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REGULARIZED TRACE FORMULA FOR DISCRETE OPERATORS WITH A PERTURBATION IN THE SCHATTEN-VON NEUMANN CLASS

KH.KH. MURTAZIN, Z.YU. FAZULLIN

Dedicated to the memory of professor
Igor' Fedorovich Krasichkov-Ternovskii

Abstract. In the paper we study a regularized trace formula for discrete self-adjoint operators with a perturbation in Schatten-von Neumann class ($\sigma_p, p \in \mathbb{N}$). We prove that the regularized trace vanishes after deducting $(p-1)$ terms of perturbation theory if there are no dilating gaps in the spectrum of the unperturbed operator.

Keywords: perturbation theory, regularized trace, discrete operator, spectrum, resolvent.

Mathematics Subject Classification: 47B10, 47B15, 47A55

1. Introduction

Let L_0 be a lower semi-bounded self-adjoint discrete operator in a separable Hilbert space H, $V = V^*$ be a bounded operator in H. By λ_k and μ_k, $k = 1, 2, \ldots$, we denote the eigenvalues of operators L_0 and $L = L_0 + V$ indexed in the ascending order counting multiplicities. By f_k we denote an orthonormalized basis in H formed by the eigenfunctions of operator L_0 associated with eigenvalues λ_k and let $N(\lambda) = \sum\limits_{\lambda_k<\lambda} 1$ be the counting function for the spectrum of operator L_0; σ_p, $p \in \mathbb{N}$, be the Schatten-von Neumann class of compact operators.

It follows from the results by M.G. Krein [1], in particular, for discrete operators, that if $V = V^* \in \sigma_1$, i.e., V is nuclear, then the relations

$$\sum_{k=1}^{\infty} (\mu_k - \lambda_k) = \operatorname{Sp} V = \sum_{k=1}^{\infty} (Vf_k, f_k) \tag{1}$$

hold true, i.e.,

$$\sum_{k=1}^{\infty} (\mu_k - \lambda_k - (Vf_k, f_k)) = 0. \tag{2}$$

These results were followed by numerous attempt to prove formula (2) for non-nuclear perturbations V. Let us mention the most essential works in this direction.

Kh.Kh. Murtazin, Z.Yu. Fazullin, Regularized trace formula for discrete operators with a perturbation in the Schatten-von Neumann class.

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In work [2], for arbitrary bounded perturbations V (not necessarily self-adjoint) it was proved for the case when the resolvent $R_0(z) = (L_0 - z)^{-1}$ is a nuclear operator that there exists a subsequence of natural numbers $\{n_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\mu_k - \lambda_k - (Vf_k, f_k)) = 0. \tag{3}$$

In work [3] formula (2) was proved for arbitrary bounded self-adjoint perturbations V under a weaker assumption, namely, as $N(\lambda) = \overline{o}(\lambda)$, $\lambda \to \infty$.

For arbitrary compact perturbations V formula (3) was proved in work [2] under the following two conditions:

1) there exists $\delta \geqslant 0$ such that operator VL_0^{δ} can be continued to a bounded one;

2) $L_0^{-(1+\delta)}$ is a nuclear operator.

In work [3] formula (3) was proved for arbitrary compact perturbations $V = V^*$ under the condition that $N(\lambda) = O(\lambda)$, $\lambda \to \infty$.

In view of the above statements for arbitrary bounded and compact perturbations, in order to prove formula (3), one has to impose a condition for counting function $N(\lambda)$ of the spectrum of unperturbed operator L_0, while for perturbations $V \in \sigma_1$ there is no need in conditions for the growth of function $N(\lambda)$ (cf. formula (2)).

The next progress in this direction was made in work [3], where an analogue of formula (1) for Hilbert-Schmidt perturbations ($V \in \sigma_2$) was obtained and it was proved that formula (3) (vanishing of regularized trace with the first term of pertubation theory deducted) is true for perturbations $V = V^* \in \sigma_2$ with no restrictions for function $N(\lambda)$. Therefore, a natural question appears: how many terms of perturbation theory one has to deduct to vanish the regularized trace in the case of a perturbation in class σ_p, $p \geqslant 3$, without imposing any restriction for the growth of function $N(\lambda)$. A particular answer was provided in work [2], where the authors proved that the regularized trace vanishes after deducting of $(p-1)$ terms of perturbation theory (see formula (4) in the present work) for perturbations in class σ_p, $p \geqslant 2$, under the existence of the system of dilating gaps in the spectrum of unperturbed operator L_0. The latter means that there exists a subsequence $\{n_m\}_{m=1}^{\infty}$ such that $\lambda_{n_m+1} - \lambda_{n_m} \to \infty$ as $m \to \infty$, and it is a rather strict condition in the perturbation theory. In the present work we succeeded to omit this condition and answer the posed question.

In order to formulate the main result of the work we introduce the notations

$$R_0(z) = (L_0 - z)^{-1}, \quad R(z) = (L - z)^{-1}, \quad r_m = \frac{\lambda_{n_m+1} + \lambda_{n_m}}{2}, \quad \Gamma_m = \{z : |z| = r_m\}.$$

Theorem. *Assume that there exists $\delta > 0$ and a sequence $\{n_m\}_{m=1}^{\infty} \subset N$ such that $\lambda_{n_m+1} - \lambda_{n_m} \geqslant \delta$. Then*

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} \left(\mu_k - \lambda_k - \sum_{l=1}^{p-1} \alpha_l^{(m)} \right) = 0, \tag{4}$$

for $V = V^* \in \sigma_p$, $3 \leqslant p$, $p \in \mathbb{N}$, where $\alpha_l^{(m)} = (2\pi i)^{-1}(-1)^l \mathrm{Sp} \oint\limits_{\Gamma_m} z(R_0(z)V)^l R_0(z) dz$ is the l-th term of the perturbation theory.

We first prove auxiliary statements concerning placing of summation brackets.

Lemma 1. *Suppose that there exists $\delta > 0$ and a subsequence $\{n_m\}_{m=1}^{\infty} \subset \mathbb{N}$ such that $\lambda_{n_m+1} - \lambda_{n_m} \geqslant \delta$. Then for each compact operator V in H*

$$\lim_{m\to\infty} \max_{|z|=r_m} \|VR_0(z)\| = 0 \tag{1.1}$$

holds true.

Proof. Let $f \in H$. Since

$$R_0(z) = \sum_{k=1}^{\infty} \frac{(f, f_k) f_k}{\lambda_k - z},$$

for each $N \in \mathbb{N}$ and $z \in \Gamma_m$ we have

$$\|R_0(z)f\|^2 = \sum_{k=1}^{N} \frac{|(f, f_k)|^2}{|\lambda_k - z|^2} + \sum_{k=N+1}^{\infty} \frac{|(f, f_k)|^2}{|\lambda_k - z|^2}. \tag{1.1}$$

Given f, since $|\lambda_k - z| \geqslant \frac{\delta}{2}$, $k \in \mathbb{N}$, and $\sum\limits_{k=1}^{\infty} |(f, f_k)| = \|f\|^2 < \infty$, by choosing appropriate N the second term can be made arbitrarily small, i.e.,

$$\sum_{k=N+1}^{\infty} \frac{|(f, f_k)|^2}{|\lambda_k - z|^2} \leqslant \frac{4}{\delta^2} \sum_{k=N+1}^{\infty} |(f, f_k)|^2 < \frac{\epsilon}{2}. \tag{1.2}$$

We fix N, then as $z \in \Gamma_m$, $m \gg N$ we get

$$\sum_{k=1}^{N} \frac{|(f, f_k)|^2}{|\lambda_k - z|^2} \leqslant \frac{C_N}{|\lambda_N - z|^2} \sum_{k=1}^{N} |(f, f_k)|^2 < \frac{\epsilon}{2}, \tag{1.3}$$

since $|\lambda_N - z| \to \infty$ as $m \to \infty$. Therefore, by (1.2) and (1.3) we conclude that for an arbitrary $f \in H$

$$\lim_{m \to \infty} \max_{|z| = r_m} \|R_0(z) f\| = 0. \tag{1.4}$$

Since V is a compact operator, it can be represented as [4, Ch. IX, Lm. 9.11]

$$V = K_{1n} + K_{2n}, \tag{1.5}$$

where K_{1n} is a finite-dimensional operator and operator K_{2n} is such that $\|K_{2n}\| \to 0$ as $n \to \infty$. Since for each finite-dimensional operator the representation

$$K_{1n} = \sum_{j=1}^{n} (\cdot, \psi_j) \varphi_j, \tag{1.2}$$

holds true [5, Ch. 1], we have

$$K_{1n} R_0(z) = \sum_{j=1}^{n} (\cdot, R_0^*(z) \psi_j) \varphi_j, \quad z \in \Gamma_m. \tag{1.3}$$

Hence,

$$\|K_{1n} R_0(z)\| \leqslant \sum_{j=1}^{n} \|R_0^*(z) \psi_j\| \, \|\varphi_j\|, \quad z \in \Gamma_m, \tag{1.4}$$

that by (1.4), (1.5) lead us to relation (1.1). The proof is complete. □

Lemma 2. *Let V be an arbitrary compact operator in H and assume that there exists a subsequence $\{n_m\}_{m=1}^{\infty} \subset \mathbb{N}$ such that $\lambda_{n_m+1} - \lambda_{n_m} \geqslant \delta$, $\delta > 0$. Then contours Γ_m contain the same number of eigenvalues of operators L_0 and $L = L_0 + V$.*

Proof. Since $D(L_0) \subseteq D(V) = H$, the family of operators $L_x = L_0 + xV$, $x \in [0, 1]$, is a holomorphic family of type (A) [6, Ch. VII]. Therefore, in accordance with the analytic perturbation theorty [6, Ch. VII], eigenvalues $\lambda_n(x)$ of operators L_x are at least continuous functions of parameter x. Let $m \gg 1$ and $z \in \Gamma_m$, then by Lemma 1

$$\|xVR_0(z)\| \leqslant \|VR_0(z)\| < 1. \tag{1.5}$$

This is why all $z \in \Gamma_m$ belong to the resolvent set of operators L_x as $m \gg 1$ since

$$R_x(z) = (L_x - zI)^{-1} = \sum_{k=0}^{\infty} (-1)^k R_0(z) \left[xVR_0(z)\right]^k, \quad z \in \Gamma_m, \tag{1.6}$$

and the series converges. Therefore, in accordance with Theorems 3.16, 3.18 in [6, Ch. IV], eigenvalues $\lambda_n(x)$ (continuous functions w.r.t. parameter x) of the family of operators L_x do not intersect contour Γ_m as $x \in [0,1]$. The proof is complete. □

Proof of Theorem. According to Lemma 1, as $m \gg 1$, $z \in \Gamma_m$ $\|VR_0(z)\| < 1$. Therefore, the resolvent of perturbed operator L satisfies the representation

$$R(z) = \sum_{l=o}^{\infty} (-1)^l (R_0(z)V)^l R_0(z). \tag{1.7}$$

Hence, for $p \geqslant 3$ we have

$$R(z) - R_0(z) - \sum_{l=1}^{p-1} (-1)^l (R_0(z)V)^l R_0(z) = G_p(z) \quad z \in \Gamma_m, \tag{1.6}$$

where

$$\begin{aligned} G_p(z) =& \sum_{l=p}^{\infty} (-1)^l (R_0(z)V)^l R_0(z) \\ =& (-1)^p (R_0(z)V)^p R(z) = (-1)^p (R_0(z)V)^{p-1} R(z) V R_0(z). \end{aligned} \tag{1.7}$$

It implies that $G_p(z)$ is a nuclear operator since $V \in \sigma_p$.

The following lemma holds true.

Lemma 3. *As $m \gg 1$,*

$$\oint_{\Gamma_m} \operatorname{Sp} G_p(z) dz = 0. \tag{1.8}$$

Proof. In view of (1.7) it is sufficient to show that

$$\oint_{\Gamma_m} \operatorname{Sp} (R_0(z)V)^l R_0(z) dz = 0 \tag{1.8}$$

for $l \geqslant p$. In order to do it, we introduce the operators

$$L_x = L_0 + xV, \quad R_x(z) = (L_x - z)^{-1}, \quad 0 \leqslant x \leqslant 1.$$

It is well-known [6, Ch. I, Sect. 4. Subsect. 5, Sect. 5, Subsect. 2] that there exists $\frac{d^s}{dx^s} R_x(z)$ in the uniform topology as $z \in \Gamma_m$ and

$$\frac{d^s}{dx^s} R_x(z) = s! [R_x(z)V]^s R_x(z), \tag{1.9}$$

at that, $R_x(z) = R_0(z)$ as $x = 0$, and $R_x(z) = R(z)$ as $x = 1$.

It is easy to see that

$$\operatorname{Sp} \oint_{\Gamma_m} R_x(z) V R_x(z) dz = 0 \tag{1.10}$$

for $m \gg 1$. Therefore, differentiating (1.10), by (1.9) we obtain that

$$\oint_{\Gamma_m} \operatorname{Sp} [R_x(z)V]^l R_x(z) dz = 0 \tag{1.11}$$

for each $l \geqslant p$. Letting $x = 0$ in (1.11), we arrive at (1.8). The proof is complete. □

Since $V\in\sigma_p$, $p\geqslant 3$, and operator G_p is nuclear, by applying operator $\mathrm{Sp}\left(-\frac{1}{2\pi i}\oint\limits_{\Gamma_m}(\,\cdot\,)z dz\right)$ to identity (1.6), we obtain

$$\sum_{k=1}^{n_m}\left(\mu_k-\lambda_k-\sum_{l=1}^{p-1}\alpha_l^{(m)}\right)=\beta_p^{(m)},$$

where $\beta_p^{(m)}=-(2\pi i)^{-1}\oint\limits_{\Gamma_m} z\mathrm{Sp}\,G_p(z)dz$. Therefore, to prove the theorem, it is sufficient to show that

$$\lim_{m\to\infty}\beta_p^{(m)}=0. \tag{1.12}$$

In order to do it, we introduce the projectors

$$Q_m=-(2\pi i)^{-1}\oint\limits_{\Gamma_m}R(z)dz,\quad Q_m^0=-(2\pi i)^{-1}\oint\limits_{\Gamma_m}R_0(z)dz,\quad Q_m^{\perp}=I-Q_m,\quad Q_m^{0\perp}=I-Q_m^0,$$

and operators

$$R_{m1}(z)=R(z)Q_m,\quad R_{m2}(z)=R(z)Q_m^{\perp},\quad R_{m1}^0(z)=R_0(z)Q_m,\quad R_{m2}^0(z)=R_0(z)Q_m^{0\perp}.$$

Let us demonstrate the proof of (1.12) as $p=3$.

Lemma 4. *If $V\in\sigma_3$, then*

$$\begin{aligned}&\oint\limits_{\Gamma_m} z\mathrm{Sp}\,\left\{(R_{ms}^0(z)V)^2R_{ms}(z)VR_{ms}^0(z)\right\}dz\\ &\quad=\oint\limits_{\Gamma_m}\mathrm{Sp}\,\left\{(R_{ms}^0(z)V)^2R_{ms}(z)VR_{ms}^0(z)\right\}dz=0,\quad s=1,2.\end{aligned}\tag{1.13}$$

Proof. Relations (1.13) for $s=2$ are implied by the fact that operator functions $R_{m2}^0(z)$ and $R_{m2}(z)$ have no singularities in contour Γ_m.

Let $f(z)=z^l\mathrm{Sp}\,\{(R_{m1}^0(z)V)^2R_{m1}(z)VR_{m1}^0(z)\}$, $l=0,1$. Since all the singularities of function $f(z)$ are located inside contour Γ_m and

$$\sum_{\lambda_j,\mu_j}\mathrm{res}\,f(z)=-\mathrm{res}_{z=\infty}f(z).$$

Thus, relations (1.13) are implied by the expansion as $z\in\Gamma_m$

$$f(z)=z^l\left\{\frac{a_4}{z^4}+\frac{a_5}{z^5}+\dots\right\},\quad l=0,1. \tag{1.9}$$

The proof is complete. □

We replace integration over contour Γ_m by the integration over straight line $z_m=\{r_m+it,t\in R\}$. By (1.7) and Lemmata 3, 4 we find that

$$\begin{aligned}\beta_3^{(m)}=&(2\pi i)^{-1}\sum_{k=1}^{n_m}\int_{-\infty}^{\infty}t(\lambda_k-r_m-it)^{-2}[(VR_{m1}^0(r_m+it)R_{m2}(r_m+it)Vf_k,f_k)\\&+(VR_{m2}^0(r_m+it)VR_{m1}(r_m+it)Vf_k,ff_k)+(VR_{m2}^0(r_m+it)VR_{m2}(r_m+it)Vf_k,f_k)]dt\\&+(2\pi i)^{-1}\sum_{k=n_m+1}^{\infty}\int_{-\infty}^{\infty}t(\lambda_k-r_m-it)^{-2}[(VR_{m1}^0VR_{m2}(r_m+it)Vf_k,f_k)\\&+(VR_{m2}^0(r_m+it)VR_{m1}(r_m+it)Vf_k,f_k)+(VR_{m1}^0(r_m+it)VR_{m1}(r_m+it)Vf_k,f_k)]dt.\end{aligned}$$

Let us prove that each of six terms tends to zero as $m \to \infty$. We restrict ourselves by proving this statement for the first term $\beta_{31}^{(m)}$; for other terms the arguments are similar.

Employing polar representation of a bounded operator and by [6]

$$U\,|V|\,U^* = |V^*| = |V|$$

and Cauchy-Schwarz-Bunyakovsky inequality, we have the estimate

$$\left|\left(VR_{m1}^0(z)VR_{m2}(z)Vf_k, f_k\right)\right|^2 \leqslant (VR_{m2}^*(z)\,|V|\,R_{m2}(z)Vf_k, f_k)\cdot\left(VR_{m1}^{*0}(z)\,|V|\,R_{m1}(z)Vf_k, f_k\right).$$

By this estimate and Hölder inequality we find that

$$\left|\beta_{31}^{(m)}\right| \leqslant \left\{\gamma_{31}^{(m)}\right\}^{\frac{1}{2}}\cdot\left\{\omega_{31}^{(m)}\right\}^{\frac{1}{2}}, \tag{1.14}$$

where

$$\gamma_{31}^{(m)} = \frac{1}{2\pi}\sum_{k=1}^{n_m}\int\limits_{-\infty}^{\infty}|t|\,((\lambda_k - r_m)^2 + t^2)^{-1}(VR_{m2}^*(z)\,|V|\,R_{m2}(z)Vf_k, f_k)dt,$$

$$\omega_{31}^{(m)} = \frac{1}{2\pi}\sum_{k=1}^{n_m}\int\limits_{-\infty}^{\infty}|t|\,((\lambda_k - r_m)^2 + t^2)^{-1}(VR_{m1}^{*0}(z)\,|V|\,R_{m1}^0(z)Vf_k, f_k)dt, \quad z = r_m + it.$$

Let us show that $\gamma_{31}^{(m)} \to 0$ as $m \to \infty$. Employing the integral representation for $z = r_m + it$

$$(VR_{m2}^*(z)\,|V|\,R_{m2}(z)Vf_k, f_k) = \int\limits_{\mu_{n_m+1}}^{\infty}ds\int\limits_{\mu_{n_m+1}}^{\infty}d\tau\frac{(V[E(s)-Q_m]\,|v|\,[E(\tau)-Q_m]Vf_k, f_k)}{(s-r_m+it)^2(\tau-r_m-it)^2}$$

and the estimate

$$|(V[E(s)-Q_m]\,|V|\,[E(\tau)-Q_m]Vf_k, f_k)| \leqslant$$
$$\leqslant \frac{1}{2}\left\{(V[E(s)-Q_m]\,|V|\,[E(s)-Q_m]Vf_k, f_k) + (V[E(\tau)-Q_m]\,|V|\,[E(\tau)-Q_m]Vf_k, f_k)\right\},$$

we find that $(z = r_m + it)$

$$|(VR_{m2}^*(z)\,|V|\,R_{m2}(z)Vf_k, f_k)| \leqslant \frac{\pi}{|t|}\int\limits_{\mu_{n_m+1}}^{\infty}\frac{(V[E(s)-Q_m]\,|V|\,[E(s)-Q_m]Vf_k, f_k)}{(s-r_m)^2+t^2}ds.$$

Therefore,

$$\gamma_{31}^{(m)} \leqslant \frac{\pi}{8}\int\limits_{\mu_{n_m+1}}^{\infty}\frac{1}{(s-r_m)^3}\mathrm{Sp}\,(V[E(s)-Q_m]\,|V|\,[E(s)-Q_m]V)ds. \tag{1.15}$$

Let $\{\alpha_i\}_{i=1}^{\infty} = \sigma(V)$ be the spectrum of operator V, $\{\psi_i\}_{i=1}^{\infty}$ be the associated sequence of eigenfunctions. Then in accordance with Hölder inequality

$$\begin{aligned}\mathrm{Sp}\,(V[E(s)-Q_m]\,|V|\,[E(s)-Q_m]V) &= \sum_{i=1}^{\infty}|\alpha_i|^2\,(|V|\,[E(s)-Q_m]\psi_i, [E(s)-Q_m]\psi_i)\\ &\leqslant (\sum_{i=1}^{\infty}|\alpha_i|^3)^{\frac{2}{3}}\cdot(\sum_{i=1}^{\infty}(|V|\,[E(s)-Q_m]\psi_i, [E(s)-Q_m]\psi_i)^3)^{\frac{1}{3}}.\end{aligned} \tag{1.16}$$

Since

$$(|V|\,[E(s)-Q_m]\psi_i, [E(s)-Q_m]\psi_i)^3 \leqslant (|V|^3\,[E(s)-Q_m]\psi_i, [E(s)-Q_m]\psi_i)$$

and

$$\mathrm{Sp}\,[E(s)-Q_m]\,|V|^3\,[E(s)-Q_m] \leqslant \mathrm{Sp}\,(Q_m^{\perp}\,|V|^3\,Q_m^{\perp}),$$

by (1.15), (1.16) we obtain that

$$\gamma_{31}^{(m)} \leqslant \frac{C}{(\mu_{n_m+1} - r_m)^2} \mathrm{Sp}\,(Q_m^{\perp}\,|V|^3\,Q_m^{\perp})^{\frac{1}{3}}, \quad C > 0.$$

Since $\mu_{n_m+1} - r_m \approx 2(\lambda_{n_m+1} - \lambda_{n_m})$ as $m \to \infty$, we conclude that $\gamma_{31}^{(m)} \to 0$ as $m \to \infty$. In the same way one can show that $\omega_{31}^{(m)} \leqslant C\mathrm{Sp}\,(Q_m^{\perp}\,|V|^3\,Q_m^{\perp})$. Therefore, in accordance with (1.14) we have proved that $\beta_{31}^{(m)} \to 0$ as $m \to \infty$. The terms $\beta_{3i}^{(m)}$, $i = \overline{2,6}$, are studied in the same way. The proof of Theorem is complete. □

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Khairulla Khabibulovich Murtazin,
Bashkir State University,
Zaki Validi str., 32,
450074, Ufa, Russia

Ziganur Yusupovich Fazullin,
Bashkir State University,
Zaki Validi str., 32,
450074, Ufa, Russia
E-mail: `fazullinzu@mail.ru`