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ON THE ORBITS OF ANALYTIC FUNCTIONS WITH RESPECT TO A POMMIEZ TYPE OPERATOR

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Dedicated to the memory of Professor
Igor' Fedorovich Krasichkov-Ternovskii

Abstract. Let Ω be a simply connected domain in the complex plane containing the origin, $A(\Omega)$ be the Fréchet space of all functions analytic in Ω. A function g_0 analytic in Ω such that $g_0(0) = 1$ defines the Pommiez type operator which acts continuously and linearly in $A(\Omega)$. In this article we describe cyclic elements of the Pommiez type operator in space $A(\Omega)$. Similar results were obtained early for functions g_0 having no zeroes in domain Ω.

Keywords: Pommiez operator, cyclic element, analytic function.

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Let Ω be a simply connected domain in $\mathbb{C}$ containing the origin; $A(\Omega)$ be the Fréchet space of all functions analytic in Ω. Let a function $g_0 \in A(\Omega)$ be such that $g_0(0) = 1$. We introduce a *Pommiez type* operator as follows. For $f \in A(\Omega)$ we denote

$$D_{0,g_0}(f)(t) := \begin{cases} \dfrac{f(t) - g_0(t)f(0)}{t}, & t \neq 0, \\ f'(0) - g_0'(0)f(0), & t = 0. \end{cases}$$

Operator D_{0,g_0} maps linearly and continuously $A(\Omega)$ into itself. For $g_0 \equiv 1$ we let $D_0 := D_{0,g_0}$. Operator D_0 is called Pommiez operator after the works by M. Pommiez [10]–[13], where successive error terms of Taylor series were studied for the functions analytic in the unit circle.

In the present work we deal with cyclic elements of operator D_{0,g_0} in $A(\Omega)$. Given a locally convex space E and a linear continuous operator $A : E \to E$, an element $x \in E$ is called a *cyclic* element of operator A if the system (orbit of x) $\{A^n(x) : n \geqslant 0\}$ is complete in E, i.e., the closure of its linear span in E coincides with E.

For $g_0 \equiv 1$, the completeness conditions for the system $\{D_{0,g_0}^n(f) : n \geqslant 0\}$ in $A(\Omega)$ were obtained by M.G. Khaplanov [5] and N.I. Nagnibida [4] (in the case Ω is a circle). In works by Y.A. Kaz'min [2] and N.E. Linchuk [3] (also for $g_0 \equiv 1$) there were obtained cyclicity critetions of $f \in A(\Omega)$ w.r.t. operator D_{0,g_0} for an arbitrary simply connected domain Ω and a finite connected domain G in $\overline{\mathbb{C}}$ containing the origin. In work [9], Yu.S. Linchuk proved necessary and sufficient conditions for the completeness in $A(\Omega)$ of the system $\{A^n(f) : n \geqslant 0\}$, where $A(g)(z) = D_0(g)(z) + g(0)\varphi(z)$, $g \in A(\Omega)$, and $\varphi \in A(\Omega)$ is a fixed function [9, Corollary]. At that, it was assumed in [9] in the corresponding criterion that function $1 - z\varphi(z)$ does not vanish in Ω. It is easy to see that $A = D_{0,g_0}$, where $g_0(z) = 1 - z\varphi(z)$.

In the present paper we prove a cyclicity criterion for a function $f \in A(\Omega)$ w.r.t. operator D_{0,g_0} for an *arbitrary* function $g_0 \in A(\Omega)$ (such that $g_0(0) = 1$) not necessary non-zero in the

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whole domain Ω. In [9], by means [9, Lm.], see Lemma 1 in what follows, the proof of the corresponding criterion is reduced to the case $g_0 \equiv 1$ studied in [3, Thm. 2] by means of the theory of characteristic functions for linear continuous operators in $A(\Omega)$ [8]. Such reduction is impossible if g_0 has zeroes in Ω.

In the present work we employ Köthe-Silva-Grothendieck duality between $A(\Omega)$ and the space $A_0(\bar{\mathbb{C}} \setminus \Omega)$ of germs of functions analytic in $\bar{\mathbb{C}} \setminus \Omega$ and vanishing at infinity [8] as well as the results on limit values for Cauchy type integrals [1, Ch. I, Sect. 4].

We fix a function $g_0 \in A(\Omega)$ such that $g_0(0) = 1$. Following [6], [7], for $z \in \Omega$ we introduce *a shift operator* for D_{0,g_0}

$$T_z(f)(t) := \begin{cases} \dfrac{tf(t)g_0(z) - zf(z)g_0(t)}{t-z}, & t \neq z, \\ zg_0(z)f'(z) - zf(z)g_0'(z) + f(z)g_0(z), & t = z, \end{cases}$$

$f \in A(\Omega)$. For each $z \in \Omega$ operator T_z maps linearly and continuously $A(\Omega)$ into itself.

We shall make use of the following abstract functional cyclicity criterion [9, Lm.]:

Lemma 1. *The following statements are equivalent:*

(i) f is a cyclic element of operator D_{0,g_0} in $A(\Omega)$.

(ii) System $\{T_z(f) : z \in \Omega\}$ is complete in $A(\Omega)$.

This result was proven in [9] *without assumption* that g_0 has zeroes in Ω.

In what follows by $A(\Omega)'$ we denote the topologically dual space for $A(\Omega)$.

Theorem 1. *Let Ω be a simply connected domain in $\mathbb{C}$, $0 \in \Omega$. The following statements are equivalent*

(i) f is not a cyclic element of operator D_{0,g_0} in $A(\Omega)$.

(ii) Functions f and g_0 have common zeroes in Ω or there exists a rational function R such that $f = g_0R$.

Proof. *(ii)* $\Rightarrow$ *(i)*: Suppose that f and g_0 have a common zero $\alpha \in \Omega$. Then $D^n_{0,g_0}(f)(\alpha) = 0$ for each $n \geqslant 0$ and each function in $F := \operatorname{span}\{D^n_{0,g_0}(f) : n \geqslant 0\Big\}$ vanishes at α and the same is true for each function in $\overline{F}$, which is the closure of F in $A(\Omega)$. Since there exists a function in $A(\Omega)$ not vanishing at α, then f is not a cyclic element of D_{0,g_0} in $A(\Omega)$.

Suppose that $f = g_0R$, where R is a rational function. Let $R = P/Q$, where P, Q are polynomials, $Q \not\equiv 0$. Let us construct a functional $\varphi \in A(\Omega)' \setminus \{0\}$ such that $\varphi(T_z(f)) = 0$ for each $z \in \Omega$. In order to do it, we construct a function $\gamma \not\equiv 0$ analytic in $\mathbb{C} \setminus \{0\}$ such that $\gamma(\infty) = 0$ and

$$\frac{1}{2\pi i} \int\limits_{|t|=r} \gamma(t)T_z(f)(t)dt = 0, \quad z \in \Omega. \tag{1}$$

Here $r > 0$ is such that the circle $\{t \in \mathbb{C} : |t| \leqslant r\}$ is contained in Ω and polynomial Q does not vanish on the circle $|t| = r$. Let $\gamma(t) := \sum\limits_{k=0}^{N} \frac{c_k}{t^{k+1}}$; numbers c_k and $N \in \mathbb{N}$ will be determined later.

Condition (1) is equivalent to

$$\frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\gamma(t)g_0(t)}{Q(t)} \frac{tP(t)Q(z) - zP(z)Q(t)}{t-z} dt = 0, \quad z \in \Omega. \tag{2}$$

If $|z| = r$, then integrand is defined naturally at $t = z$. Since P and Q are polynomials, there exist $M \in \mathbb{N}$ and polynomials b_j, $0 \leqslant j \leqslant M$, such that

$$\frac{tP(t)Q(z) - zP(z)Q(t)}{t - z} = \sum_{j=0}^{M} b_j(z)t^j.$$

The identity (2) can be rewritten as

$$\frac{1}{2\pi i} \int\limits_{|t|=r} \gamma(t) \frac{g_0(t)}{Q(t)} \sum_{j=0}^{M} b_j(z)t^j dt = 0, \quad z \in \Omega,$$

i.e.,

$$\sum_{j=0}^{M} \Big(\sum_{k=0}^{N} c_k \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{g_0(t)}{Q(t)} t^{j-k-1} dt \Big) b_j(z) = 0, \quad z \in \Omega.$$

These identities hold true if

$$\sum_{k=0}^{N} c_k a_{jk} = 0, \ 0 \leqslant j \leqslant M, \tag{3}$$

where

$$a_{jk} := \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{g_0(t)}{Q(t)} t^{j-k-1} dt.$$

We fix $N > M$. Then system (3) has a nontrivial solution c_k, $0 \leqslant k \leqslant M$. Hence, for a non-zero functional $\varphi \in A(\Omega)'$ such that

$$\varphi(f) := \frac{1}{2\pi i} \int\limits_{|t|=r} \gamma(t) f(t) dt, \quad f \in A(\Omega),$$

the identities

$$\varphi(T_z(f)) = 0, \quad z \in \Omega,$$

hold true. Thus, the system $\{T_z(f) : z \in \Omega\}$ is incomplete in $A(\Omega)$ and by Lemma 1, f is not a cyclic element of operator D_{0,g_0} in $A(\Omega)$.

(i) $\Rightarrow$ *(ii)*: Suppose that f is not a cyclic element of operator D_{0,g_0} in $A(\Omega)$. Then by [8] there exist a function $\gamma \in A_0(\overline{\mathbb{C}} \setminus \Omega)$, $\gamma \not\equiv 0$, a closed rectifiable Jordan curve Γ lying in the analyticity domain of γ and in Ω, such that

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \gamma(t) \frac{tf(t)g_0(z) - zf(z)g_0(t)}{t - z} dt = 0, \ z \in \operatorname{int}\Gamma. \tag{4}$$

Here $A_0(\overline{\mathbb{C}} \setminus \Omega)$ denotes the space of analytic functions in $\overline{\mathbb{C}} \setminus \Omega$ vanishing at infinity; $\operatorname{int}\Gamma$ is the interior of curve Γ.

Let f and g_0 have no common zeroes in Ω. We denote

$$u(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{t\gamma(t)f(t)}{t - z} dt, \quad v(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\gamma(t)g_0(t)}{t - z} dt, \quad z \in \operatorname{int}\Gamma.$$

Functions u and v can be analytically continued in Ω. It follows from (4) that

$$g_0(z)u(z) - zf(z)v(z) = 0, \quad z \in \Omega.$$

Since f and g_0 have no common zeroes in Ω, then

$$v_0 := \frac{v}{g_0} \in A(\Omega) \quad \text{and} \quad u_0 := \frac{u}{f} \in A(\Omega).$$

At that, $u_0(z) = zv_0(z)$, $z \in \Omega$. By the Cauchy integral formula

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{tf(t)v_0(t)}{t-z}dt = f(z)u_0(z), \quad z \in \operatorname{int}\Gamma.$$

Moreover,

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{tf(t)\gamma(t)}{t-z}dt = f(z)u_0(z), \quad z \in \operatorname{int}\Gamma.$$

This is why

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{tf(t)(\gamma(t)-v_0(t))}{t-z}dt = 0, \quad z \in \operatorname{int}\Gamma.$$

In the same way,

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{g_0(t)(\gamma(t)-v_0(t))}{t-z}dt = 0, \quad z \in \operatorname{int}\Gamma.$$

For $z \in \operatorname{ext}\Gamma$ we let

$$\alpha(z) := \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{tf(t)(\gamma(t)-v_0(t))}{t-z}dt, \quad \beta(z) := \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{g_0(t)(\gamma(t)-v_0(t))}{t-z}dt.$$

Here the symbol $\operatorname{ext}\Gamma$ stands for the exterior of Γ. It follows from the properties of Cauchy type integrals and its limiting values (see [1, Ch. I, Sect. 4]) that functions α and β are analytic in $\operatorname{ext}\Gamma$, can be analytically continued into some domain containing Γ, vanish at infinity, and

$$tf(t)(\gamma(t)-v_0(t)) = \alpha(t) \quad \text{and} \quad g_0(t)(\gamma(t)-v_0(t)) = \beta(t).$$

for $t \in \Gamma$. We note that $\gamma(t) - v_0(t) \not\equiv 0$ on Γ. Indeed, otherwise function $v_0(t)$ can be analytically continued into the exterior of Γ and it vanishes at infinity. Hence, it vanishes identically. At the same time, $\gamma \not\equiv 0$. It follows that

$$\frac{f(t)}{g_0(t)} = \frac{\alpha(t)}{t\beta(t)} =: \frac{\alpha(t)}{\omega(t)}$$

outside some finite set on Γ. This is why a meromorphic outside Γ function α/ω can be continued to a function f/g_0 meromorphic in Ω. The obtained function meromorphic in $\mathbb{C}$ has finitely many poles and the infinity is either its zero or pole. It implies that $R := \alpha/\omega$ is a rational function. $\square$

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