v)^2 \geqslant (1 - \cos\alpha)(P_u^2 + P_v^2).$$

Then it follows from (12) that

$$\frac{\left(\iint\limits_{R} (P_x^2 + P_y^2) dx dy \right)^{1/2}}{\sqrt{|R|} \max\limits_{R} |\nabla P|} \geqslant \frac{\sqrt{1 - \cos\alpha}}{\sin\alpha} \cdot \frac{\left(\iint\limits_{K} (P_u^2 + P_v^2) du dv \right)^{1/2}}{\sqrt{|R|} \max\limits_{K} |\nabla P|}$$

$$= \frac{\sqrt{1 - \cos\alpha}}{\sin\alpha} \cdot \frac{\left(\iint\limits_{R} (P_u^2 + P_v^2) du dv \right)^{1/2}}{\sqrt{|K|} \max\limits_{K} |\nabla P|} \sqrt{\frac{|K|}{|R|}}$$

$$= \sqrt{\frac{1 - \cos\alpha}{\sin\alpha}} \cdot \frac{\left(\iint\limits_{R} (P_u^2 + P_v^2) du dv \right)^{1/2}}{\sqrt{|K|} \max\limits_{K} |\nabla P|} \geqslant \sqrt{\frac{\pi(1 - \cos\alpha)}{2\sin\alpha}} \frac{1}{16N^2}.$$

Hence, for the rhombus with a side $a > 0$ and an acute angle $\alpha \in (0, \pi/2]$ the inequality

$$\lambda_N \geqslant \sqrt{\frac{\pi(1 - \cos \alpha)}{2 \sin \alpha}} \frac{1}{16 N^2} \tag{14}$$

holds true.

Suppose that we are given a planar domain Ω. We denote by $\alpha(\Omega) \in (0, \pi/2]$ an angle such that each point of the domain is contained in the rhombus $R \subset \Omega$ with an acute angle $\alpha(\Omega)$. For each $z_0 \in \Omega$ we find a rhombus $R \subset \overline{\Omega}$ with maximal side such that $z_0 \in R$. Let the side of this rhombus is $a(z_0) > 0$. We let

$$\Delta_1(\Omega) = \inf_{z_0 \in \Omega} a(z_0) > 0.$$

Arguing as above, we obtain the inequality

$$\lambda_N \geqslant \sqrt{\frac{\pi}{|\Omega|}} \cdot \frac{\Delta_1(\Omega) \sin(\alpha(\Omega)/2)}{16 N^2}. \tag{15}$$

Taking into consideration that the polynomials in L_N are of degree at most $N_0 + N$, we arrive at the following statement.

Theorem 3. *Let a bounded domain $\Omega \subset \mathbb{R}^2$ be such that $\Delta_1(\Omega) > 0$ and $\alpha(\Omega) > 0$. Then the following estimate*

$$\lambda_N \geqslant \sqrt{\frac{\pi}{|\Omega|}} \cdot \frac{\Delta_1(\Omega) \sin(\alpha(\Omega)/2)}{16(N + N_0)^2} \tag{16}$$

holds true.

6.   Estimate for the rate of uniform convergence

We borrow the method of estimating the solutions from work [15]. Let f be a solution to the minimal equation surface in domain $\Omega \subset \mathbb{R}^2$, $f \in C^k(\overline{\Omega})$. Let v_N^* be a solution to problem (3) satisfying (4). We let $f_N^* = \varphi + v_N^*$.

We shall assume that

$$\sup_\Omega |\nabla f| = P_0 < +\infty.$$

We shall argue as in work [15]. We let $f^t(x, y) = f_N^*(x, y) + t(f(x, y) - f_N^*(x, y))$ and $P_N^* = \sup_\Omega |\nabla f_N^*|$, $P_N = \max\{1, P_0, P_N^*\}$. It is clear that $f^*|_{\partial\Omega} = f|_{\partial\Omega}$. We observe that, generally speaking, P_N depend on N. However, if we assume that $k > 2$ and constants $\Delta_1(\Omega)$ and $\alpha(\Omega)$ for the domain, then it follows from inequalities (15) and (9) that as $N \to \infty$, quantity P_N is bounded by some constant P.

Since as $t = 0$ the function $\sigma(t) = \sigma(f^t)$ takes the minimal value, then $\sigma'(0) = 0$. Employing this identity, we obtain

$$\sigma(f_N^*) - \sigma(f) = \int_0^1 ds \int_0^s \sigma''(t) dt$$

$$= \int_0^1 ds \int_0^s dt \iint_\Omega \frac{(1 + |\nabla f^t|^2)|\nabla f_N^* - \nabla f|^2 - \langle \nabla f^t, \nabla f_N^* - \nabla f \rangle^2}{(1 + |\nabla f^t|^2)^{3/2}} dx dy \tag{17}$$

$$\geqslant \int_0^1 ds \int_0^s dt \iint_\Omega \frac{|\nabla f - \nabla f_N^*|^2}{(1 + |\nabla f^t|^2)^{3/2}} dx dy \geqslant \frac{1}{\sqrt{(1 + P^2)^3}} \iint_\Omega |\nabla f - \nabla f_N^*|^2 dx dy.$$

We employ Poincaré inequality (see, for instance, [16, Subsect. 7.8]) for the function $h(x, y) = f(x, y) - f_N^*(x, y)$, $h|_{\partial\Omega} = 0$. By (17) we obtain

$$\sigma(f_N^*) - \sigma(f) \geqslant \frac{\lambda(\Omega)}{\sqrt{(1 + P^2)^3}} \iint\limits_{\Omega} |h(x, y)|^2 dx dy,$$

where constant $\lambda(\Omega) = \pi/|\Omega|$ and $|\Omega|$ is the area of domain Ω. We let $M = \sup\limits_{\Omega} |h|$ and without loss of generality we can assume that there exists a point $z_0 = (x_0, y_0) \in \Omega$, in which $h(x_0, y_0) = M$. Let us show that $B_{M/2P}(z_0) \subset \Omega$. Indeed, let $z' \in \partial\Omega$ be such that $|z_0 - z'| = \mathrm{dist}(z_0, \partial\Omega)$. Then

$$2P|z_0 - z'| \geqslant h(z_0) - h(z') = M - h(z') = M.$$

Thus, the distance from point z_0 to boundary $\partial\Omega$ is greater than $M/2P$. Therefore, $B_{M/4P}(z_0) \subset \Omega$. Suppose now that $z = (x, y) \in B_{M/4P}(z_0)$. Then

$$h(z) \geqslant h(z_0) - 2P|z - z_0| > M - 2P\frac{M}{4P} = M/2.$$

We obtain that $B_{M/4P}(z_0) \subset D_M$, where

$$D_M = \{(x, y) \in \Omega : |h(x, y)| > M/2\} \subset\subset \Omega.$$

This is why

$$\iint\limits_{\Omega} |h(x, y)|^2 dx dy \geqslant \iint\limits_{D_M} |h(x, y)|^2 dx dy$$

$$\geqslant \iint\limits_{B_{M/4P}(z_0)} \left(\frac{M}{2}\right)^2 dx dy = \pi\frac{M^2}{4}\left(\frac{M}{4P}\right)^2 = \pi\frac{M^4}{64P^2}.$$

Thus,

$$\max_{\Omega} |f - f_N^*| \leqslant \frac{4}{\sqrt{\pi}}\left(P^5|\Omega|(\sigma(f_N^*) - \sigma(f))\right)^{1/4}.$$

Then we observe that function v_N^* solves problem (3). Hence, $\sigma(f_N^*) - \sigma(f) \leqslant \sigma(f - \varphi - g_N) - \sigma(f)$. Employing the above proven inequality

$$\sigma(f - \varphi - g_N) - \sigma(f) \leqslant \iint\limits_{\Omega} |\nabla g - \nabla g_N|^2 dx dy$$

and estimate (10) for $u = f - \varphi$, we arrive at the inequality

$$\max_{\Omega} |f - f_N^*| \leqslant \frac{4}{\sqrt{\pi}}\left(P^5|\Omega|\frac{\delta_N^2(f - \varphi)}{N^{2k-2}}\right)^{1/4}.$$

We have proven the main result of the work.

Theorem 4. *Assume that $f \in C^k(\overline{\Omega})$, $k \geqslant 3$, be a solution to the minimal surface equation (1) in domain Ω, for which $\Delta_1(\Omega) > 0$ and $\alpha(\Omega) > 0$. Assume that this solution satisfies boundary condition (2) with a function $\varphi \in C^k(\overline{\Omega})$. Consider functions $v_N^* \in L_N$ being solutions to problem (3). Suppose that $P_0 = \sup\limits_{\Omega} |\nabla f| < +\infty$ and $K = \sup\limits_{\Omega} |\nabla\varphi| < \infty$. Then the sequence of approximate solutions $f_N^* = \varphi + v_N^*$ converge uniformly to f and the estimate*

$$\max_{\overline{\Omega}} |f - f_N^*| \leqslant \frac{4}{\sqrt{\pi}}\left(P^5|\Omega|\frac{\delta_N^2(f - \varphi)}{N^{2k-2}}\right)^{1/4}$$

holds true, where

$$P = 1 + 2K + P_0 + \frac{16\sqrt{|\Omega|}(2 + \sqrt{3(1 + P_0^2)})}{\sqrt{\pi}\Delta_1(\Omega)\sin\left(\alpha(\Omega)/2\right)}\frac{\delta_N(f - \varphi)}{N^{k-1}}(N + N_0)^2.$$

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