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ON A CLASS OF INNER FUNCTIONS IN A HALF-SPACE

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Dedicated to the memory of Professor
Igor' Fedorovich Krasichkov-Ternovskii

Abstract. In the paper we obtain necessary and sufficient conditions for the weight vector function, under which a given inner function is weakly invertible in the weighted space of holomorphic functions in a tubular domain.
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1. Introduction

Let $\mathbb{C}^n$ be an n-dimensional complex space, G be a domain in $\mathbb{C}^n$, $H(G)$ be the set of analytic in G functions, $H^\infty(G)$ be the set of all bounded analytic in G functions. We suppose that X is a some topological subspace of space $H(G)$, in which $H^\infty(G)$ is a dense set, operators $S_z(f) = f(z)$, $z \in G$, and $M_\psi(f) = \psi f$, $\psi \in H^\infty$, $f \in X$, are bounded operators in X.

Definition 1. *Let $f \in X$ and there exists a sequence $f_m \in H^\infty(G)$ such that $\lim\limits_{m\to+\infty} f_m f = 1$ in the sense of the topology in space X. Then function f is called weakly invertible in space X.*

Thus, f is weakly invertible X if set $H^\infty(G)f$ is everywhere dense in space X.

We note that the issue on weak invertibility in particular functional space are related with a wide class of problems in several fields, from the theory of differential operators and their generalization till abstract harmonic analysis [1].

In the one-dimensional case the weak invertibility was studied in the classical work by M.V. Keldysh [2], where it was established that there exists a function $f \in H^\infty(D)$, $f(z) \neq 0$, $z \in D = \{z \in \mathbb{C}^1 : |z| < 1\}$ not weakly invertible in the Bergman space

$$A^p(D) = \left\{ f \in H(D) : \|f\|_{A^p(D)} = \left(\int\limits_D |f(z)|^p \, dm_2(z) \right)^{\frac{1}{p}} < +\infty \right\},$$

where m_2 is the planar Lebesgue measure. In these constructions an important role was played by the inner function $S(z) = \exp\left(-\dfrac{1+z}{1-z}\right)$, $z \in D$.

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In works by A. Berling [3] and N. Nikolskii [4] the weak invertibility of function S in the weighted space

$$A^p_\varphi = \left\{ f \in H(D) : \|f\|_{A^p_\varphi} = \left(\int\limits_D |f(z)|^p \exp\left(-\varphi\left(\frac{1}{1-|z|} \right) \right) dm_2(z) \right)^{\frac{1}{p}} < +\infty \right\}$$

was studied.

Under certain restrictions for regularity of the growth of φ, it was established in these works that the weak invertibility of function S in space A^p_φ, $1 \leqslant p < +\infty$, is equivalent to

$$\int\limits_1^{+\infty} \left(\frac{\varphi(x)}{x^3} \right)^{\frac{1}{2}} dx = +\infty. \tag{1}$$

Taking into consideration that function S is analytic everywhere except the point $z = 1$, the author and his PhD student I. Gevorkyan in [5] studied the weak invertibility of function S in the space

$$\tilde{A}^p_\varphi = \left\{ f \in H(D) : \right.$$
$$\left. \|f\|_{\tilde{A}^p_\varphi} = \left(\int\limits_D |f(z)|^p \exp\left(-\varphi\left(\frac{1}{|1-z|} \right) \right) dm_2(z) \right)^{\frac{1}{p}} < +\infty, \quad 1 \leqslant p < +\infty \right\}.$$

It was established in [5] that as opposed to (1), the criterion of the weak invertibility of function S in $\tilde{A}^p_\varphi$, $0 < p \leqslant +\infty$, reads as

$$\int\limits_1^{+\infty} \frac{\varphi(x)}{x^2} dx = +\infty. \tag{2}$$

It is obvious that condition (2) implies (1), but the converse is false.

In a recent work [6], a new proof of the above results by A. Berling, N.Nikolskii and I. Gevorkyan–F. Shamoyan was proposed for the case $p = 2$; the proof was based on the well known corona theorem.

In the present work we study the issues of this type in multi-dimensional (tubular) domains.

2. Main results and proof of auxiliary statements

To present the main results of the work we introduce the following notations.

Let $P(x) = (p_1(x_1), \ldots, p_n(x_n))$, $x = (x_1, \ldots, x_n)$, be a vector function defined on $\mathbb{R}^n_+ = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_j > 0,\ j = \overline{1, n} \right\}$, $\mathbb{C}^n_+$ be the tubular domain with the basis $\mathbb{R}^n_+$, i.e., $\mathbb{C}^n_+ = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n) \in \mathbb{R}^n_+ \right\}$.

Let

$$A^q_P\left(\mathbb{C}^n_+\right) = \left\{ f \in H\left(\mathbb{C}^n_+\right) : \|f\|_{A^q_P} = \left(\int\limits_{\mathbb{C}^n_+} |f(z)|^p \exp\left(-P(|z|)\right) dm_{2n}(z) \right)^{\frac{1}{q}} < +\infty \right\},$$

where $z = (z_1, \ldots, z_n)$, $\exp\left(-P(|z|)\right) := \prod\limits_{j=1}^{n} \exp\left(-p_j(|z_j|)\right)$; dm_{2n} is the $2n$-dimensional Lebesgue measure in $\mathbb{C}^n_+$.

In what follows we assume that $p_j(x) = \int\limits_1^x \frac{\omega_j(t)}{t}dt$, $j = \overline{1,n}$, where ω_j are defined on $\mathbb{R}_+ := \mathbb{R}_+^1$, and $\omega_j(t) \uparrow^{+\infty}$ $(t \to +\infty)$, $1 \leqslant j \leqslant n$. Such functions will be called weights, while vectors functions $P = (p_1, \ldots, p_n)$ will be called weight vector function. The set of all weight vector functions is denoted by Ω.

The main results of the paper are the two following statements.

Theorem 1. *Let* $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}_+^n$, $az = \sum\limits_{j=1}^n a_j z_j$, $S_a(z) = \exp\left(i\sum\limits_{j=1}^n a_j z_j\right)$, $z = (z_1, \ldots, z_n) \in \mathbb{C}_+^n$, $P = (p_1, \ldots, p_n) \in \Omega$. *Then*

1) the following statements are equivalent:

a) function S_a *is weakly invertible in space* A_P^q *for some* $q = q_0$, $1 \leqslant q_0 < +\infty$;

b) S_a *is weakly invertible in space* A_P^q *for all* $0 < q < +\infty$;

c)

$$\int\limits_1^{+\infty} \frac{\mathrm{p}_j(t)}{t^2}dt = +\infty, j = \overline{1,n}; \tag{3}$$

2) If at least one of the integrals in (3) converges, then function S_a *is not weakly invertible in each space* A_P^q, $0 < q < +\infty$.

Theorem 2. *Let* $P = (p_1, \ldots, p_n)$ *be a vector function in* Ω, $f \in H^\infty\left(\mathbb{C}_{-\eta}^n\right)$, *where* $\mathbb{C}_{-\eta}^n = \left\{z = (z_1, \ldots, z_n) : \operatorname{Im} z_j > -\eta, j = \overline{1,n}\right\}$, $f(z) \neq 0$, $z \in \mathbb{C}_{-\eta}^n$, $0 < s < 1$.

Let $M_m = \sup\limits_{z \in \mathbb{C}_+^n} \left\{|\ln f(z)|^m \exp(-sP(|z|))\right\}$, *where the principal branch of the logarithm is fixed. If*

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt[m]{M_m}} = +\infty, \tag{4}$$

then function f *is weakly invertible in space* $A_P^q\left(\mathbb{C}_+^n\right)$ *for all* $0 < q < +\infty$.

Remark 1. *We note that the conditions* $f \in H^\infty\left(\mathbb{C}_+^n\right)$ *and* $f(z) \neq 0$, $z \in \mathbb{C}_+^n$, *are not sufficient for the weak invertibility of function* f *in space* $A_P^q\left(\mathbb{C}_+^n\right)$.

Indeed, in view of the results of work [8], it is easy to establish that the functions $f_a(z) = \exp\left(-\sum\limits_{j=1}^n \frac{ic_j}{z_j - a_j}\right)$, $z = (z_1, \ldots, z_n) \in \mathbb{C}_+^n$, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}_+^n$, *are not weakly invertible in space* $A_P^q(\mathbb{C}_+^n)$.

Remark 2. *If series (4) diverges, and function* f *coincides with function* S_a, *then it follows from Theorem 1 that function* f *is not weakly invertible in space* $A_P^q(\mathbb{C}_+^n)$ *for each* $q > 0$, *since the convergence of series (4) is equivalent to the convergence of integrals (3) (see [12]).*

Before proving Theorems 1 and 2, we provide the following auxiliary statements.

Let $k = (k_1, \ldots, k_n)$ be a permutation of numbers $(1, 2, \ldots, n)$, $n \in \mathbb{N}$, $1 \leqslant m \leqslant n$. Then the vector with the coordinates $(k_1, \ldots, k_m)$ is called a tuple of order m. The set of all tuples of order m is denoted by K_m. It is clear that if $1 \leqslant r,\ m \leqslant n$, then the identity $(k_1, \ldots, k_r) = (s_1, \ldots, s_m)$ holds true if and only if $r = m$, $s_i = k_i$, $i = \overline{1,m}$.

Lemma 1. *Let* $f \in H\left(\mathbb{C}_+^n\right)$, $k = (k_1, \ldots, k_m) \in K_m$, $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_n) \in \mathbb{C}_+^n$, *and* $\widetilde{z}_j = z_{k_j}$, *if* $j = k_j$ *for some* $k_j \in K_m$, *and* $\widetilde{z}_j = i$, *if* $j \neq k_j$, $j = \overline{1,n}$.

Suppose that $P=(p_1,\dots,p_n)$ is a weight vector function, $P\in\Omega$. If $0<s<+\infty$, then the estimate

$$|f(\widetilde{z})|^s \exp(-P(2|\widetilde{z}|)) \leqslant \frac{c_0(s)}{\prod_{j=1}^{m} y_{k_j}^2} \int_{\widetilde{U}^n(\widetilde{z})} |f(\zeta)|^s \exp(-P(|\zeta|))\, dm_{2n}(\zeta) \tag{5}$$

holds true, where $\widetilde{U}^n(\widetilde{z})=\left\{\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}_+^n : |\zeta_j-\widetilde{z}_j|<\frac{\operatorname{Im}\widetilde{z}_j}{2}, j=\overline{1,n}\right\}$.

Proof. Without loss of generality we can assume that $j=k_j$, $1\leqslant j\leqslant m$. Then

$$\widetilde{U}^n(\widetilde{z})=\left\{\zeta=(\zeta_1,\dots,\zeta_n) : |\zeta_j-z_j|<\frac{y_j}{2},\ 1\leqslant j\leqslant m,\ |\zeta_j-i|<\frac{1}{2},\ m+1\leqslant j\leqslant n\right\}.$$

Taking into consideration the n-subharmonicity of the function $|f(\zeta)|^s$, $\zeta\in\mathbb{C}_+^n$, we obtain

$$|f(\widetilde{z})|^s \leqslant \frac{2^{2m}}{\pi^n \prod_{j=1}^{m} y_j^2} \int_{\widetilde{U}^n(\widetilde{z})} |f(\zeta)|^s\, dm_{2n}(\zeta), \tag{6}$$

where $\widetilde{z}=(z_1,\dots,z_m,i,\dots,i)$. We note that if $\zeta\in\widetilde{U}^n$, $\zeta=(\zeta_1,\dots,\zeta_n)$, then $|z_j-\zeta_j|<\frac{y_j}{2}$, $z_j=x_j+iy_j$, $j=\overline{1,m}$, and $|i-\zeta_j|<\frac{1}{2}$ if $j=\overline{m+1,n}$. Hence,

$$\frac{|z_j|}{2}\leqslant |z_j|-\frac{|y_j|}{2}\leqslant |\zeta_j|\leqslant |z_j|+\frac{|y_j|}{2}\leqslant \frac{3}{2}|z_j|,\quad j=\overline{1,m};\quad \frac{1}{2}\leqslant|\zeta_j|\leqslant\frac{3}{2},\quad j=\overline{m+1,n}.$$

Therefore,

$$\begin{aligned}\exp\left(-p_j\left(\frac{3}{2}|z_j|\right)\right) &\leqslant \exp(-p_j(|\zeta_j|))\\ &\leqslant \exp\left(-p_j\left(\frac{|z_j|}{2}\right)\right), \zeta=(\zeta_1,\dots,\zeta_n)\in\widetilde{U}^n(\widetilde{z}),\quad j=\overline{1,n}.\end{aligned} \tag{7}$$

Employing estimates (6), (7), we arrive at the inequality

$$\begin{aligned}|f(\widetilde{z})|^s \exp\left(-p_j\left(\frac{3}{2}|\widetilde{z}|\right)\right) &= |f(z_1,\dots,z_m,i,\dots,i)|^s \exp\left(-\sum_{j=1}^{m} p_j\left(\frac{3}{2}|z_j|\right)\right)\\ &\leqslant \frac{2^{2m}}{\pi^n \prod_{j=1}^{m} y_j^2} \int_{\widetilde{U}^n(\widetilde{z})} |f(\zeta)|^s \exp-(P(|\zeta|))\, dm_{2n}(\zeta)\\ &\leqslant \frac{C(m,n)}{\prod_{j=1}^{m} y_j^2} \int_{\mathbb{C}_+^n} |f(\zeta)|^s \exp(-P(|\zeta|))\, dm_{2n}(\zeta).\end{aligned} \tag{8}$$

The proof is complete. □

The next statement was proved in the work by M.M. Dzhrbashyan [9], see also [10].

Lemma 2. *Let $P=(p_1,\dots,p_n)$ be a weight vector function, $1\leqslant q<+\infty$. Then the following statements are equivalent:*

1) the set of all algebraic polynomials of $(z_1,\dots,z_n)$ is an everywhere dense in $A_P^q(\mathbb{C}_+)$ set;

2) statements (3) of Theorem 1 hold true and if one of the integrals in (3) diverges, the set of the polynomials is not dense in space $A_P^q(\mathbb{C}_+^n)$ for arbitrary $0<q<+\infty$.

Proof. Let $1 \leqslant q < +\infty$. We prove Lemma for $n = 2$, for other n the main milestones of the proof are same.

Let

$$L_P^{q'}\left(\mathbb{C}_+^2\right) := \left\{ f \in S\left(\mathbb{C}_+^n\right) : \left(\int\limits_{\mathbb{C}_+^2} |f(\zeta)|^{q'} \exp\left(-P\left(|\zeta|\right)\right) dm_4(\zeta) \right)^{\frac{1}{q'}} < +\infty \right\},$$

where S is the set of all measurable on $\mathbb{C}_+^n$ functions, and $q' = \dfrac{q}{q-1}$. Suppose that $g \in L_P^{q'}\left(\mathbb{C}_+^2\right)$ is such that

$$\int\limits_{\mathbb{C}_+^2} g(\zeta_1, \zeta_2)\, \zeta_1^{k_1} \zeta_2^{k_2} e^{-p_1(|\zeta_1|) - p_2(|\zeta_2|)} dm_4(\zeta_1, \zeta_2) = 0, k = (k_1, k_2) \in \mathbb{Z}_+^2. \tag{9}$$

Let us prove that

$$\int\limits_{\mathbb{C}_+^2} g(\zeta_1, \zeta_2)\, f(\zeta_1, \zeta_2)\, e^{-p_1(|\zeta_1|) - p_2(|\zeta_2|)} dm_4(\zeta_1, \zeta_2) = 0 \tag{10}$$

for each $f \in A_P^q\left(\mathbb{C}_+^2\right)$.

Let $\widetilde{g}(\zeta_1) = \int\limits_{\mathbb{C}_+} g(\zeta_1, \zeta_2) \exp\left(-p_2\left(|\zeta_2|\right)\right) dm_2(\zeta_2)$. It is obvious that $\widetilde{g}(\zeta_1)$ is an almost everywhere finite function. Let us prove that $\widetilde{g}(\zeta_1) \in L_{p_1}^{q'}(\mathbb{C}_+)$. By the Hölder inequality we have

$$\int\limits_{\mathbb{C}_+} |\widetilde{g}(\zeta_1)|^{q'} e^{-p_1(|\zeta_1|)} dm_2(\zeta_1) = \int_{\mathbb{C}_+} \left(\int\limits_{\mathbb{C}_+} |g(\zeta_1, \zeta_2)|\, e^{-p_2(|\zeta_2|)} dm_2(\zeta_2) \right)^{q'} e^{-p_1(|\zeta_1|)} dm_2(\zeta_1)$$

$$\leqslant \int\limits_{\mathbb{C}_+^2} |g(\zeta_1, \zeta_2)|^{q'} e^{-p_2(|\zeta_2|)} e^{-p_1(|\zeta_1|)} dm_4(\zeta_1, \zeta_2) \left(\int\limits_{\mathbb{C}_+} e^{-p_2(|\zeta_2|)} dm_2(\zeta_2) \right)^{\frac{q'}{q}}$$

$$\leqslant const \int\limits_{\mathbb{C}_+} |g(\zeta_1)|^{q'} e^{-p_1(|\zeta_1|)} dm_2(\zeta_1) < +\infty.$$

Therefore, by M.M. Dzrbashyan theorem (see [9]),

$$\int\limits_{\mathbb{C}_+} \widetilde{g}(\zeta_1)\, f(\zeta_1)\, e^{-p_1(|\zeta_1|)} dm_2(\zeta_1) = 0 \tag{11}$$

for an arbitrary $f \in A_{P_1}^q(\mathbb{C}_+)$.

Exactly in the same way one can prove that if $f \in A_{P_1}^q\left(\mathbb{C}_+^2\right)$, then the function $\tilde{f}(\zeta_1) = \int\limits_{\mathbb{C}_+} f(\zeta_1, \zeta_2) \exp\left(-p_2\left(|\zeta_2|\right)\right) dm_2(\zeta_2)$ belongs to class $A_{P_1}^q(\mathbb{C}_+)$. Hence, applying M.M. Dzrbashyan theorem, we obtain that

$$\int\limits_{\mathbb{C}_+} \widetilde{g}(\zeta_1)\, \widetilde{f}(\zeta_1)\, dm_2(\zeta_1) = 0,$$

i.e.,

$$\int\limits_{\mathbb{C}_+^2} g(\zeta_1,\zeta_2)\, f(\zeta_1,\zeta_2)\exp(-p_1(|\zeta_1|))\exp(-p_2(|\zeta_2|))\, dm_4(\zeta_1,\zeta_2) = 0.$$

This identity and Hahn-Banach theorem imply the first part of the lemma.

We proceed to the proof of the second part. It follows from Lemma 1 that if the polynomials are dense in $A_P^q(\mathbb{C}_+^n)$, then there exists a sequence of polynomials $\tilde{P}_m(z) = \sum\limits_{k=1}^{m} a_k^{(m)} z^k$, $z \in \mathbb{C}_+$ such that $\max\limits_{z\in\mathbb{C}_+}\left\{\left|\tilde{P}_m(z) - f(z)\right| \exp(-p_j(|z|))\right\} = 0$, $1 \leqslant j \leqslant n$, for each $f \in A_{P_j}^\infty(\mathbb{C}_+)\,0$. Then M.M. Dzhrbashyan theorem impliesL $\int\limits_1^{+\infty} \frac{p_j(t)}{t^2} dt = +\infty$ (see [9], [10]). □

The next lemma was proved in work [11].

Lemma 3. *Let p be a weight function such that*

$$\int\limits_0^{+\infty} \frac{\mathrm{p}(t)}{1+t^2} dt < +\infty.$$

Suppose that G is the external function in the half-plane $\mathbb{C}_+$ readings as

$$G(z) = \exp\left(-\frac{4i}{\pi}\int\limits_{-\infty}^{+\infty} \frac{tz+1}{t-z}\frac{p(t)}{1+t^2} dt\right), \quad z \in \mathbb{C}_+.$$

Then there exists a positive number c such that

$$\exp(-cp(3|z|)) \leqslant |G(z)| \leqslant \exp(-p(|z|)), \quad z \in \mathbb{C}_+. \tag{12}$$

3. Proof of the main results

Proof of Theorem 1. We begin with the first statement c) $\Rightarrow$ b).

Let $1 \leqslant q < +\infty$, $a = (a_1, \ldots, a_n)$, $a_j > 0, j = \overline{1,n}$. We denote by $E_q(S_a)$ the closure of set $H^\infty\left(\mathbb{C}_+^n\right) S_a$ in space $A_P^q\left(\mathbb{C}_+^n\right)$. To prove the desired statement, it is sufficient to show that $1 \in E_q(S_a)$.

Let Φ be a linear continuous functional orthogonal to $E_q(S_a)$. Let us prove that $\Phi(1) = 0$. We suppose that Φ is generated by some function $\Psi \in L_P^{q'}\left(\mathbb{C}_+^n\right)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\Phi(S_a F) = \int\limits_{\mathbb{C}_+^n} e^{iaz} F(z)\,\Psi(z)\, e^{-P(|z|)} dm_{2n}(z) = 0,$$

for each $F \in H^\infty\left(\mathbb{C}_+^n\right)$ as well as for $F(z_1, \ldots, z_n) = z_1^{m_1} \ldots z_n^{m_n}$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$.

Given $t \in [0,1]$, we let $e_1(t) = \int\limits_{\mathbb{C}_+^n} e^{ia_1 t z_1 + i\widetilde{a}\widetilde{z}}\Psi(z)\, e^{-P(|z|)} dm_{2n}(z)$, where $\widetilde{z} = (z_2, z_3, \ldots, z_n)$, $\widetilde{a} = (a_2, \ldots, a_n)$.

It is clear that

$$e_1^{(m)}(t) = \int\limits_{\mathbb{C}_+^n} e^{ia_1 z_1 t + i\widetilde{a}\widetilde{z}} (ia_1 z_1)^m\, \Psi(z)\, e^{\left(-p_1(|z_1|) - \widetilde{P}(|z|)\right)} dm_{2n}(z),$$

where $\exp(-\widetilde{P}(|\widetilde{z}|)) = \exp\left(-p_2\left(|z_2|\right)\ldots - p_n\left(|z_n|\right)\right)$. It is obvious that

$$e_1^{(m)}(1) = 0, \quad m \in \mathbb{Z}_+. \tag{13}$$

Let us prove that function e belongs to the quasi-analytic class on the segment $[0,1]$ (see [12]). Indeed, applying Hölder inequality, we have

$$\begin{aligned}\left|e_1^{(m)}(t)\right| &\leqslant \int\limits_{\mathbb{C}_+^n} \exp\left(-p_1\left(|z_1|\right)\right)|a_1|^m|z_1|^m|\Psi(z)|\exp\left(-\widetilde{P}\left(|\widetilde{z}|\right)\right)dm_{2n}(z)\\ &\leqslant |a_1|^m\left(\int\limits_{\mathbb{C}_+} e^{(-p_1(|z_1|))}|z_1|^{mq}\,dm_2(z_1)\right)^{\frac{1}{q}}\\ &\quad\cdot\left(\int\limits_{\mathbb{C}_+} e^{(-p_1(|z_1|))}\left(\int\limits_{\mathbb{C}_+^{n-1}}|\Psi(z)|\,e^{\left(-\widetilde{P}(|z|)\right)}dm_{2n-2}(z)\right)^{q'}dm_2(z)\right)^{\frac{1}{q'}}.\end{aligned}$$

Applying Hölder inequality once again, we arrive at the estimate

$$\begin{aligned}\left|e_1^{(m)}(t)\right| &\leqslant |a_1|^m\left(\int\limits_{\mathbb{C}_+}|z_1|^{mq}\,e^{(-p_1(|z_1|))}dm_2(z)\right)^{\frac{1}{q}}\\ &\quad\left(\int\limits_{\mathbb{C}_+} e^{(-p_1(|z_1|))}\left(\int\limits_{\mathbb{C}_+^{n-1}}|\Psi(z)|^{q'}\,e^{\left(-\widetilde{P}(|z|)\right)}dm_{2n-2}(z)\right)dm_{2n}(z)\right)\\ &\quad\cdot\left(\int\limits_{\mathbb{C}_+^{n-1}} e^{\left(-\widetilde{P}(|z|)\right)}dm_{2n-2}(z)\right)^{\frac{q'}{q}}\\ &\leqslant C_1|a_1|^m\left(\int\limits_{\mathbb{C}_+}|z_1|^{mq}\,e^{(-p_1(|z_1|))}dm_2(z_1)\right)^{\frac{1}{q}}\left(\int\limits_{\mathbb{C}_+^n}|\Psi(z)|^{q'}\,e^{(-P(|z|))}dm_{2n}(z)\right)^{\frac{1}{q'}}.\end{aligned} \tag{14}$$

Let δ be an arbitrary positive number $\delta \in (0,1)$. Then the latter estimate yields

$$\begin{aligned}\left|e_1^{(m)}(t)\right| &\leqslant C_2|a_1|^m\sup_{r>0}\left(r^{\overline{m}}e^{-\frac{\delta}{q}p_1(r)}\right)\left(\int\limits_{\mathbb{C}_+} e^{-(1-\delta)p_1(|z|)}dm_2(z)\right)^{\frac{1}{q}}\\ &= C_2|a_1|^m\sup_{r>0}\left(r^{\overline{m}}e^{-\frac{\delta}{q}p_1(r)}\right)\left(\int\limits_0^{+\infty}\int\limits_0^{\pi} e^{-(1-\delta)p_1(|\rho|)}\rho\,d\rho\,d\varphi\right)^{\frac{1}{q}} = C_3|a_1|^m\sup_{r>0}\left(r^{\overline{m}}e^{-\frac{\delta}{q}p_1(r)}\right).\end{aligned}$$

Thus, we finally obtain

$$\left|e_1^{(m)}(t)\right| \leqslant C_3|a_1|^m M_m,$$

where $M_m = \sup\limits_{r>0}\left(r^{\overline{m}}e^{-\frac{\delta}{q}p_1(r)}\right)$.

Now we employ Carleman-Ostrowski theorem (see [12]) on the quasi-analyticity of the class

$$C^\infty(M_m) = \left\{\varphi \in C^\infty[0,1] : \left|\varphi^{(m)}(t)\right| \leqslant A^m M_m\right\},$$

in accordance to which the criterion of the quasi-analyticity of class $C^{\infty}(M_m)$ is

$$\int_1^{+\infty} \frac{\ln T(r)}{r^2} dr = +\infty, \tag{15}$$

where $T(r) = \sup\limits_{r \geqslant 1} \frac{r^m}{M_m}$ (see [12]).

But by M.M. Dzhrbashyan theorem (see [9], [10]), the convergence of the integral $\int_1^{+\infty} \frac{p_1(r)}{r^2} dr$ is equivalent to condition (15). Therefore, function $e(t)$ belongs to the quasi-analytic Carleman-Ostrowsi class on the segment $[0, 1]$. In view of condition (13) we have $e_1(t) = 0$, $\forall t \in [0, 1]$, i.e., $e_1(0) = 0$. Hence,

$$\int_{\mathbb{C}_+^n} e^{-i\tilde{a}\tilde{z}} \Psi(z) e^{-P(|z|)} dm_{2n}(z) = 0.$$

We let

$$e_2(t) = \int_{\mathbb{C}_+^n} e^{-ia_2 z_2 t - i\widetilde{a}\widetilde{z}} \psi(z) e^{-p_2(|z_2|) - \tilde{P}(|z|)} dm_{2n}(z),$$

where $\widetilde{a}$ and $\widetilde{z}$ are introduced as above.

Reproducing the above arguments, we obtain $e_2(0) = 0$. Repeating these arguments $n - 1$ times, we obtain that

$$\int_{\mathbb{C}_+^n} \Psi(z) e^{-P(|z|)} dm_{2n}(z) = 0,$$

i.e., $\Phi(1) = 0$.

By Hahn-Banach theorem $1 \in E_q(S_a)$.

Thus, the implication c) $\Rightarrow$ b) is proved under the condition $q \geqslant 1$. But since for an arbitrary $f \in H^{\infty}\left(\mathbb{C}_+^n\right)$, $0 < q < 1$,

$$\|fS_a - 1\|_{A_P^q(\mathbb{C}_+^n)} \leqslant \|fS_a - 1\|_{A_P^1(\mathbb{C}_+^n)} \left(\int_{\mathbb{C}_+^n} e^{-P(|z|)} dm_{2n}(z) \right)^{\frac{1-q}{q}},$$

it completes the proof of this implication.

The implication b) $\Rightarrow$ c) is obvious. This is why we have proved that c) $\Rightarrow$ b) $\Rightarrow$ a). To prove the first statement of the theorem, it remains to establish the implication a) $\Rightarrow$ c).

It is obvious that the implication a) $\Rightarrow$ c) of the first statement is implied immediately by the second statement of the theorem. This is why we proceed to the proof of the second statement.

Suppose that there exists some $k = (k_1, \ldots, k_m) \in K_m$, such that

$$\int_1^{+\infty} \frac{p_{k_j}(t)}{t^2} dt < +\infty, \quad j = \overline{1, m}.$$

Without loss of generality we assume that $k_j = j$, $j = \overline{1, m}$.

As in Lemma 3, by means of function p_j, $j=\overline{1,m}$, we construct the set of external functions

$$G_j(z)=\exp\left(\frac{-4i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{tz+1}{t-z}\frac{p_j(3|t|)}{1+t^2}dt\right),\quad j=\overline{1,m}.$$

We also let

$$G(z)=\prod_{j=1}^{m}G_j(z_j)=\exp\left(\frac{-4i}{\pi}\sum_{j=1}^{m}\left(\int\limits_{-\infty}^{+\infty}\frac{t_jz_j+1}{t_j-z_j}\frac{p_j(3|t_j|)}{1+t_j^2}dt_j\right)\right),\quad z=(z_1,\dots,z_m)\in\mathbb{C}_+^m.$$

Employing Lemma 3, we obtain

$$\exp\Bigg(-c\sum_{j=1}^{m}p_j(9|z_j|)\Bigg)\leqslant|G(z)|\leqslant\exp\Bigg(-\sum_{j=1}^{m}p_j(3|z_j|)\Bigg),\quad z=(z_1,\dots,z_m)\in\mathbb{C}_+^m,\quad(16)$$

for some positive c.

Suppose that on the contrary to the second statement, there exists a sequence $\{f_k\}_{k=1}^{+\infty}$, $f_k\in H^\infty(\mathbb{C}_+^n)$, such that

$$\lim_{k\to+\infty}\|f_kS_a-1\|_{A_P^q(\mathbb{C}_+^n)}=0.\quad(17)$$

Employing Lemma 1, we obtain

$$\begin{aligned}|f_k(z_1,\dots,z_m,i,\dots,i)S_a(z_1,\dots,z_m,i,\dots,i)-1|^q\exp\Bigg(-\sum_{j=1}^{m}p_j(2|z_j|)\Bigg)&\\ \leqslant\frac{c_0(q)}{\prod\limits_{j=1}^{m}y_j^2}\|f_kS_a-1\|^q_{A_P^q(\mathbb{C}_+^n)},\quad z=(z_1,\dots,z_m,i,\dots,i)\in\mathbb{C}_+^n.&\end{aligned}\quad(18)$$

It follows immediately from estimates (16) and (18) that

$$\begin{aligned}|f_k(z_1,\dots,z_m,i,\dots,i)S_a(z_1,\dots,z_m,i,\dots,i)-1|^q|G(z_1,\dots,z_m)|&\\ \leqslant\frac{c(q)}{\prod\limits_{j=1}^{m}y_j^2}\|f_kS_a-1\|^q_{A_P^q(\mathbb{C}_+^n)},\quad z=(z_1,\dots,z_m)\in\mathbb{C}_+^n.&\end{aligned}\quad(19)$$

In particular, it follows immediately from estimate (19) that

$$\begin{aligned}|f_k(x_1+i,x_2+i,\dots,x_m+i,i,\dots,i)S_a(x_1+i,x_2+i,\dots,x_m+i,i,\dots,i)-1|^q&\\ \cdot|G(x_1+i,x_2+i,\dots,x_m+i)|\leqslant1,&\end{aligned}$$

as $k\geqslant k_0$. Therefore,

$$\begin{aligned}|f_k(x_1+i,\dots,x_m+i,i,\dots,i)|^q|G(x_1+i,\dots,x_m+i)||S_a(x_1+i,\dots,x_m+i,i,\dots,i)|^q&\\ \leqslant|G(x+i)||f_k(\widetilde{x+i})S_a(\widetilde{x+i})-1|^q+|G(x+i)|\leqslant1+|G(x+i)|,&\end{aligned}\quad(20)$$

where $x+i=(x_1+i,x_2+i,\dots,x_m+i,i,i\dots,i)\in\mathbb{C}_+^m$.

It is obvious that estimate (16) implies that $|G(z)|\leqslant1$ for each $z\in\mathbb{C}_+^m$, moreover,

$$|S_a(x+i)|^q=\left|\exp i\sum_{j=1}^{n}a_j(x_j+i)\right|^q=\exp\left(-q\sum_{j=1}^{n}a_j\right)\leqslant1.$$

Letting $A = \exp\left(-q\sum_{j=1}^{n} a_j\right)$, by (20) we obtain that

$$\left|f_k\left(\widetilde{x+i}\right)\right|^q |G(x+i)| \leqslant 2A,$$

i.e.,

$$\left|f_k\left(\widetilde{x+i}\right)\right| |G(x+i)|^{\frac{1}{q}} \leqslant (2A)^{\frac{1}{q}}. \tag{21}$$

Since the function

$$F_k(z) = f_k(\tilde{z}+i)\left(G(z+i)\right)^{\frac{1}{q}}, k = 1, 2, \ldots,$$

can be represented by the Poisson integral (see [13], [14]) in the half-space $\mathbb{C}_+^m$ and $F_k \in H^\infty\left(\mathbb{C}_+^m\right)$, we obtain estimate (21) in half-space $\mathbb{C}_+^m$, i.e.,

$$|f_k(\tilde{z}+i)|^q |G(z+i)| \leqslant 2A, \tag{22}$$

for each $z = (z_1, \ldots, z_m) \in \mathbb{C}_+^m$.

Taking into consideration that

$$\lim_{k\to+\infty} f_k(z_1, \ldots, z_m, i, \ldots, i) = e^{-\sum\limits_{j=1}^{m} ia_j z_j + \sum\limits_{j=m+1}^{n} a_j}, \quad (z_1, \ldots, z_m) \in \mathbb{C}_+^m,$$

and passing to the limit in inequality (22), we finally obtain

$$\left(\exp\sum_{j=1}^{m} qa_j y_j + q\sum_{j=m+1}^{n} a_j\right) \leqslant 2A\prod_{j=1}^{m}\exp(cp_j(3|z_j|)), \quad (z_1, \ldots, z_m) \in \mathbb{C}_+^m. \tag{23}$$

It follows from (3) that

$$\lim_{y\to+\infty}\frac{p_j(3y)}{y} = 0, \quad j = \overline{1, m}.$$

But it is impossible in view of estimate (23). The proof is complete. □

Proof of Theorem 2. We first prove that if $f \in H^\infty\left(\mathbb{C}_{-\eta}^n\right)$, $f(z) \neq 0, z \in \mathbb{C}_+^n \cup \mathbb{R}^n$, for some $\eta > 0$, then the function $(\ln f)^m$ with the principal branch of the logarithm belongs to class $A_P^q\left(\mathbb{C}_+^n\right)$, $1 \leqslant q < +\infty$, $P \in \Omega$. Indeed, without loss of generality we can assume that $|f(z)| \leqslant 1$, $z \in \mathbb{C}_{-\eta}^n$. Hence, function $\Psi(z) = -i\ln f(z - i\delta)$ satisfies condition $\operatorname{Im}\Psi(z) \geqslant 0$, and $\Psi \in H\left(\mathbb{C}_{\frac{-\eta}{2}}^n\right)$. We let $\delta = \dfrac{\eta}{2}$ and apply Schwarz kind formulae for function Ψ in $\mathbb{C}_+^n$ (see [13]). We obtain

$$\Psi(z) = \frac{2i^n}{(2\pi)^n}\int\limits_{R^n}\prod_{j=1}^{n}\left(\frac{i+z_j}{z_j-t_j}\right)\frac{1}{i+t_j} \times \ln\frac{1}{|f(t)|}dt + i\arg f(i).$$

Hence,

$$|\ln f\,|z-i\delta|| \leqslant \frac{2}{(2\pi)^n}\int\limits_{R^n}\prod_{j=1}^{n}\left(\left|\frac{i+z_j}{z_j-t_j}\right|\frac{1}{i+t_j}\right) \times \ln\frac{1}{|f(t-i\delta)|}dt + c_0.$$

Employing the elementary estimate

$$\sup_{t\in R}\left|\frac{i-t}{z-t}\right| = \frac{|z-i|+|z+i|}{2\operatorname{Im} z},$$

where $z \in \mathbb{C}_+$ (see [12]), we obtain

$$|\ln|f(z-i\delta)|| \leqslant \frac{2}{(2\pi)^n}\int\limits_{R^n}\prod_{j=1}^{n}\left(\frac{|i+z_j|}{|i-t_j|}\frac{|i-t_j|}{|z_j-t_j|(i+t_j)}\right)\times\ln\frac{1}{|f(t-i\delta)|}$$
$$\leqslant \frac{2}{(2\pi)^n}\int\limits_{R^n}\prod_{j=1}^{n}(1+|z_j|)\int\limits_{R^n}\frac{\ln\frac{1}{|f(t-i\delta)|}}{\prod_{j=1}^{n}\left(1+t_j^2\right)}\times\prod_{j=1}^{n}\sup_{t_j\in R}\left|\frac{i-t_j}{z_j-t_j}\right|$$
$$\leqslant \frac{2}{(2\pi)^n}\prod_{j=1}^{n}\left[(1+|z_j|)\left(\frac{|z_j-1|+|z_j+1|}{2\mathrm{Im}\, z_j}\right)\right]\times\int\limits_{R^n}\frac{\ln\frac{1}{|f(t-i\delta)|}}{\prod_{j=1}^{n}\left(1+t_j^2\right)}dt_1\dots dt_n,$$

$t=(t_1,\dots,t_n)$. Thus, we finally get

$$|\ln|f(z-i\delta)|| \leqslant const\int\limits_{R^n}\frac{\ln\frac{1}{|f(t-i\delta)|}}{\prod_{j=1}^{n}\left(1+t_j^2\right)}dt\times\prod_{j=1}^{n}\frac{\left(1+|z_j|^2\right)}{\mathrm{Im}\, z_j}\leqslant const\prod_{j=1}^{n}\frac{\left(1+|z_j|^2\right)}{\mathrm{Im}\, z_j}.$$

We let $\zeta=z-i\delta$ in this inequality. If $\mathrm{Im}\,\zeta_j\geqslant 0$, then $\mathrm{Im}\, z_j\geqslant\delta$, $1\leqslant j\leqslant n$. Therefore,

$$|\ln|f(\zeta)||\leqslant const\prod_{j=1}^{n}\frac{\left(1+|z_j|^2\right)}{\mathrm{Im}\, z_j}\leqslant const\prod_{k=1}^{n}\frac{\left(1+|z_k|^2\right)}{\delta},\zeta\in\mathbb{C}_+^n. \tag{24}$$

Here we have employed the estimate [13]

$$\int\limits_{R^n}\frac{\ln\frac{1}{|f(t-i\delta)|}}{\prod_{j=1}^{n}\left(1+t_j^2\right)}dt_1\dots dt_n<+\infty,\quad t=(t_1,\dots,t_n).$$

By estimate (24) we obtain that the function $\Psi_m(z)=(\ln f(z))^m$ belongs to class $A_P^q\left(\mathbb{C}_+^n\right)$ for each $1\leqslant q<+\infty$.

Now we follow the lines of the proof of Theorem 1. Let $1\leqslant q<+\infty$.

We let once again

$$e(t)=\int\limits_{\mathbb{C}_+^n}f^t(\zeta)\,\Psi(\zeta)\,e^{-P(|\zeta|)}dm_{2n}(\zeta)\,,0\leqslant t\leqslant 1,$$

where the principal branch of the power function is used, and Ψ is an arbitrary function in $A_P^{q'}\left(\mathbb{C}_+^n\right)$, $\dfrac{1}{q}+\dfrac{1}{q'}=1$, and

$$\int\limits_{\mathbb{C}_+^n}\Psi(\zeta)\,F(\zeta)\,e^{-P(|\zeta|)}dm_{2n}(\zeta)=0,$$

for arbitrary $F\in E_q(f)$. We recall that $E_q(f)$ is the closure of set $H^\infty\left(\mathbb{C}_+^n\right)f$ in space $A_P^q\left(\mathbb{C}_+^n\right)$. It is clear that

$$e^{(m)}(t)=\int\limits_{\mathbb{C}_+^n}f^t(\zeta)\,(\ln f(\zeta))^m\,\Psi(\zeta)\,e^{-P(|\zeta|)}dm_{2n}(\zeta)\,.$$

As in the proof of Theorem 1, let us prove that $e^{(m)}(1)=0$, $m=0,1\ldots$ Indeed,

$$e^{(m)}(1)=\int\limits_{\mathbb{C}_+^n} f(\zeta)(\ln f(\zeta))^m e^{-P(|\zeta|)}dm_{2n}(\zeta).$$

Hence, for an arbitrary sequence $\{f_k\}\in H^\infty\left(\mathbb{C}_+^n\right)$ we have

$$\|f_k f - f\Psi_m\|_{A_P^q\left(\mathbb{C}_+^n\right)} \leqslant \|f\|_\infty \|f_k-\Psi_m\|_{A_P^q\left(\mathbb{C}_+^n\right)},$$

where $\Psi_m=(\ln f)^m$, $m\in\mathbb{Z}^+$. As it was established above, $\Psi_m\in A_P^q\left(\mathbb{C}_+^n\right)$, and this is why we can choose a sequence $\{f_k\}^\infty\in H^\infty\left(\mathbb{C}_+^n\right)$ such that $\|f_k-\Psi_m\|_{A_P^q\left(\mathbb{C}_+^n\right)}\to 0$ as $k\to+\infty$, $m=1,2\ldots.$

Thus, $f(\ln f)^m\in E_q(f)$. We proceed to the estimate $e^{(m)}(t)$ on the segment $[0,1]$.

We have

$$\left|e^{(m)}(t)\right| \leqslant \int\limits_{\mathbb{C}_+^n} \left|f^t(\zeta)\right| |\ln f(\zeta)|^m |\Psi(\zeta)| e^{-P(|\zeta|)}dm_{2n}(\zeta).$$

Now we employ the estimate

$$\left|f^t(\zeta)\right| \leqslant (|f(\zeta)|+1)\leqslant 2,\ \zeta\in\mathbb{C}_+^n, t\in[0,1].$$

Then

$$\left|e^{(m)}(t)\right| \leqslant 2\int\limits_{\mathbb{C}_+^n} |\ln f(\zeta)|^m |\Psi(\zeta)| e^{-P(|\zeta|)}dm_{2n}(\zeta).$$

Applying Hölder inequality, we arrive at the estimate

$$\left|e^{(m)}(t)\right| \leqslant 2\left(\int\limits_{\mathbb{C}_+^n} |\ln f(\zeta)|^{qm} e^{-qP(|\zeta|)}dm_{2n}(\zeta)\right)^{\frac{1}{q}} \left(\int\limits_{\mathbb{C}_+^n} |\Psi(\zeta)|^{q'} e^{-q'P(|\zeta|)}dm_{2n}(\zeta)\right)^{\frac{1}{q'}}.$$

Therefore, if $0<s<1$, then

$$\begin{aligned}\left|e^{(m)}(t)\right| \leqslant & 2\left(\int\limits_{\mathbb{C}_+^n} \left(|\ln f(\zeta)|^m e^{-sP(|\zeta|)}\right)^q e^{(-q(1-s)P(|\zeta|))}dm_{2n}(\zeta)\right)^{\frac{1}{q}} \\ & \cdot\left(\int\limits_{\mathbb{C}_+^n} |\Psi(\zeta)|^{q'} e^{-q'P(|\zeta|)}dm_{2n}(\zeta)\right)^{\frac{1}{q'}} \\ \leqslant & 2M_m\left(\int\limits_{\mathbb{C}_+^n} e^{-(q(1-s)P(|\zeta|))}dm_{2n}(\zeta)\right)^{\frac{1}{q}} \left(\int\limits_{\mathbb{C}_+^n} |\Psi(\zeta)|^{q'} e^{-q'P(|\zeta|)}dm_{2n}(\zeta)\right)^{\frac{1}{q'}},\end{aligned}$$

where $s\in(0,1)$. In view of $P\in\Omega$ we finally we obtain

$$\left|e^{(m)}(t)\right| \leqslant A^m M_m,\quad m\in\mathbb{Z}_+,\quad t\in[0,1].$$

Now we employ the condition $e^{(m)}(1)=0$, $m=0,1,\ldots.$ At that, the convergence of the series implies that e belongs to Carleman-Ostrowski quasi-analytic class (see [12]). Hence, $e(0)=0$. The proof is complete. □

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