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ON SOME PROPERTIES OF SINC-APPROXIMATIONS OF CONTINUOUS FUNCTIONS ON INTERVAL

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Abstract. We study approximation properties of various operators being the modifications of sinc approximations of continuous functions on an interval.

Keywords: sinc approximation, interpolation functions, uniform approximation.

1. Introduction

E. Borel and E.T. Whittaker introduced independently the notion of a cardinal and a truncated cardinal function, whose restriction on the segment $[0, \pi]$ reads as follows:

$$L_n(f, x) = \sum_{k=0}^{n} \frac{\sin(nx - k\pi)}{nx - k\pi} f\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n} \frac{(-1)^k \sin nx}{nx - k\pi} f\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n} l_{k,n}(x) f\left(\frac{k\pi}{n}\right). \quad (1)$$

At present, the problem on sinc-approximation of a function decaying exponentially at infinity and analytic in a strip containing the real axis is studied in great details. The most complete survey of the results obtained in this direction by 1993 as well as many important applications of sinc-approximations can be found in [1]. Interesting historical surveys of studieds in this field were also provided in [2], [3].

Sinc-approximations have wide applications in constructing various numerical methods in mathematical physics and the approximation theory for the functions of both one and several variables [1], [4], in the theory of quadrature formulae [1], in the theory of wavelet-transforms or wavelets [5, Ch. 7, Sect. 4, Subsect. 2], [6, Ch. 2], [7], [8].

Interesting tests for the uniform convergence on the axis for Whittaker cardinal functions were provided in [9], [10].

Another important sufficient condition for convergence of sinc-approximations was obtained in paper [11]. It was established that for some subclasses of functions absolutely continuous together with their derivatives on the interval $(0, \pi)$ and having a bounded variation on the whole axis $\mathbb{R}$, Kotel'nikov series (or cardinal Whittaker functions) converge uniformly inside the interval $(0, \pi)$. In [12] an upper bound for the best possible approximation of continuous functions vanishing at the end-points of $[0, \pi]$ by linear combinations of sincs was obtained by an original approach. In works [13], [14], [15] there were obtained estimates for the error of uniform approximations of uniformly continuous and bounded on $\mathbb{R}$ functions by the values of various operators being combinations of sincs. We note that the structure of some operators considered in [13], [14] are similar with the operators studied in the present work.

Unfortunately, while approximating continuous functions on a segment by means of (1) and many other operators, Gibbs phenomenon arises in the vicinity of the segment end-points, see, for instance, [16] and [17].

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In [18], [19], [20] and [17] various estimates for the error of approximation of analytic in a circle functions by sinc-approximations (1) were obtained. To the best of the author's knowledge, before works [18], [19], [20] and [17], approximation by cardinal Whittaker functions on a segment or a bounded interval was made just for particular classes of analytic functions by reducing to the axis via a conformal mapping.

In paper [20] sharp estimates were established for the functions and Lebesgue constants of operator (1) and an analogue of G.P. Névai formula was obtained, which was appropriate for studying approximative properties of operator (1). Works [21], [22] were devoted to obtaining necessary and sufficient conditions of pointwise and uniform in the interval $(0,\pi)$ convergence of sinc-approximations (1) for continuous on $[0,\pi]$ functions. The authors of an interesting paper [23] employed the results of work [21] to study the convergence of the algorithms of multi-level sinc-approximations of functions with a minimal smoothness.

In [24] there was constructed an example of a continuous function vanishing at the end-points of the segment $[0,\pi]$ for which the sequence of the values of operators (1) diverges unboundedly everywhere on the interval $(0,\pi)$. The results of [24] show that while attempting to approximate non-smooth continuous functions by the values of operators (1) a "resonance" can appear that produces an unbounded growth of approximation error everywhere on the interval $(0,\pi)$. In the same work there was proven the absence of the equiconvergence of the values of operators (1) and Fourier series or integral on the class of continuous functions.

Work [25] was denoted to studying approximative properties of interpolation operators constructed by means of solutions to the Cauchy problems with second order differential expressions. In [26] the results of work [25] were applied for studying approximative properties of classical Lagrange interpolation processes with the matrix of interpolation nodes, whose each row consists of zeroes of Jacobi polynomials $P_n^{\alpha_n,\beta_n}$ with the parameters depending on n. Papers [27] and [28] were devoted to applications of the considered in [25] operators to studying Lagrange-Sturm-Liouville interpolation processes.

This short historical background does not, of course, pretend for a complete survey of all works devoted to sampling theorem and its generalization. Moreover, we do not cite the papers in a huge series of works devoted to numerous applications of this direction in the mathematical analysis to the related fields of natural sciences.

In the present work we follow the lines of publications [29]–[36] and study the issues on approximations of continuous on the segment $[0,\pi]$ functions by means of linear combinations of sincs $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ and linear functions. At that, as an information on the approximated function, we employ only its values at the nodes $x_{k,n}=\frac{\pi k}{n}$ $0\leqslant k\leqslant n$, $n=1,2,3,\ldots$. The main attention in the present study is paid to the following issues. The first one is how to compensate the appearance of the undesirable "resonance" while approximating non-smooth functions of fractal kind. The second issue is whether one can suggest operators having no Gibbs phenomenon (Wilbraham-Gibbs phenomenon) in the vicinity of the end-points of the segment $[0,\pi]$ and whether there is a chance to keep at that the interpolation property for the new operators.

To each function f taking finite values on the set $x_{k,n}=\frac{\pi k}{n}$, $n\in\mathbb{N}$, $0\leqslant k\leqslant n$, we associate an entire function LT_n by the rule:

$$\begin{aligned} LT_n(f,x)=&\sum_{k=1}^{n-1}\left\{f(x_{k,n})-\frac{\big(f(\pi)-f(0)\big)k}{n}-f(0)\right\}\frac{\cos nx\sin nx}{nx-k\pi}\\ &-\sum_{k=0}^{n-1}\left\{\frac{f(x_{k+1,n})+f(x_{k,n})}{2}-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)\right\}\frac{\sin nx\cos nx}{nx-(k+\frac{1}{2})\pi}\\ &+\frac{f(\pi)-f(0)}{\pi}x+f(0). \end{aligned} \tag{2}$$

It should be stressed that as an information on function f, operator (2) employs only its values at the nodes $x_{k,n} = \frac{\pi k}{n}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$. Moreover, $\cos nx_{k,n} = (-1)^k$ as $n \in \mathbb{N}$, $0 \leqslant k \leqslant n$, and this is why the first term in the definition of operator (2) is in fact a slightly "corrected" operator of sinc-approximation (1). Second term in (2) compensates the undesirable resonance if it appears in approximating non-smooth functions. Hence, operator (2) has the same approximative properties as operators (13) despite the values of this operator are smooth enough and interpolate an approximated function, i.e., $f(x_{k,n}) = LT_n(f, x_{k,n})$ for each $n \in \mathbb{N}$, $0 \leqslant k \leqslant n$. The trick used in constructing operator $T_\lambda(f, \cdot)$ [25, Formula (1.9)] allows us to get rid of the Gibbs effect in the vicinity of the end-points of the segment $[0, \pi]$ while approximating functions be means of operator (2).

For the numerical mathematics, an equivalent more compact representation of operator (2) can be useful:

$$LT_n(f,x) \equiv \sum_{k=1}^{n-1}\left(f(x_{k,n}) - \frac{\big(f(\pi)-f(0)\big)k}{n} - f(0)\right)\left\{\frac{\pi^2 \sin 2nx}{2(nx-k\pi)\big(\pi^2-4(nx-k\pi)^2\big)}\right\} + \frac{f(\pi)-f(0)}{\pi}x + f(0).$$

Theorem 1. *For each function f continuous on the segment $[0, \pi]$ the relation*

$$\lim_{n\to\infty} \|f - LT_n(f, \cdot)\|_{C[0,\pi]} = 0.$$

holds true.

We denote by $C_0[0, \pi]$ the space of continuous function vanishing at the end-points of the segment with the Chebyshev norm, i.e., $C_0[0, \pi] = \{f : f \in C[0, \pi],\ f(0) = f(\pi) = 0\}$.

The results of the present work allow us to state the completeness of the elements $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ in normed spaces $C[0, \pi]$ and $C_0[0, \pi]$.

Corollary 1. *The system* $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ *is complete in* $C_0[0, \pi]$ *that is in agreement with the results of work* [12]. *A system of functions* $\{1, x\} \cup \{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ *is complete in* $C[0, \pi]$.

Moreover, by none of linear combinations of the functions in the system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ one can approximate an arbitrary element in space $C[0, \pi]$.

Theorem 2. *Linear spans of the systems of functions*

$$\{l_{k,n}\}_{k=0}^{n}, n \in \mathbb{N} \tag{3}$$

are not dense in $C[0, \pi]$.

2. Auxiliary statements

We begin with some auxiliary statements, which we shall employ in what follows.

Proposition 1. [20, Thm. 2] *If a function f is continuous on the segment $[0, \pi]$, then for each $x \in [0, \pi]$ the identities*

$$\lim_{n\to\infty}\left(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\big(f(x_{k+1,n}) - f(x_{k,n})\big)l_{k,n}(x)\right) = 0 \tag{4}$$

hold true, where

$$l_{k,n}(x) = \frac{(-1)^k \sin nx}{nx - k\pi}.$$

The convergence in (4) is pointwise on the segment $[0, \pi]$ and is uniform inside the interval $(0, \pi)$, i.e., it is uniform on each compact set contained in this interval.

Under the assumption $\rho_\lambda \geqslant 0$, for each nonnegative λ we assume that function q_λ is such that

$$V_0^\pi[q_\lambda] \leqslant \rho_\lambda, \quad \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln \lambda}\right), \quad \text{as } \lambda \to \infty, \quad q_\lambda(0) = 0. \tag{5}$$

Then for each potential $q_\lambda \in V_{\rho_\lambda}[0,\pi]$ the zeroes of the solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 1, \\ y'(0,\lambda) = h(\lambda), \end{cases} \tag{6}$$

as $\lambda \to +\infty$, or under the additional condition $h(\lambda) \neq 0$, the zeroes of the solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 0, \\ y'(0,\lambda) = h(\lambda), \end{cases} \tag{7}$$

locating in $[0,\pi]$ and taken in the ascending order are denoted by

$$0 \leqslant x_{0,\lambda} < x_{1,\lambda} < \ldots < x_{n(\lambda),\lambda} \leqslant \pi \quad (x_{-1,\lambda} < 0, x_{n(\lambda)+1,\lambda} > \pi). \tag{8}$$

Here $x_{-1,\lambda} < 0$, $x_{n(\lambda)+1,\lambda} > \pi$ stand for the zeroes of the continuation of the solution to Cauchy problem (6) or (7) with some continuation of function q_λ outside the segment $[0,\pi]$ keeping a bounded variation. For Cauchy problem (7) we also assume that function $h(\lambda)$ is non-zero, i.e.,

$$V_0^\pi[q_\lambda] \leqslant \rho_\lambda, \quad \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln \lambda}\right), \text{ as } \lambda \to \infty, \quad q_\lambda(0) = 0, \quad h(\lambda) \neq 0. \tag{9}$$

In [25] the approximative properties were studided for Lagrange type operators constructed by the solutions to Cauchy problems (6) or (7) and mapping each function f on $[0,\pi]$ into a continuous function interpolating it at the nodes $\{x_{k,\lambda}\}_{k=0}^n$ so that

$$S_\lambda(f,x) = \sum_{k=0}^n \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})} f(x_{k,\lambda}) = \sum_{k=0}^n s_{k,\lambda}(x) f(x_{k,\lambda}). \tag{10}$$

In particular, the following proposition was proved.

Proposition 2. [25, Prop. 9] *Let $y(x,\lambda)$ be solutions to Cauchy problem* (6) *or* (7). *Cauchy problem* (6) *satisfies relations* (5), *while Cauchy problem* (7) *satisfies* (9).

If $f \in C_0[0,\pi]$, then the relation

$$\lim_{\lambda\to\infty}\Big(f(x) - S_\lambda(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\{f(x_{k+1,\lambda}) - f(x_{k,\lambda})\}s_{k,\lambda}(x)\Big) = 0$$

holds true uniformly on $x \in [0,\pi]$ and in all $q_\lambda \in V_{C_\lambda}[0,\pi]$, where $s_{k,\lambda}(x) = \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})}$.

Remark 1. [25, Prop. 9] *In the same one can that the following proposition holds true under assumption 2. If $f \in C_0[0,\pi]$, then the relations*

$$\lim_{\lambda\to\infty}\Big(f(x) - S_\lambda(f,x) - \frac{1}{2}\sum_{k=1}^{n}\{f(x_{k-1,\lambda}) - f(x_{k,\lambda})\}s_{k,\lambda}(x)\Big) = 0,$$

$$\lim_{\lambda\to\infty}\Big(f(x) - S_\lambda(f,x) - \frac{1}{4}\sum_{k=1}^{n-1}\{f(x_{k+1,\lambda}) - 2f(x_{k,\lambda}) + f(x_{k-1,\lambda})\}s_{k,\lambda}(x)\Big) = 0$$

hold true uniformly in $x \in [0,\pi]$ and all $q_\lambda \in V_{C_\lambda}[0,\pi]$.

Corollary 2. *If $f \in C_0[0,\pi]$, then Proposition 2 holds true uniformly in $x \in [0,\pi]$ as $\lambda_n = n^2$, $h(\lambda) \neq 0$, $q_\lambda \equiv 0$, $S_{\lambda_n}(f,x) \equiv L_n(f,x)$, and $s_{k,\lambda_n}(x) \equiv l_{k,n}(x)$.*

Proof. In the case of Cauchy problem (7), as $\lambda_n = n^2$, $h(\lambda) \neq 0$, $q_\lambda \equiv 0$, operator (10) turns into (1), $l_{k,n}(x) \equiv s_{k,\lambda_n}(x)$. □

For approximating non-smooth continuous functions, for instance, functions f having a fractal character, we introduce new operators. For instance, operators $A_n(f,x)$ and $\widetilde{A}_n(f,x)$ maps each continuous on segment $[0,\pi]$ function f into a linear combination of sincs by the rules

$$A_n(f,x) = \sum_{k=1}^{n} \frac{l_{k,n}(x) + l_{k-1,n}(x)}{2} f(x_{k,n}), \tag{11}$$

$$\widetilde{A}_n(f,x) = \sum_{k=0}^{n-1} \frac{f(x_{k,n}) + f(x_{k+1,n})}{2} l_{k,n}(x). \tag{12}$$

We observe that the values of $A_n(f,x)$ and $\widetilde{A}_n(f,x)$ in space $C_0[0,\pi]$ coincide and they behave similarly in $C[0,\pi]$ at the internal points in $(0,\pi)$. Here we provide the results in terms of both operators in order not to recheck these facts while using (11) and (12) in applications.

The modification of these operators by the trick, which allows to get rid of the Gibbs phenomenon in the vicinity of the end-points of the segment $[0,\pi]$, are denoted by

$$\begin{aligned} AT_n(f,x) = &\sum_{k=1}^{n} \frac{l_{k,n}(x) + l_{k-1,n}(x)}{2} \left\{ f(x_{k,n}) - \frac{\big(f(\pi) - f(0)\big)k}{n} - f(0) \right\} \\ &+ \frac{f(\pi) - f(0)}{\pi} x + f(0) \\ = &\sum_{k=0}^{n-1} \left\{ \frac{f(x_{k+1,n}) + f(x_{k,n})}{2} - \frac{\big(f(\pi) - f(0)\big)(2k+1)}{2n} - f(0) \right\} l_{k,n}(x) \\ &+ \frac{f(\pi) - f(0)}{\pi} x + f(0). \end{aligned} \tag{13}$$

Proposition 3. *Let $f \in C[0,\pi]$. Then*

$$\lim_{n\to\infty} AT_n(f,x) = f(x) \tag{14}$$

uniformly on $[0,\pi]$.

Proof. We first note that in accordance with Corollary 2 of Proposition 2 for $f \in C_0[0,\pi]$, the identity

$$\begin{aligned} &\lim_{n\to\infty} \Big(f(x) - L_n(f,x) - \frac{1}{2} \sum_{k=0}^{n-1} \Big(f(x_{k+1,n}) - f(x_{k,n}) \Big) l_{k,n}(x) \Big) \\ &\qquad = \lim_{n\to\infty} f(x) - A_n(f,x) = \lim_{n\to\infty} f(x) - \widetilde{A}_n(f,x) = 0 \end{aligned}$$

holds true uniformly on $[0,\pi]$.

In order to prove (14), we note that function $f(x) - \frac{f(\pi)-f(0)}{\pi} x - f(0)$ belongs to space $C_0[0,\pi]$. And therefore,

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=0}^{n-1} \left\{ \frac{f(x_{k+1,n}) + f(x_{k,n})}{2} - \frac{\big(f(\pi) - f(0)\big)(2k+1)}{2n} - f(0) \right\} l_{k,n}(x) \\ &\qquad = f(x) - \frac{f(\pi) - f(0)}{\pi} x - f(0), \end{aligned}$$

uniformly on segment $[0,\pi]$, i.e., (14) holds true. □

It is also possible to consider operators similar to (11), (12), (13):

$$B_n(f,x) = \sum_{k=0}^{n-1} \frac{l_{k,n}(x) + l_{k+1,n}(x)}{2} f(x_{k,n}),$$

$$\widetilde{B}_n(f,x) = \sum_{k=1}^{n} \frac{f(x_{k-1,n}) + f(x_{k,n})}{2} l_{k,n}(x),$$

$$BT_n(f,x) = \sum_{k=0}^{n-1} \frac{l_{k,n}(x) + l_{k+1,n}(x)}{2} \left\{ f(x_{k,n}) - \frac{\big(f(\pi) - f(0)\big)k}{n} - f(0) \right\} + \frac{f(\pi) - f(0)}{\pi} x + f(0)$$
$$= \sum_{k=1}^{n} \left\{ \frac{f(x_{k-1,n}) + f(x_{k,n})}{2} - \frac{\big(f(\pi) - f(0)\big)(2k-1)}{2n} - f(0) \right\} l_{k,n}(x)$$
$$+ \frac{f(\pi) - f(0)}{\pi} x + f(0).$$

Finally, to get rid of the asymmetry in the introduced operators, we let

$$C_n(f,x) = \sum_{k=1}^{n-1} \frac{l_{k+1,n}(x) + 2l_{k,n}(x) + l_{k-1,n}(x)}{4} f(x_{k,n}), \tag{15}$$

$$\widetilde{C}_n(f,x) = \sum_{k=1}^{n-1} \frac{f(x_{k+1,n}) + 2f(x_{k,n}) + f(x_{k-1,n})}{4} l_{k,n}(x). \tag{16}$$

We denote the modification of these operators obtained by applying the trick allowing to get rid of the Gibbs phenomenon in the vicinity of the end-points of the segment by

$$CT_n(f,x) = \sum_{k=1}^{n-1} \frac{l_{k+1,n}(x) + 2l_{k,n}(x) + l_{k-1,n}(x)}{4} \left\{ f(x_{k,n}) - \frac{\big(f(\pi) - f(0)\big)k}{n} - f(0) \right\} l_{k,n}(x)$$
$$+ \frac{f(\pi) - f(0)}{\pi} x + f(0),$$

$$\widetilde{CT}_n(f,x) = \sum_{k=1}^{n-1} \left\{ \frac{f(x_{k+1,n}) + 2f(x_{k,n}) + f(x_{k-1,n})}{4} - \frac{\big(f(\pi) - f(0)\big)k}{n} - f(0) \right\} l_{k,n}(x)$$
$$+ \frac{f(\pi) - f(0)}{\pi} x + f(0).$$

Remark 2. *Following the lines the proof of Proposition 3 we can also prove the following statement. Let* $f \in C[0,\pi]$. *Then*

$$\lim_{n\to\infty} BT_n(f,x) = f(x), \qquad \lim_{n\to\infty} CT_n(f,x) = \lim_{n\to\infty} \widetilde{CT}_n(f,x) = f(x)$$

uniformly on $[0,\pi]$.

Unfortunately, the proposed operators do not possess such interpolation properties like L_n, i.e., generally speaking, the value of operators A_n, AT_n, B_n, BT_n, C_n, CT_n, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{CT}_n$ not necessarily coincide with the approximated function at points $x_{k,n} = \frac{k\pi}{n}$, $0 \leqslant k \leqslant n$, $n \in \mathbb{N}$. On the other hand, their approximative properties are much less sensitive to the smoothness properties of the approximated function. By their means we can approximate arbitrary element in space $C[0,\pi]$.

Remark 3. *In the theory of approximating functions by algebraic polynomials, Bernstein processes by Chebyshev nodes matrix are well known* [37, see formula (11) and the previous one], *which are in some sense identical to construction* $\widetilde{A}_n$ *(12) and* $\widetilde{C}_n$ *(16). We also note* *that an operator similar to C_n (15) was employed by V.P. Sklyarov in the proof of Theorem 1 in* [12]. *Being continued to the whole axis it becomes the Blackman-Harris operator as $m = 1$, $a_0 = a_1 = 0,5$* [13, Formula (9)]. *The methods for studying approximative properties of the operators considered by S.N. Bernstein, V.P. Sklyarov, by the authors of* [13] *and the approach proposed in the present work differ essentially.*

Remark 4. *If together with operators (11), (12), (15), (16) we consider, for instance, the operators*

$$\sum_{k=1}^{n-1} \frac{f(x_{k+1,n}) + f(x_{k-1,n})}{2} l_{k,n}(x)$$

or

$$\sum_{k=1}^{n-1} \frac{l_{k+1,n}(x) + l_{k-1,n}(x)}{2} f(x_{k,n}),$$

then to guarantee the convergence of their value to the approximated function f, one needs adequate necessary and sufficient conditions (for instance, the conditions proposed in [21, Thms. 1, 2]*).*

3. Study of the completeness of the sincs system in $C_0[0,\pi]$ and $C[0,\pi]$

The results of the previous section allows to conclude on the completeness of the system of elements $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ in normed spaces $C[0,\pi]$ and $C_0[0,\pi]$.

Proof of Corollary 1. Corollary 2 and Proposition 3 imply Corollary 1. □

Proof of Theorem 2. Let us show that the linear spans of the systems of functions (3) are not dense in $C[0,\pi]$. System (3) is Chebyshev system [38], [39], i.e., the linear spans of functions (3) are Chebyshev spaces [38, Ch. 1, Sect. 2]. Indeed, first, these are continuous functions. Second, each generalized polynomial

$$\sum_{k=0}^{n} a_{k,n} l_{k,n}(x) = \frac{\sin nx}{\omega_n(x)} \sum_{k=0}^{n} \frac{a_{k,n}\omega_n'(x_{k,n})}{(-1)^k n} \frac{\omega_n(x)}{\omega_n'(x_{k,n})(x - x_{k,n})},$$

where $\omega_n(x) = \prod_{k=0}^{n}(x - x_{k,n})$, can have at most n zeroes as the product of a polynomial of degree n by the entire function $\frac{\sin nx}{\omega_n(x)}$, non-vanishing on the segment $[0,\pi]$. By Haar theorem [38, Ch. 1, Sect. 2] or by Bernstein theorem [39, Ch. IX, Sect. 1], for each element $f \in C[0,\pi]$ there exists the unique best approximation element

$$\Big\| f - \sum_{k=0}^{n} p_{k,n} l_{k,n} \Big\|_{C[0,\pi]} = \inf_{a_{k,n}\in\mathbb{R}} \Big\| f - \sum_{k=0}^{n} a_{k,n} l_{k,n} \Big\|_{C[0,\pi]} = E_n(f).$$

Let us consider function $f \equiv 1$. Then as $n \geqslant 2$,

$$\left| \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n}\Big) - \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n} + \frac{2\pi}{n}\Big) \right| \leqslant 2E_n(1).$$

By the biorthogonality of systems (3) and $\{x_{k,n}\}_{k=0}^{n,}$ $n \in \mathbb{N}$, the relations $1 - E_n(1) \leqslant p_{k,n} \leqslant 1 + E_n(1)$ hold true for all $0 \leqslant k \leqslant n$, $n \in \mathbb{N}$. If there exists a sequence $n_i \nearrow \infty$ as $i \to \infty$ such that $E_{n_i}(1) \geqslant 1$, then it proves the theorem. Otherwise we estimate the difference

$$\begin{aligned} 2E_n(1) \geqslant & \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n}\Big) - \sum_{k=0}^{n} p_{k,n} l_{k,n}\Big(\frac{\pi}{2n} + \frac{2\pi}{n}\Big) \\ = & \frac{8}{\pi}\left\{ p_{0,n}\frac{1}{5} + p_{1,n}\frac{1}{3} - p_{2,n}\frac{1}{3} - \sum_{j=0}^{n-3} \frac{(-1)^j p_{j+3,n}}{(2j+5)(2j+1)} \right\} \end{aligned}$$

$$\geqslant \frac{8}{\pi}\Big\{\big(1-E_n(1)\big)\frac{1}{5}+\big(1-E_n(1)\big)\frac{1}{3}-\big(1+E_n(1)\big)\frac{1}{3}$$
$$+\big(1-E_n(1)\big)\sum_{m=0}^{[\frac{n-3}{2}]}\frac{1}{(4m+7)(4m+3)}-\big(1+E_n(1)\big)\sum_{m=0}^{[\frac{n-3}{2}]+1}\frac{1}{(4m+1)(4m+5)}\Big\}.$$

Suppose that

$$E_n(1)\to 0,\ \text{as } n\to\infty. \tag{17}$$

Taking into consideration that

$$\sum_{m=0}^{\infty}\frac{1}{(4m+1)(4m+5)}=\frac{1}{4},\quad \sum_{m=0}^{\infty}\frac{1}{(4m+3)(4m+7)}=\frac{1}{12},$$

see [40, Sect. 5.1.11, Subsects. 4, 14]) and passing to the limit as $n\to\infty$ we obtain a contradiction to Proposition (17). Therefore, it is impossible to approximate uniformly on $[0,\pi]$ even function $f\equiv 1$ by any no linear combination of functions in system (3). The proof is complete. □

Lemma 1. [21, Lm. 1] *For all $x\in[0,\pi]$ and $n\in\mathbb{N}$ the inequality*

$$\sum_{k=1}^{n}\big|l_{k,n}(x)+l_{k-1,n}(x)\big|\leqslant 4\left(1+\frac{1}{\pi}\right)$$

holds true, where

$$l_{k,n}(x)=\frac{(-1)^k\sin nx}{nx-k\pi}.$$

Lemma 1 implies the boundedness of the sequence of the Lebesgue constants for operators A_n defined by (11):

$$\|A_n\|_{C[0,\pi]\to C[0,\pi]}\leqslant 2\left(1+\frac{1}{\pi}\right)\quad \text{for each}\quad n\in\mathbb{N}.$$

Unfortunately, this fact does not imply, for instance, relation (18). The reason is that due to Banach-Steinhaus theorem [41, Ch. 4, Thm. 2], we also need to prove the existence of a subset M_0 in the set of continuous functions vanishing at the end-points of the segment $[0,\pi]$, whose linear combinations are dense in $C_0[0,\pi]$ and such that for each $f\in M_0$

$$\lim_{n\to\infty}A_n(f,x)=f(x)\quad\text{uniformly on}\quad[0,\pi].$$

However, the following proposition is true.

Proposition 4. *Let $f\in C[0,\pi]$. Then the relations*

$$\lim_{n\to\infty}A_n(f,x)=\lim_{n\to\infty}\widetilde{A}_n(f,x)=f(x) \tag{18}$$

hold true uniformly on $(0,\pi)$, i.e., uniformly on any compact subset of the interval $(0,\pi)$. The convergence in (18) is uniform in $[0,\pi]$ if and only if $f\in C_0[0,\pi]$.

Proof. Let us prove (18) for an arbitrary function f continuous on $[0,\pi]$. We rewrite the left hand side of (4) in accordance with definitions (11) and (12) as follows,

$$\begin{aligned}&\lim_{n\to\infty}\left(f(x)-L_n(f,x)-\frac{1}{2}\sum_{k=0}^{n-1}\Big(f(x_{k+1,n})-f(x_{k,n})\Big)l_{k,n}(x)\right)\\&=\lim_{n\to\infty}\big(f(x)-\widetilde{A}_n(f,x)-f(\pi)l_{n,n}(x)\big)\\&=\lim_{n\to\infty}\left(f(x)-A_n(f,x)-\frac{f(\pi)}{2}l_{n,n}(x)-\frac{f(0)}{2}l_{0,n}(x)\right).\end{aligned}\tag{19}$$

We take an arbitrary segment $[a,b] \subset (0,\pi)$. In accordance with Proposition 1, relation (4) holds true uniformly on $[a,b]$, i.e., in the sense of the uniform convergence in $[a,b]$ the limits (19) are zero. But for each $x \in [a,b]$

$$\big|f(\pi)l_{n,n}(x)\big| \leqslant \|f\|_{C[0,\pi]}\frac{1}{n(\pi-b)} \to 0 \quad \text{as} \quad n\to\infty,$$
$$\big|f(0)l_{0,n}(x)\big| \leqslant \|f\|_{C[0,\pi]}\frac{1}{na} \to 0 \quad \text{as} \quad n\to\infty.$$

Let $f \in C_0[0,\pi]$. Then in view of Proposition 2, the identities

$$\lim_{n\to\infty}\Big(f(x)-L_n(f,x)-\frac{1}{2}\sum_{k=0}^{n-1}\Big(f(x_{k+1,n})-f(x_{k,n})\Big)l_{k,n}(x)\Big)$$
$$=\lim_{n\to\infty} f(x)-A_n(f,x)=\lim_{n\to\infty} f(x)-\widetilde{A}_n(f,x)=0$$

hold true uniformly on $[0,\pi]$. Hence, the belonging of function f to space $C_0[0,\pi]$ is sufficient for the uniform convergence in (18).

Theorem 2 implies that the belonging of function f to space $C_0[0,\pi]$ is necessary for the convergence in (18) to be uniform on $[0,\pi]$. □

Remark 5. *In the same way we establish that given $f\in C[0,\pi]$, the relations*

$$\lim_{n\to\infty} B_n(f,x)=\lim_{n\to\infty}\widetilde{B}_n(f,x)=f(x) \tag{20}$$
$$\lim_{n\to\infty} C_n(f,x)=\lim_{n\to\infty}\widetilde{C}_n(f,x)=f(x) \tag{21}$$

hold true uniformly on $(0,\pi)$. The convergence in (20) and (21) is uniform on $[0,\pi]$ if and only if $f\in C_0[0,\pi]$.

Remark 6. *We note that while constructing operators AT_n, BT_n, CT_n, $\widetilde{CT}_n$, instead of functions $\{1,x\}$, system $\{l_{k,n}\}_{k=0,n=1}^{n,\ \infty}$ can be completed by another convenient pair of linearly independent functions, for instance, $\{l_{0,1},l_{1,1}\}$.*

Before proving Theorem 1, let us prove one auxiliary statement.

Lemma 2. *For each continuous on the segment $[0,\pi]$ function f we have the following representation for the approximation error by means of operators LT_n*

$$\begin{aligned}|f(x)-LT_n(f,x)| = \Bigg|f(x)-\Bigg[&\sum_{k=0}^{n-1}\Big\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)k}{n}-f(0)\Big\}\frac{\sin 2nx}{2(nx-k\pi)}\\
&+\sum_{k=0}^{n-1}\Big\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\Big\}\Big(\frac{\sin 2nx}{2(nx-k\pi)}-\frac{\sin 2nx}{2nx-(2k+1)\pi}\Big)\\
&-\sum_{k=0}^{n-1}\Big\{\frac{f(x_{2k+2,2n})+f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)\Big\}\frac{\sin 2nx}{2nx-(2k+1)\pi}\\
&+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]\\
&+\sum_{k=0}^{n-1}\Big\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\Big\}\Big(\frac{\sin 2nx}{2(nx-k\pi)}-\frac{\sin 2nx}{2nx-(2k+1)\pi}\Big)\Bigg|.\end{aligned}$$

Proof. We choose arbitrary function continuous on the segment $[0,\pi]$. Since as $k=0$, $f(x_{k,n})-\frac{\big(f(\pi)-f(0)\big)k}{n}-f(0)=0$, we obtain the identity

$$|f(x)-LT_n(f,x)|=\Bigg|f(x)-\Bigg[\cos nx\sum_{k=0}^{n-1}\Big\{f(x_{k,n})-\frac{\big(f(\pi)-f(0)\big)k}{n}-f(0)\Big\}\frac{(-1)^{2k}\sin nx}{nx-k\pi}$$
$$+\sin nx\sum_{k=0}^{n-1}\Big\{\frac{f(x_{k+1,n})+f(x_{k,n})}{2}-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)\Big\}\frac{(-1)^{2k+1}\cos nx}{nx-(k+\frac{1}{2})\pi}$$
$$+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]\Bigg|$$
$$=\Bigg|f(x)-\Bigg[\cos nx\sum_{k=0}^{n-1}\Big\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\Big\}\frac{(-1)^{2k}\sin nx}{nx-k\pi}$$
$$+\sin nx\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+2,2n})-\frac{\big(f(\pi)-f(0)\big)(2k+2)}{2n}-f(0)+f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)}{2}\Bigg\}$$
$$\cdot\frac{(-1)^{2k+1}\cos nx}{nx-(k+\frac{1}{2})\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]\Bigg|.$$

To the obtained representation we add the term

$$\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\big(f(\pi)-f(0)\big)2k}{2n}+f(0)}{2}\Bigg\}$$
$$\cdot\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)$$

and deduct it. Then we have the relation

$$|f(x)-LT_n(f,x)|$$
$$=\Bigg|f(x)-\Bigg[\cos nx\sum_{k=0}^{n-1}\Big\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\Big\}\frac{(-1)^{2k}\sin nx}{nx-k\pi}$$
$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\big(f(\pi)-f(0)\big)2k}{2n}+f(0)}{2}\Bigg\}$$
$$\cdot\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)$$
$$+\sin nx\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+2,2n})-\frac{\big(f(\pi)-f(0)\big)(2k+2)}{2n}-f(0)+f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)}{2}\Bigg\}$$
$$\cdot\frac{(-1)^{2k+1}\cos nx}{nx-(k+\frac{1}{2})\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$
$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\big(f(\pi)-f(0)\big)2k}{2n}+f(0)}{2}\Bigg\}$$

$$\cdot\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.$$

By further calculations we obtain the representation

$$|f(x)-LT_n(f,x)| = \Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\left\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\right\}\frac{\sin 2nx}{2nx-2k\pi}$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(x_{2k,2n})+\frac{(f(\pi)-f(0))2k}{2n}}{2}\right\}$$
$$\cdot\left(\frac{\sin 2nx}{2nx-2k\pi}-\frac{\sin 2nx}{2nx-(2k+1)\pi}\right)$$
$$-\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+2,2n})-\frac{(f(\pi)-f(0))(2k+2)}{2n}+f(x_{2k,2n})-\frac{(f(\pi)-f(0))2k}{2n}-2f(0)}{2}\right\}$$
$$\cdot\frac{\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\right\}\left(\frac{\sin 2nx}{2nx-2k\pi}-\frac{\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.$$

It completes the proof. □

Proof of Theorem 1. Given an arbitrary continuous on the segment $[0,\pi]$ function f, let us estimate the absolute value of its deviation from the value of operator (2) for each natural n. By Lemma 2, the deviation of operator LT_n from function f can be represented as

$$|f(x)-LT_n(f,x)| = \Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\left\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\right\}\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{(f(\pi)-f(0))2k}{2n}+f(0)}{2}\right\}$$
$$\cdot\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+2,2n})-\frac{(f(\pi)-f(0))(2k+2)}{2n}-f(0)+f(x_{2k,2n})-\frac{(f(\pi)-f(0))2k}{2n}-f(0)}{2}\right\}$$
$$\cdot\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$
$$+\sum_{k=0}^{n-1}\left\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\right\}\left(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right)\Bigg|.$$

We open brackets in the second term of the sum enclosed in square brackets and to the numerator of the first factor in the third term of this sum we add $f(x_{2k+1,2n})-\frac{(f(\pi)-f(0))(2k+1)}{2n}-f(0)$

and deduct it. After reordering of the terms we obtain the representation

$$|f(x)-LT_n(f,x)|=\Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\Bigg\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\Bigg\}\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\left(f(\pi)-f(0)\right)2k}{2n}+f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\left(f(\pi)-f(0)\right)2k}{2n}+f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}$$

$$-\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\left(f(\pi)-f(0)\right)2k}{2n}+f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+2,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+2)}{2n}-f(0)+f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\Bigg\}\Bigg(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\Bigg)\Bigg|.$$

Then we obtain the equivalent representation

$$|f(x)-LT_n(f,x)|=\Bigg|f(x)-\Bigg[\sum_{k=0}^{n-1}\Bigg\{f(x_{2k,2n})-\frac{\big(f(\pi)-f(0)\big)2k}{2n}-f(0)\Bigg\}\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)-f(x_{2k,2n})+\frac{\left(f(\pi)-f(0)\right)2k}{2n}+f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+2,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+2)}{2n}-f(0)+f(x_{2k+1,2n})-\frac{\left(f(\pi)-f(0)\right)(2k+1)}{2n}-f(0)}{2}\Bigg\}$$

$$\cdot\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}+\frac{f(\pi)-f(0)}{\pi}x+f(0)\Bigg]$$

$$+\sum_{k=0}^{n-1}\Bigg\{\frac{f(x_{2k+1,2n})-f(x_{2k,2n})}{2}-\frac{\big(f(\pi)-f(0)\big)}{4n}\Bigg\}\Bigg(\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\Bigg)\Bigg|$$

$$= \Bigg| f(x) - \Bigg[\sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - \frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n} - f(0) + f(x_{2k,2n}) - \frac{\big(f(\pi)-f(0)\big)2k}{2n} - f(0)}{2} \Bigg\}$$

$$\cdot \frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi}$$

$$+ \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+2,2n}) - \frac{\big(f(\pi)-f(0)\big)(2k+2)}{2n} - f(0) + f(x_{2k+1,2n}) - \frac{\big(f(\pi)-f(0)\big)(2k+1)}{2n} - f(0)}{2} \Bigg\}$$

$$\cdot \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} + \frac{f(\pi) - f(0)}{\pi} x + f(0) \Bigg]$$

$$+ \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - f(x_{2k,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)}{4n} \Bigg\} \Bigg(\frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi} + \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} \Bigg) \Bigg|.$$

Combining the first and second sum in the square brackets, we obtain

$$|f(x) - LT_n(f, x)|$$

$$= \Bigg| f(x) - \Bigg[\sum_{j=0}^{2n-1} \Bigg\{ \frac{f(x_{j+1,2n}) - \frac{\big(f(\pi)-f(0)\big)(j+1)}{2n} - f(0) + f(x_{j,2n}) - \frac{\big(f(\pi)-f(0)\big)j}{2n} - f(0)}{2} \Bigg\}$$

$$\cdot \frac{(-1)^{j} \sin 2nx}{2nx - j\pi} + \frac{f(\pi) - f(0)}{\pi} x + f(0) \Bigg]$$

$$+ \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - f(x_{2k,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)}{4n} \Bigg\} \Bigg(\frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi} + \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} \Bigg) \Bigg|$$

$$= \Bigg| f(x) - \Bigg[\sum_{j=0}^{2n-1} \Bigg\{ \frac{f(x_{j+1,2n}) + f(x_{j,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)(2j+1)}{4n} - f(0) \Bigg\} \frac{(-1)^{j} \sin 2nx}{2nx - j\pi}$$

$$+ \frac{f(\pi) - f(0)}{\pi} x + f(0) \Bigg]$$

$$+ \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - f(x_{2k,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)}{4n} \Bigg\} \Bigg(\frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi} + \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} \Bigg) \Bigg|.$$

The definition of operator (13) implies the representation

$$|f(x) - LT_n(f, x)| = \Bigg| f(x) - AT_{2n}(f, x)$$

$$+ \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - f(x_{2k,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)}{4n} \Bigg\} \Bigg(\frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi} + \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} \Bigg) \Bigg|.$$

To obtain an uniform on the segment $[0, \pi]$ estimate for the error of approximation of an arbitrary continuous function f by the values of operator (2), we employ the triangle inequality

$$|f(x) - LT_n(f, x)| \leqslant |f(x) - AT_{2n}(f, x)|$$

$$+ \Bigg| \sum_{k=0}^{n-1} \Bigg\{ \frac{f(x_{2k+1,2n}) - f(x_{2k,2n})}{2} - \frac{\big(f(\pi) - f(0)\big)}{4n} \Bigg\} \Bigg(\frac{(-1)^{2k} \sin 2nx}{2nx - 2k\pi} + \frac{(-1)^{2k+1} \sin 2nx}{2nx - (2k+1)\pi} \Bigg) \Bigg|$$

$$\leqslant |f(x) - AT_{2n}(f, x)|$$

$$
\begin{aligned}
&+\left\{\frac{1}{2}\omega\Big(f,\frac{\pi}{2n}\Big)+\left|\frac{\big(f(\pi)-f(0)\big)}{4n}\right|\right\}\sum_{k=0}^{n-1}\left|\frac{(-1)^{2k}\sin 2nx}{2nx-2k\pi}+\frac{(-1)^{2k+1}\sin 2nx}{2nx-(2k+1)\pi}\right|\\
\leqslant&|f(x)-AT_{2n}(f,x)|\\
&+\left\{\frac{1}{2}\omega\Big(f,\frac{\pi}{2n}\Big)+\left|\frac{\big(f(\pi)-f(0)\big)}{4n}\right|\right\}\sum_{k=0}^{2n-1}\left|\frac{(-1)^{k+1}\sin 2nx}{2nx-(k+1)\pi}+\frac{(-1)^{k}\sin 2nx}{2nx-k\pi}\right|.
\end{aligned}
$$

By Lemma 1 and Proposition 3 we get the relation

$$
|f(x)-LT_n(f,x)|\leqslant|f(x)-AT_{2n}(f,x)|+\left\{\frac{1}{2}\omega\Big(f,\frac{\pi}{2n}\Big)+\left|\frac{\big(f(\pi)-f(0)\big)}{4n}\right|\right\}4\Big(1+\frac{1}{\pi}\Big)=o(1).
$$

The proof is complete. □

Let us consider the operator mapping each function f with finite values on the set $x_{k,2n}=\frac{\pi k}{2n}$, $n\in\mathbb{N}$, $0\leqslant k\leqslant 2n$, into an entire function Q_n by the rule

$$
Q_n(f,x)=\sum_{i=0}^{n}\frac{\cos nx\sin nx}{nx-i\pi}f\Big(\frac{i\pi}{n}\Big)-\sum_{i=0}^{n-1}\frac{\sin nx\cos nx}{nx-(i+\frac{1}{2})\pi}f\Big(\frac{(2i+1)\pi}{2n}\Big). \tag{22}
$$

As opposed to (2), this operator possesses the following interpolation property $f(x_{k,2n})=Q_n(f,x_{k,2n})$ for all $n\in\mathbb{N}$, $0\leqslant k\leqslant 2n$. On the face of it, operator (22) should possesses better approximative properties than (2). However, its values, as the values of sinc approximations (1), approximate only sufficiently smooth functions. For instance, as $\lambda_n=n^2$, $q_{\lambda_n}\equiv 0$, $h(\lambda_n)\neq 0$ Theorem 2 in [21] implies

Corollary 3. *Let $f\in C_0[0,\pi]$. For each natural n the identity*

$$
\lim_{n\to\infty}\left|f(x)-Q_n(f,x)-\frac{\sin 2nx}{2\pi}\sideset{}{'}\sum_{m=1}^{[\frac{2n-1}{2}]}\frac{f\big(\frac{\pi(2m+1)}{2n}\big)-2f\big(\frac{2\pi m}{2n}\big)+f\big(\frac{\pi(2m-1)}{2n}\big)}{[\frac{2nx}{\pi}]-2m}\right|=0
$$

holds true uniformly $[0,\pi]$.

The identity

$$
\lim_{n\to\infty}\|f-Q_n(f,\cdot)\|_{C_0[0,\pi]}=0
$$

holds true if and only if

$$
\lim_{n\to\infty}\max_{0\leqslant p\leqslant 2n}\left|\sideset{}{'}\sum_{m=1}^{[\frac{2n-1}{2}]}\frac{f\big(\frac{\pi(2m+1)}{2n}\big)-2f\big(\frac{2\pi m}{2n}\big)+f\big(\frac{\pi(2m-1)}{2n}\big)}{p-2m}\right|=0,
$$

where the prime at the sums indicates the absence of the terms with vanishing denominator.

Proof. We make the following equivalent transformations

$$
\begin{aligned}
Q_n(f,x)=&\sum_{i=0}^{n}\frac{\cos nx\sin nx}{nx-i\pi}f\Big(\frac{i\pi}{n}\Big)-\sum_{i=0}^{n-1}\frac{\sin nx\cos nx}{nx-(i+\frac{1}{2})\pi}f\Big(\frac{(2i+1)\pi}{2n}\Big)\\
=&\sum_{i=0}^{n}\frac{\sin 2nx}{2nx-2i\pi}f\Big(\frac{2i\pi}{2n}\Big)-\sum_{i=0}^{n-1}\frac{\sin 2nx}{2nx-(2i+1)\pi}f\Big(\frac{(2i+1)\pi}{2n}\Big)\\
=&\sum_{i=0}^{n}\frac{(-1)^{2i}\sin 2nx}{2nx-2i\pi}f\Big(\frac{2i\pi}{2n}\Big)+\sum_{i=0}^{n-1}\frac{(-1)^{2i+1}\sin 2nx}{2nx-(2i+1)\pi}f\Big(\frac{(2i+1)\pi}{2n}\Big)\\
=&\sum_{k=0}^{2n}\frac{(-1)^{k}\sin 2nx}{2nx-k\pi}f\Big(\frac{k\pi}{2n}\Big)=L_{2n}(f,x).
\end{aligned}
$$

Then as $\lambda_n = 4n^2$, $q_{\lambda_n} \equiv 0$, $h(\lambda_n) \neq 0$, Theorem 2 in [21], Theorem 2 and Corollary 1 imply the desired corollary. □

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