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SHARP BOUNDS OF LOWER TYPE FOR ENTIRE FUNCTION OF ORDER $\rho \in (0, 1)$ WITH ZEROES OF PRESCRIBED AVERAGE DENSITIES

G.G. BRAICHEV

Dedicated to the memory of professor
Igor' Fedorovich Krasichkov-Ternovskii

Abstract. We provide sharp two-sided estimates for lower type of entire functions of order $\rho \in (0, 1)$. The zeroes of these functions have prescribed upper and lower average densities and are arbitrarily distributed in the complex plane or on a ray. We analyze the obtained results and compare them with known facts for entire functions of usual type.

Keywords: type and lower type of an entire function, the upper and lower average densities of the sequence of zeroes.

Mathematics Subject Classification: 30D15

1. Introduction

Studying the dependence of the growth of an entire functions on the distribution of its zeroes in the complex plane is important both for the theory of entire functions and for its numerous applications like the theory of interpolation and approximation by exponentials, the problem on finding completeness radius for exponentials systems and general functional systems, spectral theory of operators, probability theory, non-harmonic analysis, issues on analytic continuation of power series and Dirichlet series. This dependence had been studied quite in details by the middle of the previous century in the case of “regularly” growing functions with “properly” distributed zeroes (see works by B.Ya. Levin [1], A. Pfluger [2], [3]). Here asymptotic formulae for an entire function and for its zeroes determine each other. Once such regularity is absent, asymptotic laws do not work anymore and what is come to the foreground is the problem on identifying sharp boundaries of range for growth characteristics of a function subject to the boundaries of zeroes variation speed. Classical characteristics of growth of entire functions are type and lower type, while the speed of zeroes variation is measured by their distribution densities. Let us introduce exact definitions.

Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers tending to infinity and taken in the order of non-decreasing absolute values. Let $n_\Lambda(r) = \sum\limits_{|\lambda_n| \leqslant r} 1$ be a counting function (taking multiplicities into consideration) of this sequence, and $N_\Lambda(r) := \int\limits_0^r \frac{n_\Lambda(t)}{t}\, dt$ be its integral or averaged counting function. Without loss of generality we assume that $0 \notin \Lambda$.

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We take $\rho > 0$. Upper and lower density with exponent ρ (ρ-densities) of sequence Λ are defined respectively by the identities

$$\overline{\Delta}_\rho(\Lambda) := \varlimsup_{r\to+\infty} \frac{n_\Lambda(r)}{r^\rho}, \qquad \underline{\Delta}_\rho(\Lambda) := \varliminf_{r\to+\infty} \frac{n_\Lambda(r)}{r^\rho}.$$

Upper and lower averaged ρ-densities of sequence Λ are introduced by similar identities

$$\overline{\Delta}^*_\rho(\Lambda) := \varlimsup_{r\to+\infty} \frac{N_\Lambda(r)}{r^\rho}, \qquad \underline{\Delta}^*_\rho(\Lambda) := \varliminf_{r\to+\infty} \frac{N_\Lambda(r)}{r^\rho}.$$

The type of entire function $f(z)$ with exponent ρ (briefly, ρ-type) is the quantity

$$\sigma_\rho(f) := \varlimsup_{r\to+\infty} r^{-\rho} \ln \max_{|z|=r} |f(z)| \, .$$

Replacing the upper limit in this identity by the lower one leads us to the definition of lower ρ-type of entire function, which we denote by $\underline{\sigma}_\rho(f)$. The following sharp estimates are well-known (see, for instance, [1, Ch. 4, Sect. 1], [4]):

$$\frac{\overline{\Delta}_\rho(\Lambda)}{\rho\, e} \leqslant \sigma_\rho(f) \leqslant \frac{\pi}{\sin \pi\rho}\, \overline{\Delta}_\rho(\Lambda), \tag{1}$$

$$\overline{\Delta}^*_\rho(\Lambda) \leqslant \sigma_\rho(f) \leqslant \frac{\pi\rho}{\sin \pi\rho}\, \overline{\Delta}^*_\rho(\Lambda). \tag{2}$$

Upper bounds in (1), (2) are valid for $\rho \in (0, 1)$ and they are attained in the case when all the zeroes are located at the same ray and in the definition of ρ-densities there exist usual limits for them (such sequences are called measurable). The lower bound are valid for each $\rho > 0$ and are attained on a rather complicated sequence of complex numbers with the arguments uniformly distributed in $[0, 2\pi]$.

Does the lower bound for ρ-type of an entire function increases if we take into consideration not only the upper ρ-density of its zeroes but also the lower one? The positive answer to this question is implied by the general inequality provided in book [5]:

$$\sigma_\rho(f) \geqslant \frac{\overline{\Delta}_\rho(\Lambda)}{\rho} \exp\left\{ \frac{\underline{\Delta}_\rho(\Lambda)}{\overline{\Delta}_\rho(\Lambda)} - 1 \right\}. \tag{3}$$

Estimate (3) was assumed to be sharp for a long time, but only recently A.Yu. Popov constructed an example [6, Thm. 2.1] providing identity in (3) and showing that lower estimate (2) can not be improved by employing the lower averaged ρ-density of zeroes. The described facts allow us to give the answer for the following extremal problems.

We fix numbers $\rho > 0$, $\beta > 0$, $\alpha \in [0, \beta]$, $\beta^* > 0$, $\alpha^* \in [0, \beta^*]$. Then the identities

$$s_{\mathbb{C}}(\beta;\, \rho) := \inf \left\{ \sigma_\rho(f) :\ \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}_\rho(\Lambda) = \beta \right\} = \frac{\beta}{\rho\, e}, \tag{4}$$

$$s^*_{\mathbb{C}}(\beta^*;\, \rho) := \inf \left\{ \sigma_\rho(f) :\ \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}^*_\rho(\Lambda) = \beta^* \right\} = \beta^*, \tag{5}$$

$$s_{\mathbb{C}}(\alpha,\, \beta;\, \rho) := \inf \left\{ \sigma_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}_\rho(\Lambda) = \beta,\ \underline{\Delta}_\rho(\Lambda) \geqslant \alpha \right\} = \frac{\beta}{\rho\, e}\, e^{\frac{\alpha}{\beta}}, \tag{6}$$

$$s^*_{\mathbb{C}}(\alpha^*, \beta^*;\, \rho) := \inf \left\{ \sigma_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{C},\ \overline{\Delta}^*_\rho(\Lambda) = \beta^*,\ \underline{\Delta}^*_\rho(\Lambda) \geqslant \alpha^* \right\} = \beta^*. \tag{7}$$

hold true.

As it was mentioned above, the upper bounds in (1), (2) are attained by entire functions with measurable zeroes located at a single ray. How can we precise lower bound in (1)– (3) if the zeroes of an entire function are also located at a single ray? Namely, given fixed numbers $\rho \in (0, 1)$, $\beta > 0$, $\alpha \in [0, \beta]$, $\beta^* > 0$, $\alpha^* \in [0, \beta^*]$, we need to calculate the quantities

$$s_{\mathbb{R}_+}(\beta;\, \rho) := \inf \left\{ \sigma_\rho(f) :\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \overline{\Delta}_\rho(\Lambda) = \beta \right\}, \tag{8}$$

$$s_{\mathbb{R}_+}(\alpha,\,\beta;\,\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f=\Lambda\subset\mathbb{R}_+,\ \underline{\Delta}_\rho(\Lambda)\geqslant\alpha,\ \overline{\Delta}_\rho(\Lambda)=\beta\right\}, \tag{9}$$

$$s^*_{\mathbb{R}_+}(\beta^*;\,\rho) := \inf\left\{\sigma_\rho(f):\ \Lambda_f=\Lambda\subset\mathbb{R}_+,\ \overline{\Delta}^*_\rho(\Lambda)=\beta^*\right\}, \tag{10}$$

$$s^*_{\mathbb{R}_+}(\alpha^*,\,\beta^*;\,\rho) := \inf\left\{\sigma_\rho(f):\Lambda_f=\Lambda\subset\mathbb{R}_+,\ \underline{\Delta}^*_\rho(\Lambda)\geqslant\alpha^*,\ \overline{\Delta}^*_\rho(\Lambda)=\beta^*\right\}. \tag{11}$$

Extremal problems (8)–(11) were solved very recently.

Theorem A (A.Yu. Popov [7]). *For each $\rho\in(0,1)$ and $\beta>0$ the identity*

$$s_{\mathbb{R}_+}(\beta;\,\rho)=\beta\, C(\rho)$$

holds true, where $C(\rho)=\max\limits_{a>0}\dfrac{\ln(1+a)}{a^\rho}$. The lower bound $s_{\mathbb{R}_+}(\beta;\,\rho)$ is attained on some increasing sequence of positive numbers.

For problem (8) see also [8], [9].

Theorem B (V.B. Sherstyukov [10]). *For arbitrary $\rho\in(0,1)$ and each number $\beta>0$ and $\alpha\in[0,\beta]$ the identity*

$$s_{\mathbb{R}_+}(\alpha,\,\beta;\,\rho)=\frac{\pi\alpha}{\sin\pi\rho}+\max_{a>0}\int\limits_{a(\alpha/\beta)^{1/\rho}}^{a}\frac{\beta a^{-\rho}-\alpha\tau^{-\rho}}{\tau+1}\,d\tau$$

holds true. Lower bound $s_{\mathbb{R}_+}(\alpha,\,\beta;\,\rho)$ is attained at some increasing sequence $\widetilde{\Lambda}\subset\mathbb{R}_+$ such that $\underline{\Delta}_\rho(\widetilde{\Lambda})=\alpha$ and $\overline{\Delta}_\rho(\widetilde{\Lambda})=\beta$.

Passage from usual densities to the averaged ones required to involve the results of Tauberian type that the appearance roots of some transcendental equation (see [11]).

Theorem C (G.G. Braichev [12], [13]).

I. *For fixed $\rho\in(0,1)$ and $\beta^*>0$ extremal quantity (10) can be found by the formula*

$$s^*_{\mathbb{R}_+}(\beta^*;\,\rho)=C(\rho)\rho e\beta^*,$$

where function $C(\rho)$ is defined in Theorem A.

II. *For fixed $\rho\in(0,1)$, $\beta^*>0$, $\alpha^*\in[0,\beta^*]$ extremal quantity (11) is calculated by the formula*

$$s^*_{\mathbb{R}_+}(\alpha^*,\,\beta^*;\,\rho)=\rho\left(\frac{\pi\alpha^*}{\sin\pi\rho}+\max_{a>0}\int\limits_{aa_1^{1/\rho}}^{aa_2^{1/\rho}}\frac{\beta^*a^{-\rho}-\alpha^*\tau^{-\rho}}{\tau+1}d\tau\right), \tag{12}$$

where $a_1=a_1(\alpha^*,\beta^*)$ and $a_2=a_2(\alpha^*,\beta^*)$, $a_1\leqslant 1\leqslant a_2$ are the roots of the equation

$$a\ln\frac{e}{a}=\alpha^*/\,\beta^*. \tag{13}$$

Quantity $s^*_{\mathbb{R}_+}(\alpha^*,\,\beta^*;\,\rho)$ is attained for some entire function with zeroes $\Lambda_f=\widetilde{\Lambda}$ such that $\underline{\Delta}^*_\rho(\widetilde{\Lambda})=\alpha^*$ and $\overline{\Delta}^*_\rho(\widetilde{\Lambda})=\beta^*$.

For problems (10), (11) see also [14]–[17]. We notice that as opposed to the general distribution of zeroes $\Lambda\subset\mathbb{C}$, usage of lower averaged ρ-density changes essentially the minimal possible ρ-type of entire function as $\Lambda\subset\mathbb{R}_+$.

Summarizing, we can say that the influence of basic density characteristics of zeroes sequences Λ_f in $\mathbb{C}$ or in $\mathbb{R}_+$ of an entire function $f(z)$ for its type was completely studied. On the other hand, as it was shown in paper by V.S. Azarin [18] (see also survey [19]), an entire function with a measurable sequence of zeroes can have no a complete regular growth of the absolute value, i.e., its type and lower type can differ. For a full description of the behavior of such functions $f(z)$ with a known density of zeroes one should know not only the exact range of type $\sigma_\rho(f)$, but also that of lower type $\underline{\sigma}_\rho(f)$. Papers by A.A. Goldberg, B.Ya. Levin and I.V. Ostrovskii [20], [21], A.A. Kondratyuk [22], V.S. Azarin [23] were devoted to the studying of extremal problems involving lower indicator and lower type of entire functions under a given range of density characteristics of zeroes. We also mention the results by I.F. Krasichkov-Ternovskii related with lower bounds for entire functions of finite order in terms of a close averaged characteristics of zeroes distribution called concentration index (see, for instance, [24]). However, much less attention was paid to the most natural problem related with the lower type under fixed densities. The lack of facts in the general theory encourages one for looking particular answers in each particular situation. The evidence of such situation can be found in the known monograph by A.F. Leontiev [25, Ch. VI, Sect. 2].

The present work is devoted to two-sided estimates for the lower type of an entire function with positive or arbitrarily located in the plane zeroes of prescribed averaged densities. We begin with a known relation

$$\underline{\sigma}_\rho(f) \geqslant \underline{\Delta}^*_\rho(\Lambda) \geqslant \frac{\underline{\Delta}_\rho(\Lambda)}{\rho}, \tag{14}$$

implied immediately by Jensen formula and inequalities relating usual and averaged densities. Some preliminary estimates for lower ρ-type of a function $f(z)$ with $\Lambda_f \subset \mathbb{R}_+$ were given in work [17]. Upper estimates for lower ρ-type in terms of lower ρ-densities similar to estimates in (1), (2) are absent in mathematical literature. The explanation of this fact was provided in [26]. Namely, it was proven that it is impossible in principle to estimate lower ρ-type from above only in terms of lower ρ-density. However, upper estimates for lower ρ-type in terms of both ρ-densities are possible and in [26] such precise result was proven, which stated in addition that the greatest possible lower ρ-type is independent of the distribution of zeroes of an entire function on the plane.

Theorem D (G.G. Braichev, O.V. Sherstyukov [26]). *For arbitrary order $\rho \in (0,1)$ and arbitrary numbers $\alpha \geqslant 0$ and $\beta > 0$ ($\alpha \leqslant \beta$) the following identities hold true:*

I.

$$\begin{aligned}&\sup\left\{\underline{\sigma}_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}_\rho(\Lambda) \geqslant \alpha,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\}\\ &\quad = \sup\left\{\underline{\sigma}_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \underline{\Delta}_\rho(\Lambda) \geqslant \alpha,\ \overline{\Delta}_\rho(\Lambda) = \beta\right\} =: \underline{S}(\alpha,\, \beta;\, \rho).\end{aligned}$$

II.

$$\underline{S}(\alpha,\, \beta;\, \rho) = \frac{\pi\beta}{\sin\pi\rho} - \sup_{a>0} \int\limits_a^{a(\beta/\alpha)^{1/\rho}} \frac{\beta\tau^{-\rho} - \alpha a^{-\rho}}{\tau + 1}\, d\tau.$$

Upper bound $\underline{S}(\alpha,\, \beta;\, \rho)$ is attained on some increasing sequence $\widetilde{\Lambda}$ obeying $\underline{\Delta}_\rho(\widetilde{\Lambda}) = \alpha$ and $\overline{\Delta}_\rho(\widetilde{\Lambda}) = \beta$.

In the present work we end up the complete description of the growth of an entire function of order $\rho \in (0,1)$ under irregular behavior of its zeroes and we fill the lack of information on estimates for lower ρ-type. More precisely, we find best possible upper and lower bounds for lower ρ-type in terms of averaged ρ-densities of zeroes by solving extremal problems in two principal cases: the roots of a function are located at the single ray or the roots of a function are arbitrarily distributed in the plane. We mean the following extremal problems.

Given numbers $\rho > 0$, $\beta^* > 0$ and $\alpha^* \in [0, \beta^*]$, find extremal values

$$\underline{s}^*_{\mathbb{C}}(\alpha^*;\, \rho) := \inf\left\{\underline{\sigma}_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}^*_\rho(\Lambda) = \alpha^*\right\}, \tag{15}$$

$$\underline{s}^*_{\mathbb{C}}(\alpha^*,\, \beta^*;\, \rho) := \inf\left\{\underline{\sigma}_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}^*_\rho(\Lambda) = \alpha^*,\ \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\}, \tag{16}$$

$$\underline{s}^*_{\mathbb{R}_+}(\alpha^*;\rho) := \inf\left\{\underline{\sigma}_\rho(f):\ \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \underline{\Delta}^*_\rho(\Lambda) = \alpha^*\right\}, \tag{17}$$

$$\underline{s}^*{}_{\mathbb{R}_+}(\alpha^*,\,\beta^*;\,\rho) := \inf\left\{\underline{\sigma}_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{R}_+,\ \underline{\Delta}^*_\rho(\Lambda) = \alpha^*, \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\}, \tag{18}$$

$$\underline{S}^*{}_{\mathbb{C}}(\alpha^*,\,\beta^*;\,\rho) := \sup\left\{\underline{\sigma}_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{C},\ \underline{\Delta}^*_\rho(\Lambda) = \alpha^*,\ \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\}, \tag{19}$$

$$\underline{S}^*{}_{\mathbb{R}_+}(\alpha^*,\beta^*;\rho) := \sup\left\{\underline{\sigma}_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{R}_+, \underline{\Delta}^*_\rho(\Lambda) = \alpha^*, \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\}. \tag{20}$$

In Section 2 we solve extremal problems (15)–(18). Our results yield that the lowest possible ρ-type in each of the mentioned case of location of zeroes in the plane are independent of upper averaged ρ-density. It is implied immediately by Theorems 1, 2.

Theorem 1. *Let $\rho > 0$. For each fixed numbers $\alpha^* \geqslant 0$ and $\beta^* \geqslant \alpha^*$ the identities*

$$\underline{s}^*{}_{\mathbb{C}}(\alpha^*;\,\rho) = \underline{s}^*{}_{\mathbb{C}}(\alpha^*\,,\beta^*;\,\rho) = \alpha^*$$

hold true. For each value $\beta^* \geqslant \alpha^*$ there exists a sequence $\widetilde{\Lambda} \subset \mathbb{C}$ with averaged ρ-densities $\underline{\Delta}^*_\rho(\widetilde{\Lambda}) = \alpha^*$ and $\overline{\Delta}^*_\rho(\widetilde{\Lambda}) = \beta^*$, on which the lower bounds are attained.

Theorem 2. *Let $\rho \in (0,1)$. For each fixed numbers $\alpha^* \geqslant 0$ and beta$^* \geqslant \alpha^*$ the identities*

$$\underline{s}^*{}_{\mathbb{R}_+}(\alpha^*;\,\rho) = \underline{s}^*{}_{\mathbb{R}_+}(\alpha^*,\,\beta^*;\,\rho) = \frac{\pi\,\rho}{\sin\pi\,\rho}\,\alpha^*$$

hold true. For each value $\beta^* \geqslant \alpha^*$ there exists an increasing sequence $\widetilde{\Lambda} \subset \mathbb{R}_+$ with averaged ρ-densities $\underline{\Delta}^*_\rho(\widetilde{\Lambda}) = \alpha^*$ and $\overline{\Delta}^*_\rho(\widetilde{\Lambda}) = \beta^*$, on which the lower bounds are attained.

In Section 3 we solve extremal problems (19), (20), which imply that, as for usual ρ-densities, the greatest possible lower ρ-type of an entire function is independent of zeroes distribution in the plane, but depend on both averaged ρ-densities. Here we prove the following theorem.

Theorem 3. *For each $\rho \in (0,1)$ and each fixed numbers $\alpha^* \geqslant 0$ and $\beta^* \geqslant \alpha^*$ the identities*

$$\underline{S}^*{}_{\mathbb{C}}(\alpha^*\,,\beta^*;\,\rho) = \underline{S}^*{}_{\mathbb{R}_+}(\alpha^*,\beta^*;\,\rho) = \rho\,\beta^*\left(\frac{\pi}{\sin\pi\rho} - \sup_{b>0}\ \Phi(b)\right)$$

hold true, where

$$\Phi(b) = \int\limits_{ba_2^{-\frac{1}{\rho}}}^{b} \frac{\tau^{-\rho} - a_2 b^{-\rho}}{\tau+1}\,d\tau + \int\limits_{b}^{ba_1^{-\frac{1}{\rho}}} \frac{\tau^{-\rho} - a_1 b^{-\rho}}{\tau+1}\,d\tau = \rho\int\limits_{ba_2^{-\frac{1}{\rho}}}^{ba_1^{-\frac{1}{\rho}}} \tau^{-\rho-1}\ln\frac{\tau+1}{b+1}\,d\tau,$$

and a_1, a_2 are the roots of the equation $a\ln\dfrac{e}{a} = \alpha^*/\,\beta^*$ $(0 \leqslant a_1 \leqslant 1 \leqslant a_2 \leqslant e)$. The upper bound are attained on some increasing sequence $\widetilde{\Lambda}$ such that $\underline{\Delta}^*_\rho(\widetilde{\Lambda}) = \alpha^*$ and $\overline{\Delta}^*_\rho(\widetilde{\Lambda}) = \beta^*$.

In the last section we analyze extremal quantity in Theorem 3 and provide simple two-sided estimate of this quantity. We also establish that it is impossible to obtain an upper bound of lower ρ-type of a function just in terms of lower averaged ρ-density of its zeroes.

2. Lower bound for lower ρ-type of entire function

Proof of Theorem 1. Let $\rho > 0$, $\alpha^* \geqslant 0$ and $\beta^* \geqslant \alpha^*$ be fixed numbers and $f(z)$ be an entire function of order ρ with arbitrarily located in complex plane zeroes $\Lambda_f = \Lambda$ of averaged ρ-densities $\underline{\Delta}^*_\rho(\Lambda) = \alpha^*$, $\overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*$. Classical Jensen formula implies the inequality

$$\ln\max_{|z|=r}|f(z)| \geqslant \frac{1}{2\pi}\int\limits_0^{2\pi}\ln\left|f(re^{i\varphi})\right|\,d\varphi = N_\Lambda(r).$$

We divide it by r^ρ and pass to the limit as $r \to +\infty$ to obtain the estimate

$$\underline{\sigma}_\rho(f) \geqslant \underline{\Delta}^*_\rho(\Lambda) = \alpha^*. \tag{21}$$

To complete the proof of the theorem, for all values of parameters $\rho > 0$, $\beta^* > 0$ and $\alpha^* \in [0, \beta^*]$ we need to construct an entire function $\tilde{f}(z)$, whose lower ρ-type should satisfy identity $\underline{\sigma}_\rho(\tilde{f}) = \alpha^*$.

If $\alpha^* = 0$, by the known inequality $\underline{\Delta}_\rho(\Lambda)/\rho \leqslant \underline{\Delta}^*_\rho(\Lambda) = \alpha^* = 0$, we have $\underline{\Delta}_\rho(\Lambda) = 0$. But as it was shown in work [26], each entire function with zero lower ρ-density of zeroes has a zero lower ρ-type.

Suppose that $\alpha^* > 0$. It follows from Theorem 2.1 in work by A.Yu. Popov [6] that for each $\rho > 0$ and $k \in (0, 1]$ there exists an entire function $f_0(z)$ with zero set Λ_0 satisfying the conditions

$$\underline{\Delta}^*_\rho(\Lambda_0) = \frac{k}{\rho}, \quad \overline{\Delta}^*_\rho(\Lambda_0) = \frac{e^{k-1}}{\rho}.$$

Moreover, it was shown in the proof of this theorem (see [6, Eq.(2.23)]) that an unbounded set of values r_s satisfy the relation

$$\frac{\ln \max\limits_{|z|=r_s} |f_0(z)|}{r_s^\rho} = \frac{k}{\rho} + o(1), \qquad r_s \to +\infty.$$

By (21) it implies the identity $\underline{\sigma}_\rho(f_0) = \dfrac{k}{\rho}$.

Since on the segment $(0, 1]$ function $\dfrac{e^{x-1}}{x}$ decreases from $+\infty$ to 1, we find $k \in (0, 1]$ by the restriction $\dfrac{e^{k-1}}{k} = \dfrac{\beta^*}{\alpha^*}$. For the number $q = \dfrac{\alpha^* \rho}{k}$ we consider the function $\tilde{f}(z) = f_0(q^{1/\rho} z)$. Its zero set $\tilde{\Lambda} = q^{-1/\rho}\Lambda_0$ has the averaged counting function $N_{\tilde{\Lambda}}(r) = N_{\Lambda_0}\left(q^{1/\rho} r\right)$. This is why the identities

$$\underline{\Delta}^*_\rho(\tilde{\Lambda}) = q\,\underline{\Delta}^*_\rho(\Lambda_0) = q\,\frac{k}{\rho} = \alpha^*, \quad \overline{\Delta}^*_\rho(\tilde{\Lambda}) = q\,\overline{\Delta}^*_\rho(\Lambda_0) = q\,\frac{e^{k-1}}{\rho} = q\,\frac{k}{\rho}\,\frac{e^{k-1}}{k} = \alpha^* \frac{\beta^*}{\alpha^*} = \beta^*$$

hold true. It is easy to calculate also lower ρ-type of function $\tilde{f}(z)$:

$$\underline{\sigma}_\rho(\tilde{f}) = q\,\underline{\sigma}_\rho(f_0) = q\,\frac{k}{\rho} = \alpha^*.$$

Summarizing, we conclude that function $\tilde{f}(z)$ has the growth characteristics

$$\underline{\Delta}^*_\rho(\tilde{\Lambda}) = \alpha^*, \qquad \overline{\Delta}^*_\rho(\tilde{\Lambda}) = \beta^*, \qquad \underline{\sigma}_\rho(\tilde{f}) = \alpha^*,$$

i.e., this function is extremal for problems (15), (16). The proof is complete. □

Proof of Theorem 2. We fix numbers $\rho \in (0, 1)$, $\alpha^* \geqslant 0$ and $\beta^* \geqslant \alpha^*$. Suppose that an entire function $f(z)$ has a lower ρ-type $\underline{\sigma}_\rho(f)$ and positive roots $\Lambda_f = \Lambda$ with averaged ρ-densities $\underline{\Delta}^*_\rho(\Lambda) = \alpha^*$, $\overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*$.

We first prove the inequality

$$\underline{\sigma}_\rho(f) \geqslant \frac{\pi\rho}{\sin \pi\rho}\,\alpha^*. \tag{22}$$

We shall make use of the following representation obtained in [15]:

$$r^{-\rho} \ln \max_{|z|=r} |f(z)| = \int\limits_0^{+\infty} \varphi_r(t) \frac{t^\rho}{(1+t)^2}\,dt, \tag{23}$$

where $\varphi_r(t) := \dfrac{N_\Lambda(rt)}{(rt)^\rho}$. We fix $\varepsilon > 0$. It follows from the definition of averaged ρ-densities for sequence Λ that $\varphi_r(t)$ is bounded and there exists a number $c > 0$ such that for all $t \geqslant \dfrac{c}{r}$ the inequality $\varphi_r(t) \geqslant \alpha^*_\varepsilon := \dfrac{\alpha^*}{1+\varepsilon}$ is satisfied. We rewrite and estimate the integral in the right hand side of (23):

$$\int\limits_0^{+\infty} \varphi_r(t) \frac{t^\rho}{(1+t)^2}\,dt \geqslant \alpha^*_\varepsilon \int\limits_0^{+\infty} \frac{t^\rho}{(1+t)^2}\,dt + \int\limits_0^{c/r} (\varphi_r(t) - \alpha^*_\varepsilon) \frac{t^\rho}{(1+t)^2}\,dt$$
$$= \alpha^*_\varepsilon \int\limits_0^{+\infty} \frac{t^\rho}{(1+t)^2}\,dt + o(1) = \frac{\pi\rho}{\sin\pi\rho}\,\alpha^*_\varepsilon + o(1), \quad r \to +\infty.$$

Here we have used the known identity (see [29, Subsect. 2.29, no. 24])

$$\int\limits_0^{+\infty} \frac{t^\rho}{(1+t)^2}\,dt = \frac{\pi\rho}{\sin\pi\rho}.$$

Thus, by (23) and the proven inequalities we obtain

$$r^{-\rho} \ln \max_{|z|=r} |f(z)| \geqslant \frac{\pi\rho}{\sin\pi\rho}\,\alpha^*_\varepsilon + o(1), \quad r \to +\infty.$$

Passing to the lower limit as $r \to +\infty$, and then as $\varepsilon \to 0$, we arrive at inequality (22).

Let us show that there exists a function $\tilde f(z)$ with positive zeroes $\Lambda_f = \tilde\Lambda$ realizing the identity in (22). In order to do it, for any $\rho \in (0,1)$, $\beta^* > 0$ and $\alpha^* \in [0, \beta^*]$ it is sufficient to construct an averaged counting function $N_{\tilde\Lambda}(r)$ uniquely determining the sequence of zeroes such that

$$\underline{\Delta}^*_\rho(\tilde\Lambda) = \alpha^*, \qquad \overline{\Delta}^*_\rho(\tilde\Lambda) = \beta^*, \qquad \underline{\sigma}_\rho(\tilde f) = \frac{\pi\rho}{\sin\pi\rho}\,\alpha^*.$$

In work [16], there was constructed an example of an entire function with zeroes of prescribed averaged ρ-densities located in the angle 2θ and having the least possible ρ-type. As $\theta = 0$, the zeroes of such function are located at a ray. Let us show that in this case it provides not only the least possible ρ-type, but also the least possible lower ρ-type. Let us describe the main idea of constructing counting function $N_{\tilde\Lambda}(r)$ for sequence of zeroes $\tilde\Lambda$ in work [16, Sect. 3] (another approach for constructing such examples was employed earlier in [12], [15]).

Let ξ_n be a sequence of positive numbers obeying the condition

$$\xi_n = o(\xi_{n+1}), \quad n \to +\infty, \tag{24}$$

and a_1, a_2 are the roots of the equation $a \ln \dfrac{e}{a} = \alpha^*/\beta^*$ $(0 \leqslant a_1 \leqslant 1 \leqslant a_2 \leqslant e)$. On each segment $[\xi_n a_2^{-1/\rho}, \xi_n a_1^{-1/\rho}]$, the quantity $N_{\tilde\Lambda}(t)$ is determined by the formulae

$$N_{\tilde\Lambda}(t) = \beta^* \xi_n^\rho + \rho\,\beta^*\,\xi_n^\rho \ln \frac{t}{\zeta_n}, \qquad t \in [\xi_n a_2^{-1/\rho}, \xi_n a_1^{-1/\rho}], \quad n \in \mathbb{N},$$

while outside these segments it is a continuous function quite fast approaching function $y = \alpha^* t^\rho$, $t > 0$, and satisfying the estimate

$$N_{\tilde\Lambda}(t) \leqslant \alpha^* t^\rho, \qquad t \notin \bigcup_{n\in\mathbb{N}} [\xi_n a_2^{-1/\rho}, \xi_n a_1^{-1/\rho}].$$

We denote

$$\varphi(a) = \int\limits_{aa_1^{1/\rho}}^{aa_2^{1/\rho}} \frac{\beta^* a^{-\rho} - \alpha^* \tau^{-\rho}}{\tau + 1}\, d\tau, \quad a > 0.$$

It is easy to show that function $\varphi(a)$ possesses the properties

$$\varphi(0+) = \varphi(+\infty) = 0. \tag{25}$$

In [16, Sect. 3] the relation

$$\tilde{\sigma}(r) := r^{-\rho} \ln \max_{|z|=r} \left|\tilde{f}(z)\right| \leqslant \rho \left(\frac{\pi\, \alpha^*}{\sin \pi\rho} + \varphi\left(\frac{r}{\xi_n}\right) + \varphi\left(\frac{r}{\xi_{n+1}}\right)\right) + o(1)$$

was obtained, which is valid for $r \in [\xi_n, \xi_{n+1}]$, $n \to \infty$. As opposed to the estimate in work [16], here we let $r = r_n = \sqrt{\xi_n\, \xi_{n+1}}$ that implies

$$\tilde{\sigma}(r_n) \leqslant \rho \left(\frac{\pi\, \alpha^*}{\sin \pi\rho} + \varphi\left(\sqrt{\frac{\xi_{n+1}}{\xi_n}}\right) + \varphi\left(\sqrt{\frac{\xi_n}{\xi_{n+1}}}\right)\right) + o(1), \quad n \to \infty.$$

Employing properties (24), (25), we obtain

$$\underline{\sigma}_\rho(\tilde{f}) \leqslant \varliminf_{n\to\infty} \tilde{\sigma}(r_n) \leqslant \frac{\pi\, \rho}{\sin \pi\rho}\, \alpha^*.$$

These estimate and inequality (22) valid for each entire function $f(z)$ with positive zeroes $\Lambda_f = \Lambda$ of averaged ρ-densities $\underline{\Delta}^*_\rho(\Lambda) = \alpha^*$, $\overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*$ lead us to the desired result

$$\underline{\sigma}_\rho(\tilde{f}) = \frac{\pi\, \rho}{\sin \pi\rho}\, \alpha^*.$$

The proof is complete. □

3. Upper bounds for lower ρ-type of an entire function

Proof of Theorem 3. We begin by proving the first statement of the theorem. We take numbers $\rho \in (0, 1)$, $\alpha^* \geqslant 0$ and $\beta^* \geqslant \alpha^*$. Inequality

$$\begin{aligned}
&\underline{S}^*{}_{\mathbb{C}}(\alpha^*, \beta^*; \rho) \geqslant \underline{S}^*{}_{\mathbb{R}_+}(\alpha^*, \beta^*; \rho), \text{ or} \\
&\sup\left\{\underline{\sigma}_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{C},\, \underline{\Delta}^*_\rho(\Lambda) = \alpha^*,\, \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\} \\
&\geqslant \sup\left\{\underline{\sigma}_\rho(f) : \Lambda_f = \Lambda \subset \mathbb{R}_+,\, \underline{\Delta}^*_\rho(\Lambda) = \alpha^*,\, \overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*\right\}
\end{aligned}$$

is obvious since the set of the functions over which we take the first supremum is wider than that for the second supremum. Let us check the opposite inequality. Each entire function $f(z)$ of order $\rho \in (0, 1)$, whose zero set coincides with $\Lambda = (\lambda_n)_{n=1}^\infty = \Lambda_f \subset \mathbb{C} \setminus \{0\}$, is represented by the infinite product

$$f(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{\lambda_n}\right), \qquad z \in \mathbb{C}.$$

We denote $|\Lambda| = (|\lambda_n|)_{n=1}^\infty$ and $f_+(z) = \prod\limits_{n=1}^\infty \left(1 - \dfrac{z}{|\lambda_n|}\right)$. Then

$$\max_{|z|=r} |f(z)| = \max_{|z|=r} \prod_{n=1}^\infty \left|1 - \frac{z}{\lambda_n}\right| \leqslant \max_{|z|=r} \prod_{n=1}^\infty \left(1 + \frac{|z|}{|\lambda_n|}\right) = \max_{|z|=r} |f_+(z)|\,.$$

It yields

$$\underline{\sigma}_\rho(f) = \varliminf_{r\to+\infty} r^{-\rho} \ln \max_{|z|=r} |f(z)| \leqslant \varliminf_{r\to+\infty} r^{-\rho} \ln \max_{|z|=r} |f_+(z)| = \underline{\sigma}_\rho(f_+),$$

and
$$|\Lambda| \subset \mathbb{R}_+, \qquad \underline{\Delta}^*_\rho(\Lambda) = \underline{\Delta}^*_\rho(|\Lambda|), \qquad \overline{\Delta}^*_\rho(\Lambda) = \overline{\Delta}^*_\rho(|\Lambda|).$$
Hence,
$$\underline{S}^*{}_{\mathbb{C}}(\alpha^*,\, \beta^*;\, \rho) \leqslant \underline{S}^*{}_{\mathbb{R}_+}(\alpha^*, \beta^*; \rho),$$
and the desired identity $\underline{S}^*{}_{\mathbb{C}}(\alpha^*\,, \beta^*\,, \rho) = \underline{S}^*{}_{\mathbb{R}_+}(\alpha^*, \beta^*, \rho)$ is proved. The first statement of the theorem reduces the issue on finding the most possible lower ρ-type of an entire function of order less than one by its upper and lower averaged ρ-density of its roots distributed arbitrarily in $\mathbb{C}$ to the case of its distribution in a single ray. The common value of these extremal quantities will be denoted briefly by
$$\underline{S}^*(\alpha^*,\, \beta^*;\, \rho) := \underline{S}^*{}_{\mathbb{C}}(\alpha^*,\, \beta^*;\, \rho) = \underline{S}^*{}_{\mathbb{R}_+}(\alpha^*, \beta^*; \rho).$$
To find this extremal quantity, let us prove first for each $\beta^* > 0$ and $\alpha^* \in [0, \beta^*]$ lower ρ-type of each entire function $f(z)$ of order less than one with zeroes $\Lambda_f = \Lambda$ of averaged ρ-densities $\underline{\Delta}^*_\rho(\Lambda) = \alpha^*$, $\overline{\Delta}^*_\rho(\Lambda) \leqslant \beta^*$ satisfy inequality
$$\underline{\sigma}_\rho(f) \leqslant \rho\beta^* \left(\frac{\pi}{\sin \pi\rho} - \sup_{b>0} \Phi(b) \right), \tag{26}$$
where
$$\Phi(b) = \int\limits_{ba_2^{-1/\rho}}^{b} \frac{\tau^{-\rho} - a_2 b^{-\rho}}{\tau + 1}\, d\tau \; + \int\limits_{b}^{ba_1^{-1/\rho}} \frac{\tau^{-\rho} - a_1 b^{-\rho}}{\tau + 1}\, d\tau,$$
and a_1, a_2 are the roots of equation (13).

Let us consider three cases 1) $\alpha^* = 0$; 2) $\alpha^* = \beta^*$; 3) $\alpha^* \in (0, \beta^*)$. In the first case the roots of equation (13) are numbers $a_1 = 0$, $a_2 = e$ and estimate (26) becomes
$$\begin{aligned}
\underline{\sigma}_\rho(f) \leqslant& \rho\beta^* \left(\frac{\pi}{\sin \pi\rho} - \sup_{b>0} \left\{ \int\limits_{be^{-1/\rho}}^{b} \frac{\tau^{-\rho} - e b^{-\rho}}{\tau + 1}\, d\tau \; + \int\limits_{b}^{+\infty} \frac{\tau^{-\rho}}{\tau + 1}\, d\tau \right\} \right) \\
=& \rho\, \beta^* \left(\frac{\pi}{\sin \pi\rho} - \sup_{b>0} \left\{ \int\limits_{be^{-1/\rho}}^{+\infty} \frac{\tau^{-\rho}}{\tau + 1}\, d\tau \; - e b^{-\rho} \int\limits_{be^{-1/\rho}}^{b} \frac{d\tau}{\tau + 1} \right\} \right) \\
=& \rho\, \beta^* \inf_{b>0} \left(\int\limits_{0}^{be^{-1/\rho}} \frac{\tau^{-\rho}}{\tau + 1}\, d\tau + e b^{-\rho} \ln \frac{1 + b}{1 + b e^{-1/\rho}} \right) \\
\leqslant& \rho\, \beta^* \lim_{b \to +0} \left(\int\limits_{0}^{be^{-1/\rho}} \frac{\tau^{-\rho}}{\tau + 1}\, d\tau + e b^{-\rho} \ln \frac{1 + b}{1 + b e^{-1/\rho}} \right) = 0.
\end{aligned}$$
Thus, in the case $\alpha^* = 0$ we need to prove that $\underline{\sigma}_\rho(f) = 0$. But the implication
$$\alpha^* = 0 \quad \Rightarrow \quad \underline{\sigma}_\rho(f) = 0$$
was proved in Theorem 1.

We proceed to the second case $\alpha^* = \beta^*$, when the sequence of zeroes of an entire function is measurable. Now both roots a_1 and a_2 of equation (13) coincides with one and the integrals in the definition of function $\Phi(b)$ disappear. Relation (26) is reduced to the inequality
$$\underline{\sigma}_\rho(f) \leqslant \frac{\pi\rho}{\sin \pi\rho}\, \beta^*, \tag{27}$$

which is contained in known estimate (2). At that, if sequence Λ is measurable, has averaged ρ-density Δ_ρ^* and is located at a single ray, then, as it is known, the identity

$$\underline{\sigma}_\rho(f) = \sigma_\rho(f) = \frac{\pi\rho}{\sin\pi\rho}\,\Delta_\rho^*$$

holds true. Thus, as $\alpha^* = \beta^*$ the proof of Theorem 3 is complete. □

It should be also mentioned that if a measurable sequence of zeroes of an entire function is located arbitrarily in the complex plane, then inequality (27) can be strict. We are convinced of this fact by the example of function $f_0(z)$ in Theorem 1, where for $\alpha^* = \beta^*$ we have $k = 1$ and

$$\underline{\sigma}_\rho(f_0) = \sigma_\rho(f_0) = \beta^* < \beta^*\,\frac{\pi\rho}{\sin\pi\rho}.$$

It remains to consider the central case of the theorem, when sequence of zeroes $\Lambda_f = \Lambda$ with averaged counting function $N_\Lambda(r) = N(r)$ is such that $0 < \alpha^* < \beta^*$. It follows from the definition of averaged ρ-densities of Λ that for an arbitrary $\varepsilon > 0$ there exists a number $c > 0$ such that for all values $r \geqslant c$ the inequalities

$$\alpha^*(1-\varepsilon)\,r^\rho < N(r) < \beta^*(1+\varepsilon)\,r^\rho$$

hold true, and for some sequence $r_k \nearrow +\infty$ we have

$$N(r_k) < \alpha^*(1+\varepsilon)\,r_k^\rho.$$

It is more convenient to pass to the counting functions $n_1(x) := n(e^x)$ and $N_1(x) := N(e^x)$. It is obvious that $N_1(x)$ satisfies the relations

$$\alpha^*(1-\varepsilon)\,e^{\rho x} < N_1(x) < \beta^*(1+\varepsilon)\,e^{\rho x}, \qquad x > \ln c, \tag{28}$$

$$N_1(x_k) < \alpha^*(1+\varepsilon)\,e_k^{\rho x_k}, \qquad k \in \mathbb{N}, \qquad x_k = \ln r_k. \tag{29}$$

For further estimates it is convenient to denote $A = \alpha^*(1+\varepsilon)$, $B = \beta^*(1+\varepsilon)$, $y_A(x) = Ae^{\rho x}$, $y_B(x) = Be^{\rho x}$ and to employ the following auxiliary statement.

Lemma 1. *Let $A < B$ and we draw the tangent lines to the graph of function $y_B(x)$ to the left and to the right of the point $(x_0,\, Ae^{\rho x_0})$. Then the abscissas of the left and right tangency points x_l and x_r are given by the formulae*

$$x_l = x_0 + \frac{1}{\rho}\ln a_1, \quad x_r = x_0 + \frac{1}{\rho}\ln a_2, \tag{30}$$

where a_1 and a_2 are the roots of equation(13)*.*

Proof. To find the required points, we calculate slopes for the tangent lines to the graph of G_B passing through the point $(x_0,\, e^{Ax_0})$. For instance, for the right tangency point (for the left point the calculations are similar)

$$\frac{Be^{\rho\,x_r} - Ae^{\rho\,x_0}}{x_r - x_0} = B\,\rho\,e^{\rho\,x_r}.$$

We divide this identity by $\dfrac{Be^{\rho\,x_r}}{x_r - x_0}$ and we denote $y_r = x_r - x_0$. We obtain $1 - \frac{A}{B}e^{-\rho\,y_r} = \rho\,y_r$, or $(1-\rho\,y_r)\,e^{\rho\,y_r} = \frac{A}{B}$, that can be written as

$$e^{\rho\,y_r}\,\ln\frac{e}{e^{\rho\,y_r}} = \frac{A}{B} = \frac{\alpha^*}{\beta^*}.$$

In view of the relation $y_r = x_r - x_0 > 0$ we conclude that $e^{\rho\,y_r} = a_2$, i.e. $y_r = x_r - x_0 = \frac{1}{\rho}\ln a_2$, where $a_2 \in (1, e)$. Finally $x_r = x_0 + \frac{1}{\rho}\ln a_2$ and it completes the proof. □

We continue estimating averaged counting function $N_1(x)$ taking as x_0 any of points x_k appearing in (29). Taking relation (29) into consideration, we can state that the graph of $N_1(x)$ on the segment $[x_0,\, x_r]$ does not intersect the right tangent line since otherwise it would intersect also the graph of the function $y_B(x) = Be^{\rho x}$, while in accordance with (28) it is located below this graph. We write the equation of the considered tangent line as $y = Be^{\rho\, x_r} + \rho Be^{\rho\, x_r}(x - x_r)$ to obtain that

$$N_1(x) \leqslant Be^{\rho\, x_r} + \rho Be^{\rho\, x_r}(x - x_r), \quad x \in [x_0,\, x_r].$$

Since all what was said above is true for the left tangent line as well, we also have the inequality

$$N_1(x) \leqslant Be^{\rho\, x_l} + \rho Be^{\rho\, x_l}(x - x_l), \quad x \in [x_l,\, x_0].$$

We denote $y_0 = e^{x_0}$, $y_l = e^{x_l}$, $y_r = e^{x_r}$. By Lemma 1 we have $y_l = y_0 a_1^{1/\rho}$, $y_r = y_0 a_2^{1/\rho}$. Returning back to the original counting function $N(y) = N_1(\ln y)$, we write inequalities obtained for function $N_1(x)$ as

$$N(y) \leqslant \begin{cases} By_l^\rho\,(1 + \rho(\ln y - \ln y_l)) = By_0^\rho a_1 \left(1 + \rho \ln \dfrac{y}{y_0 a_1^{1/\rho}}\right), & y \in [y_0 a_1^{1/\rho},\, y_0], \\[2ex] By_0^\rho a_2 \left(1 + \rho \ln \dfrac{y}{y_0 a_2^{1/\rho}}\right), & y \in [y_0,\, y_0 a_2^{1/\rho}], \\[2ex] By^\rho, & y \notin [y_0 a_1^{1/\rho},\, y_0 a_2^{1/\rho}], \quad y > c. \end{cases}$$

We recall that here $a_1 \leqslant 1 \leqslant a_2$ are roots of equation (13) and as x_0 we choose an arbitrary point x_k in (29). We fix numbers $b > 0$, $k \in \mathbb{N}$, and in the previous inequalities we let $y = r_k\tau$, $r_k = by_0 = be^{x_k}$. Then

$$N(r_k\tau) \leqslant \begin{cases} B\left(\dfrac{r_k}{b}\right)^\rho a_1 \left(1 + \ln \dfrac{(b\tau)^\rho}{a_1}\right), & \tau \in \left[\dfrac{a_1^{1/\rho}}{b},\, \dfrac{1}{b}\right], \\[2ex] B\left(\dfrac{r_k}{b}\right)^\rho a_2 \left(1 + \ln \dfrac{(b\tau)^\rho}{a_2}\right), & \tau \in \left[\dfrac{1}{b},\, \dfrac{a_2^{1/\rho}}{b}\right], \\[2ex] B(r_k\tau)^\rho, & \tau \notin \left[\dfrac{a_1^{1/\rho}}{b},\, \dfrac{a_2^{1/\rho}}{b}\right], \ \tau > \dfrac{c}{r_k}. \end{cases} \tag{31}$$

Taking into consideration that a_1 and a_2 are roots of equation (13), we rewrite

$$a_i \left(1 + \ln \frac{(b\tau)^\rho}{a_i}\right) = a_i \ln \frac{e}{a_i} + a_i \ln (b\tau)^\rho = \frac{A}{B} + a_i \ln (b\tau)^\rho, \qquad i = 1, 2.$$

Now by formula (23) for the function $\varphi_r(\tau) = \dfrac{N(r\tau)}{(r\tau)^\rho}$ we obtain the estimate

$$\varphi_{r_k}(\tau) \leqslant \psi(\tau) = \begin{cases} B(\tau b)^{-\rho}\left(\dfrac{A}{B} + a_1 \ln (b\tau)^\rho\right), & \tau \in \left[\dfrac{a_1^{1/\rho}}{b},\, \dfrac{1}{b}\right], \\[2ex] B(\tau b)^{-\rho}\left(\dfrac{A}{B} + a_2 \ln (b\tau)^\rho\right), & \tau \in \left[\dfrac{1}{b},\, \dfrac{a_2^{1/\rho}}{b}\right], \\[2ex] B, & \tau \notin \left[\dfrac{a_1^{1/\rho}}{b},\, \dfrac{a_2^{1/\rho}}{b}\right] \end{cases} \tag{32}$$

as $\tau > \frac{c}{r_k}$. Employing the first part of the theorem, we assume that all the zeroes of the function are positive and we use representation (23) once again not for all $r > 0$ but only for the values

$r = r_k$, $k = 1, 2, \ldots$, (see (31)). As a result we get

$$\sigma(r_k) = r_k^{-\rho} \ln \max_{|z|=r_k} |f(z)| = \int_0^{+\infty} \varphi_{r_k}(\tau) \frac{t^\rho}{(1+\tau)^2}\, d\tau = \int_0^{+\infty} \psi(\tau) \frac{\tau^\rho}{(1+\tau)^2}\, d\tau$$

$$+ \int_0^{\frac{c}{r_k}} \left(\varphi_{r_k}(\tau) - \psi(\tau)\right) \frac{\tau^\rho}{(1+\tau)^2}\, d\tau + \int_{\frac{c}{r_k}}^{+\infty} \left(\varphi_{r_k}(\tau) - \psi(\tau)\right) \frac{\tau^\rho}{(1+\tau)^2}\, d\tau$$

$$\leqslant \int_0^{+\infty} (\psi(\tau) - B) \frac{\tau^\rho}{(1+\tau)^2}\, d\tau + B \int_0^{+\infty} \frac{\tau^\rho}{(1+\tau)^2}\, d\tau + O(1) \frac{c}{r_k}$$

$$= B \frac{\pi\rho}{\sin \pi\rho} + B \int_{b^{-1} a_1^{1/\rho}}^{b^{-1}} \frac{b^{-\rho}\left(\frac{A}{B} + a_1 \ln(b\tau)^\rho\right) - \tau^\rho}{(1+\tau)^2}\, d\tau$$

$$+ B \int_{b^{-1}}^{b^{-1} a_2^{1/\rho}} \frac{b^{-\rho}\left(\frac{A}{B} + a_2 \ln(b\tau)^\rho\right) - \tau^\rho}{(1+\tau)^2}\, d\tau + o(1) =: B\left(\frac{\pi\rho}{\sin \pi\rho} + I_1 + I_2\right) + o(1).$$

Thus, the inequality

$$\sigma(r_k) \leqslant B\left(\frac{\pi\rho}{\sin \pi\rho} + I_1 + I_2\right) + o(1), \quad k \to \infty. \tag{33}$$

holds true. We simplify integrals I_1 and I_2 integrating by parts. For I_1 we have

$$I_1 = \left. \frac{b^{-\rho}\left(\frac{A}{B} + a_1 \ln(b\tau)^\rho\right) - \tau^\rho}{(1+\tau)^2} \right|_{b^{-1} a_1^{1/\rho}}^{b^{-1}} + \rho \int_{b^{-1} a_1^{1/\rho}}^{b^{-1}} \frac{a_1 b^{-\rho} - \tau^\rho}{\tau(\tau+1)}\, d\tau.$$

In the integral we make the change of variable $\tau = t^{-1}$, while the calculation of the off-integral term leads us to the expression

$$-\frac{b^{-\rho}\left(\frac{A}{B} - 1\right)}{b^{-1}+1} + \frac{b^{-\rho}\left(\frac{A}{B} + a_1 \ln a_1 - a_1\right)}{b^{-1}+1} = -\frac{b^{-\rho}\left(\frac{A}{B} - 1\right)}{b^{-1}+1}.$$

Finally we obtain

$$I_1 = -\frac{b^{-\rho}\left(\frac{A}{B} - 1\right)}{b^{-1}+1} + \rho \int_b^{b a_1^{-1/\rho}} \frac{a_1 b^{-\rho} - t^{-\rho}}{t+1}\, dt. \tag{34}$$

Similar calculations for integral I_2 give

$$I_2 = \frac{b^{-\rho}\left(\frac{A}{B} - 1\right)}{b^{-1}+1} + \int_{b a_2^{-1/\rho}}^{b} \frac{a_2 b^{-\rho} - t^{-\rho}}{t+1}\, dt. \tag{35}$$

Taking into consideration (33)–(35), as $k \to \infty$ we can write

$$\sigma(r_k) \leqslant B\rho \left(\frac{\pi\rho}{\sin \pi\rho} + \int_b^{b a_1^{-1/\rho}} \frac{a_1 b^{-\rho} - t^{-\rho}}{t+1}\, dt + \int_{b a_2^{-1/\rho}}^{b} \frac{a_2 b^{-\rho} - t^{-\rho}}{t+1}\, dt\right) + o(1).$$

Changing signs in the integrands and passing to the lower limit as $k \to \infty$, we get

$$\underline{\sigma}_\rho(f) \leqslant B\rho \left(\frac{\pi}{\sin \pi\rho} - \left\{ \int\limits_{ba_2^{-1/\rho}}^{b} \frac{t^{-\rho} - a_2 b^{-\rho}}{t+1}\, dt + \int\limits_{b}^{ba_1^{-1/\rho}} \frac{t^{-\rho} - a_1 b^{-\rho}}{t+1}\, dt \right\} \right).$$

The estimate is valid for each $\varepsilon > 0$ and $b > 0$. Letting ε to tend to zero and taking the supremum over $b > 0$, we finally obtain

$$\underline{\sigma}_\rho(f) \leqslant \beta^* \rho \left(\frac{\pi}{\sin \pi\rho} - \sup_{b>0} \left\{ \int\limits_{ba_2^{-1/\rho}}^{b} \frac{t^{-\rho} - a_2 b^{-\rho}}{t+1}\, dt + \int\limits_{b}^{ba_1^{-1/\rho}} \frac{t^{-\rho} - a_1 b^{-\rho}}{t+1}\, dt \right\} \right).$$

In order to complete the proof of the theorem, given numbers $\rho \in (0,1)$, $\beta^* > 0$ and $\alpha^* \in [0, \beta^*]$, we need to find an entire function $\tilde{f}(z)$ providing the identity in the obtained estimate for the lower type and having positive zeroes $\Lambda_f = \tilde{\Lambda}$ of averaged ρ-densities $\underline{\Delta}^*_\rho(\tilde{\Lambda}) = \alpha^*$, $\overline{\Delta}^*_\rho(\tilde{\Lambda}) \leqslant \beta^*$. Let us construct the averaged counting function $N_{\tilde{\Lambda}}(r) =: N(r)$ for such extremal sequence of zeroes. We order the terms of the required sequence in the ascending order:

$$\widetilde{\Lambda} = (\lambda_n)_{n=1}^{\infty}, \quad 0 < \lambda_1 = \ldots = \lambda_{n_1} < \lambda_{n_1+1} = \ldots = \lambda_{n_2} < \lambda_{n_2+1} = \ldots \tag{36}$$

As $r \in \left[\lambda_{n_k},\, \lambda_{n_{k+1}}\right)$, for the counting function we have $n_{\tilde{\Lambda}}(r) = n(r) = n_k$, while the averaged counting function $N(r)$ reads as

$$N(r) = N(\lambda_{n_k}) + n_k \ln \frac{r}{\lambda_{n_k}}, \qquad k \in \mathbb{N}.$$

It is again convenient to pass to the exponential independent variable. We consider the functions

$$\omega(x) := n(e^x), \qquad \Omega(x) := N(e^x).$$

The graph of the function $\Omega(x) = \int\limits_0^x \omega(t)\, dt$ is a polyline, each segment of which is described by a linear equation with a natural slope:

$$\Omega(x) = \Omega(\ln \lambda_{n_k}) + n_k(x - \ln \lambda_{n_k}), \qquad x \in [\ln \lambda_{n_k},\, \ln \lambda_{n_{k+1}}), \quad k \in \mathbb{N}.$$

As opposed to the above consideration, we denote $A = \alpha^*$, $B = \beta^*$ and consider the function $y_B(x) = B\, e^{\rho x}$, $x > 0$. We consider tangent lines l_j to the graph G_B of this function with the slopes being equal to successive natural numbers $j \in \mathbb{N}$. The abscissas x_j of the tangency points can be easily calculated:

$$x_j = \frac{1}{\rho} \ln \frac{j}{\rho B}, \quad j \in \mathbb{N}.$$

Let $y = \Omega_1(x)$ be the equation of the polyline, whose j-th segment is the segment of tangent line l_j containing the tangency point $(x_j,\, Be^{\rho x_j})$. By the convexity of function $y_B(x)$, the graph of this polyline is located below graph G_B. Moreover, by Lemma in book [11], on each segment $[x_j, x_{j+1}]$, $j \to \infty$, we have the relation

$$0 \leqslant y_B(x) - \Omega_1(x) \leqslant \frac{1}{4}(x_{j+1} - x_j) = \frac{1}{4\rho} \ln \frac{j+1}{j} < \frac{1}{4\rho\, j} = O\left(e^{-\rho x}\right). \tag{37}$$

Thus,

$$0 \leqslant y_B(x) - \Omega_1(x) \to 0, \quad x \to +\infty.$$

It follows that there exists the limit

$$\lim_{x \to +\infty} \frac{\Omega_1(x)}{e^{\rho x}} = B. \tag{38}$$

We modify polyline $y = \Omega_1(x)$. In order to do it, we choose a strictly increasing sequence of natural numbers m_n satisfying the condition

$$\frac{m_{n+1}}{m_n} \nearrow +\infty, \qquad n \to \infty. \tag{39}$$

For each $j = m_n$, $n \in \mathbb{N}$, we extend segment l_j of polyline $y = \Omega_1(x)$ until it touches graph G_A of function $y_A(x) = A\, e^{\rho x}$ at the point $\left(\xi_j, A\, e^{\rho\, \xi_j}\right)$. Then from this point we draw the tangent line l'_j to G_B at the point $\left(\xi'_j, B\, e^{\rho\, \xi'_j}\right)$. In accordance with Lemma 1, the abscissas of the mentioned points are given by the formulae

$$\xi_j = x_j - \frac{1}{\rho}\ln a_1, \quad \xi'_j = \xi_j + \frac{1}{\rho}\ln a_2 = x_j + \frac{1}{\rho}\ln\frac{a_2}{a_1}, \quad j = m_n, \quad n \in \mathbb{N},$$

where a_1, a_2 are the roots of equation (13).

We denote $\left[x_{m_n},\, x_{m_n} + \dfrac{1}{\rho}\ln\dfrac{a_2}{a_1}\right] =: I_n$, $n \in \mathbb{N}$. If $\dfrac{a_2}{a_1} j \in \mathbb{N}$, then on each segment I_n we assume that new polyline $y = \Omega(x)$ is defined by the equations of the above described semi-tangent lines ($j = m_n$):

$$\Omega(x) = Be^{\rho x_j} + \rho Be^{\rho x_j}(x - x_j) = Be^{\rho x_j}\left(1 - \rho(x - x_j)\right), \qquad x \in [x_j,\, \xi_j], \tag{40}$$

$$\Omega(x) = Be^{\rho\xi'_j} + \rho Be^{\rho\xi'_j}(x - \xi'_j) = Be^{\rho\xi'_j}\left(1 - \rho(x - \xi'_j)\right), \qquad x \in \left[\xi_j,\, \xi'_j\right]. \tag{41}$$

On the set $\mathbb{R} \setminus \bigcup\limits_{n=1}^{\infty} I_n =: J$ we keep polyline $y = \Omega_1(x)$ unchanged, i.e., we let $\Omega(x) = \Omega_1(x)$, $x \in J$.

If $\dfrac{a_2}{a_1} j \notin \mathbb{N}$, then we draw the right tangent line with the slope $\left[\dfrac{a_2}{a_1} j\right]$ (square brackets stand for the integer part). It intersects the left tangent line not at a point of graph G_A, but at some point, which we denote by $(\xi_j,\, \Omega(\xi_j))$. This point is located above this graph. Not complicated but careful calculations show that the condition

$$\frac{\Omega(\xi_j)}{e^{\rho\xi_j}} \to A, \qquad j = m_n \to \infty, \tag{42}$$

holds true. Moreover, since on set J we have let $\Omega(x) = \Omega_1(x)$, relation (38) holds true for $x \in J$, i.e.,

$$\lim_{J \ni x \to +\infty} \frac{\Omega(x)}{e^{\rho x}} = B. \tag{43}$$

In view of conditions (42), (43) we conclude that the limiting relations

$$\varliminf_{x \to +\infty} \frac{\Omega(x)}{e^{\rho x}} = A, \quad \varlimsup_{x \to +\infty} \frac{\Omega(x)}{e^{\rho x}} = B \tag{44}$$

hold true. The formula $N(e^x) = \Omega(x)$ determines the averaged counting function of sequence $\widetilde{\Lambda}$ like (36) as follows: the abscissas of the vertices of the constructed polyline (more precisely, their logarithms) define the terms of the sequences, each of them appears $\widetilde{\Lambda}$ with the multiplicity being equal to the difference of the slope of the segments having a joint vertex. Let us prove that the constructed sequence is extremal. Indeed, conditions (44) mean that

$$\underline{\Delta}^*_\rho(\widetilde{\Lambda}) = \varliminf_{r \to +\infty} \frac{N(r)}{r^\rho} = \varliminf_{x \to +\infty} \frac{\Omega(x)}{e^{\rho\, x}} = A, \quad \overline{\Delta}^*_\rho(\widetilde{\Lambda}) = \varlimsup_{r \to +\infty} \frac{N(r)}{r^\rho} = \varlimsup_{x \to +\infty} \frac{\Omega(x)}{e^{\rho\, x}} = B.$$

Letting

$$x = \ln t, \quad \zeta_n = e^{\xi_{m_n}} = \left(\frac{m_n}{\rho B a_1}\right)^{1/\rho}, \quad t_n^{(i)} = \zeta_n a_i^{1/\rho}, \qquad n \in \mathbb{N},$$

in (40), (41), (37) we obtain

$$N(t) = B\,\zeta_n^\rho\, a_1\,\left(1+\rho\,\ln\frac{t}{\zeta_n a_1^{1/\rho}}\right) = B\,\zeta_n^\rho\, a_1\,\ln\frac{e}{a_1}\left(\frac{tr}{\zeta_n}\right)^\rho, \quad t\in[t_n^{(1)},\,\zeta_n], \tag{45}$$

$$N(t) = B\,\zeta_n^\rho\, a_2\,\left(1+\rho\,\ln\frac{t}{\zeta_n a_2^{1/\rho}}\right) = B\,\zeta_n^\rho\, a_2\,\ln\frac{e}{a_2}\left(\frac{tr}{\zeta_n}\right)^\rho, \quad t\in[\zeta_n,\,t_n^{(2)}], \tag{46}$$

$$B\,t^\rho \geqslant N(t) \geqslant B\,t^\rho\,-\,O\left(t^{-\rho}\right), \quad t\notin \bigcup_{n=1}^{\infty}[t_n^{(1)},\,t_n^{(2)}] \;=: T. \tag{47}$$

For the logarithm of the maximal absolute value of the canonical product $\widetilde{f}(z)$ constructed by sequence $\widetilde{\Lambda}$ we again employ representation (23):

$$\sigma(r) = r^{-\rho}\ln\max_{|z|=r}\left|\widetilde{f}(z)\right| = \int\limits_0^{+\infty}\varphi_r(t)\frac{t^\rho}{(1+t)^2}\,dt, \tag{48}$$

where $\varphi_r(t) := \dfrac{N_{\widetilde{\Lambda}}(rt)}{(rt)^\rho}$. According to (45)–(47), we have

$$\varphi_r(t) = \begin{cases} Ba_1\left(\dfrac{\zeta_n}{rt}\right)^\rho\,\ln\dfrac{e}{a_1}\left(\dfrac{tr}{\zeta_n}\right)^\rho, & t\in\left[\dfrac{t_n^{(1)}}{r},\,\dfrac{\zeta_n}{r}\right], \quad n\in\mathbb{N},\\[2ex] Ba_2\left(\dfrac{\zeta_n}{rt}\right)^\rho\,\ln\dfrac{e}{a_2}\left(\dfrac{tr}{\zeta_n}\right)^\rho, & t\in\left[\dfrac{\zeta_n}{r},\,\dfrac{t_n^{(2)}}{r}\right], \quad n\in\mathbb{N},\\[2ex] B\,-\,O\left((tr)^{-2\rho}\right), & t\notin T=\bigcup\limits_{n=1}^{\infty}[t_n^{(1)},\,t_n^{(2)}]. \end{cases} \tag{49}$$

We rewrite the integrals in the right hand side of (48) as follows

$$\begin{aligned}\sigma(r) =& B\int\limits_0^{+\infty}\frac{t^\rho}{(1+t)^2}\,dt\,-\,\int\limits_0^{+\infty}(B-\varphi_r(t))\,\frac{t^\rho}{(1+t)^2}\,dt\\ =& B\frac{\pi\rho}{\sin\pi\rho}\,-\int\limits_{\mathbb{R}_+\setminus T}(B-\varphi_r(t))\,\frac{t^\rho}{(1+t)^2}\,dt-\int\limits_T(B-\varphi_r(t))\,\frac{t^\rho}{(1+t)^2}\,dt\\ =& B\frac{\pi\rho}{\sin\pi\rho}\,-\,\left(I_1(r)\,+\,I_2(r)\right).\end{aligned}$$

As $r\to+\infty$, integral $I_1(r)$ can be simply estimated:

$$I_1(r) = O\left(r^{-2\rho}\right)\int\limits_{\mathbb{R}_+\setminus T}\frac{dt}{t^\rho(1+t)^2}\;\leqslant\;O\left(r^{-2\rho}\right)\int\limits_0^{+\infty}\frac{dt}{t^\rho(1+t)^2} = O\left(r^{-2\rho}\right).$$

Calculation of integral $I_2(r)$ requires much more efforts. Employing two first lines in (49), we have

$$I_2(r) = \int\limits_T(B-\varphi_r(t))\,\frac{t^\rho}{(1+t)^2}\,dt = \sum_{n=1}^{\infty}\int\limits_{t_n^{(1)}/r}^{t_n^{(2)}/r}(B-\varphi_r(t))\,\frac{t^\rho}{(1+t)^2}\,dt$$

$$
= \sum_{n=1}^{\infty} \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \left(B - Ba_1 \left(\frac{\zeta_n}{rt}\right)^{\rho} \ln \frac{e}{a_1} \left(\frac{tr}{\zeta_n}\right)^{\rho} \right) \frac{t^\rho}{(1+t)^2}\, dt
$$

$$
+ \sum_{n=1}^{\infty} \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \left(B - Ba_2 \left(\frac{\zeta_n}{rt}\right)^{\rho} \ln \frac{e}{a_2} \left(\frac{tr}{\zeta_n}\right)^{\rho} \right) \frac{t^\rho}{(1+t)^2}\, dt = B\, S(r),
$$

i.e., $I_2(r) = B\, S(r)$, where we denote $S(r) = \sum\limits_{n=1}^{\infty} S_n(r)$ and

$$
S_n(r) = \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \frac{t^\rho - a_1 \left(\frac{\zeta_n}{r}\right)^{\rho} \ln \frac{e}{a_1} \left(\frac{tr}{\zeta_n}\right)^{\rho}}{(1+t)^2}\, dt \; + \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho - a_2 \left(\frac{\zeta_n}{r}\right)^{\rho} \ln \frac{e}{a_2} \left(\frac{tr}{\zeta_n}\right)^{\rho}}{(1+t)^2}\, dt.
$$

A preliminary result of transformations of the integral in (48) is the asymptotic identity

$$
\sigma(r) = B \frac{\pi\rho}{\sin \pi\rho} \; - \sum_{n=1}^{\infty} S_n(r) \; + \; o(1), \qquad r \to +\infty. \tag{50}
$$

Our next task is to estimate the sum in this formula. We integrate by parts in each term:

$$
\begin{aligned}
S_n(r) = &- \left. \frac{t^\rho - a_1 \left(\frac{\zeta_n}{r}\right)^{\rho} \ln \frac{e}{a_1} \left(\frac{tr}{\zeta_n}\right)^{\rho}}{(1+t)} \right|_{t_n^{(1)}/r}^{\zeta_n/r} + \; \rho \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \frac{t^\rho - a_1 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt \\
&- \left. \frac{t^\rho - a_2 \left(\frac{\zeta_n}{r}\right)^{\rho} \ln \frac{e}{a_2} \left(\frac{tr}{\zeta_n}\right)^{\rho}}{(1+t)} \right|_{\zeta_n/r}^{t_n^{(2)}/r} + \; \rho \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho - a_2 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt \\
&+ \rho \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \frac{t^\rho - a_1 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt + \rho \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho - a_2 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt.
\end{aligned}
$$

While calculating off-integral terms, we have used that a_1, a_2 are the roots of equation (13). Hence,

$$
S_n(r) = \rho \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \frac{t^\rho - a_1 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt \; + \; \rho \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho - a_2 \left(\frac{\zeta_n}{r}\right)^{\rho}}{t(t+1)}\, dt. \tag{51}
$$

We fix $j \in \mathbb{N}$ and estimate $S_n(r)$ for $r \in [\zeta_j, \zeta_{j+1}]$. Neglecting negative terms, we obtain two estimates

$$
S_n(r) \; \leqslant \; \rho \int\limits_{t_n^{(1)}/r}^{t_n^{(2)}/r} \frac{t^\rho\, dt}{t(t+1)} \; \leqslant \; \rho \int\limits_{t_n^{(1)}/r}^{t_n^{(2)}/r} t^{\rho-1}\, dt = \left(\frac{t_n^{(2)}}{r}\right)^{\rho} - \left(\frac{t_n^{(1)}}{r}\right)^{\rho} \; \leqslant \; \left(\frac{\zeta_n}{\zeta_j}\right)^{\rho} (a_2 - a_1), \tag{52}
$$

$$
\begin{aligned}
S_n(r) \leqslant & \rho \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho\, dt}{t(t+1)} \leqslant \rho \int\limits_{t_n^{(1)}/r}^{t_n^{(2)}/r} t^{\rho-2}\, dt \ = \frac{\rho}{\rho-1}\left(\left(\frac{t_n^{(2)}}{r}\right)^{\rho-1} - \left(\frac{t_n^{(1)}}{r}\right)^{\rho-1}\right) \\
\leqslant & \frac{\rho}{1-\rho}\left(\frac{\zeta_{j+1}}{\zeta_n}\right)^{1-\rho}(a_1^{1-1/\rho} - a_2^{1-1/\rho}).
\end{aligned}
\tag{53}
$$

It follows from condition (39) that

$$
\frac{\zeta_{j+1}}{\zeta_j} = \left(\frac{m_{j+1}}{m_j}\right)^{1/\rho} \to \infty, \quad j \to \infty. \tag{54}
$$

It implies easily that the identities

$$
\lim_{j\to\infty} \frac{\sum\limits_{n=1}^{j-1} \zeta_n^\rho}{\zeta_j^\rho} = 0, \qquad \lim_{j\to\infty} \frac{\sum\limits_{n=j+2}^{\infty} \zeta_n^{\rho-1}}{\zeta_{j+1}^{\rho-1}} = 0
$$

hold true (see [16, Sect. 3]). Now by means (52) and (53) we obtain that the relations

$$
\sum_{n=1}^{j-1} S_n(r) \leqslant (a_2 - a_1) \sum_{n=1}^{j-1} \left(\frac{\zeta_n}{\zeta_j}\right)^\rho = (a_2 - a_1)\, \frac{\sum\limits_{n=1}^{j-1} \zeta_n^\rho}{\zeta_j^\rho} \longrightarrow 0, \quad j \to \infty, \tag{55}
$$

$$
\begin{aligned}
\sum_{n=j+2}^{\infty} S_n(r) \leqslant & \ \frac{\rho}{1-\rho}\,(a_1^{1-1/\rho} - a_2^{1-1/\rho}) \sum_{n=j+2}^{\infty} \left(\frac{\zeta_{j+1}}{\zeta_n}\right)^{1-\rho} \\
\leqslant & \ \frac{\rho}{1-\rho}\,(a_1^{1-1/\rho} - a_2^{1-1/\rho})\, \frac{\sum\limits_{n=j+2}^{\infty} \zeta_n^{\rho-1}}{\zeta_{j+1}^{\rho-1}} \longrightarrow 0, \quad j \to \infty.
\end{aligned}
\tag{56}
$$

hold true uniformly in $r \in [\zeta_j, \zeta_{j+1}]$.

Thus, we have established that for sufficiently large j, the main terms in the sum

$$
S(r) = \sum_{n=1}^{\infty} S_n(r) = \sum_{n=1}^{j-1} S_n(r) + \sum_{n=j+2}^{\infty} S_n(r) + [S_j(r) + S_{j+1}(r)]
$$

are those with indices j and $j+1$. More precisely, the identity

$$
S(r) = o(1) + [S_{j+1}(r) + S_{j+2}(r)], \qquad j \to \infty, \tag{57}
$$

holds true uniformly in $r \in [\zeta_j,\, \zeta_{j+1}]$. Let us transform the sum in the square brackets. In order to do it we make the change of variables $t = 1/\tau$ in formula (51) expressing $S_n(r)$:

$$
\begin{aligned}
S_n(r) = & \rho \int\limits_{t_n^{(1)}/r}^{\zeta_n/r} \frac{t^\rho - a_1\left(\frac{\zeta_n}{r}\right)^\rho}{t(t+1)}\, dt \ + \ \rho \int\limits_{\zeta_n/r}^{t_n^{(2)}/r} \frac{t^\rho - a_2\left(\frac{\zeta_n}{r}\right)^\rho}{t(t+1)}\, dt \\
= & \rho \int\limits_{r/\zeta_n}^{a_1^{-1/\rho} r/\zeta_n} \frac{\tau^{-\rho} - a_1\left(\frac{r}{\zeta_n}\right)^{-\rho}}{\tau+1}\, d\tau \ + \ \rho \int\limits_{a_2^{-1/\rho} r/\zeta_n}^{r/\zeta_n} \frac{\tau^{-\rho} - a_2\left(\frac{r}{\zeta_n}\right)^{-\rho}}{\tau+1}\, d\tau.
\end{aligned}
$$

We write the obtained identity as

$$
S_n(r) = \rho\, \Phi\left(\frac{r}{\zeta_n}\right), \tag{58}
$$

where the function

$$\Phi(b) = \int\limits_b^{ba_1^{-1/\rho}} \frac{\tau^{-\rho} - a_1 b^{-\rho}}{\tau+1}\, d\tau + \int\limits_{ba_2^{-1/\rho}}^{b} \frac{\tau^{-\rho} - a_2 b^{-\rho}}{\tau+1}\, d\tau, \qquad b>0,$$

is the same as in formula (26). We represent it as

$$\Phi(b) = \Phi_1(b) - \Phi_2(b), \quad \text{where} \quad \Phi_k(b) = \int\limits_b^{ba_k^{-1/\rho}} \frac{\tau^{-\rho} - a_k b^{-\rho}}{\tau+1}\, d\tau, \qquad k=1,\, 2.$$

We shall need some properties of function $\Phi(b)$. Let us show first that

$$\Phi(0+) = \Phi(+\infty) = 0. \tag{59}$$

It is sufficient to check that each function $\Phi_k(b)$, $k = 1, 2$, satisfies these conditions. We integrate by parts to obtain

$$\Phi_k(b) = (a_k - 1)\frac{\ln(b+1)}{b^\rho} + \rho \int\limits_b^{ba_k^{-1/\rho}} \frac{\ln(\tau+1)}{\tau^{\rho+1}}\, d\tau, \quad k=1,\, 2.$$

Let us make sure that here each term satisfies (59). Indeed, the quantity $\dfrac{\ln(b+1)}{b^\rho}$ tends to zero both as $b \to 0+$ and as $b \to +\infty$. Then

$$\int\limits_b^{ba_k^{-1/\rho}} \frac{\ln(\tau+1)}{\tau^{\rho+1}}\, d\tau \sim \int\limits_b^{ba_k^{-1/\rho}} \tau^{-\rho}\, d\tau = O\left(b^{1-\rho}\right) \longrightarrow 0, \quad b \to 0+,$$

$$\int\limits_b^{ba_k^{-1/\rho}} \frac{\ln(\tau+1)}{\tau^{\rho+1}}\, d\tau \sim \int\limits_b^{ba_k^{-1/\rho}} \frac{\ln \tau}{\tau^{\rho+1}}\, d\tau \leqslant \frac{1}{2b^\rho}\left(\ln^2 ba_k^{-1/\rho} - \ln^2 b\right) \to 0, \quad b \to +\infty.$$

Thus, (59) is satisfied. Since $\Phi(b)$ can have positive values for some $b > 0$ (we shall check this fact in order not to interrupt the proof here), we can state that a continuous on $\mathbb{R}_+$ function $\Phi(b)$ attains its maximum at some point b_0, i.e.,

$$\max_{b>0} \Phi(b) = \Phi(b_0) > 0. \tag{60}$$

By (57), (58) we obtain the uniform in $r \in [\zeta_j, \zeta_{j+1}]$ relation

$$S(r) = S_j(r) + S_{j+1}(r) + o(1) = \rho\left[\Phi\left(\frac{r}{\zeta_j}\right) + \Phi\left(\frac{r}{\zeta_{j+1}}\right)\right] + o(1),\ j \to \infty. \tag{61}$$

Let us estimate $S(r)$ from above. Suppose first that $r \in [\zeta_j,\ \sqrt{\zeta_j\zeta_{j+1}}]$. Then by (60) we have $\Phi\left(\frac{r}{\zeta_j}\right) \leqslant \Phi(b_0)$, $j \in \mathbb{N}$. Since

$$0 < \frac{\zeta_j}{\zeta_{j+1}} \leqslant \frac{r}{\zeta_{j+1}} \leqslant \sqrt{\frac{\zeta_j}{\zeta_{j+1}}} \longrightarrow 0, \qquad j \to \infty,$$

thanks to (59) we find that

$$\max_{\zeta_j \leqslant r \leqslant \sqrt{\zeta_j\zeta_{j+1}}} \Phi\left(\frac{r}{\zeta_{j+1}}\right) \longrightarrow 0, \qquad j \to \infty.$$

Suppose now that $r\in[\sqrt{\zeta_j\zeta_{j+1}},\ \zeta_{j+1}]$. Then $\Phi\left(\frac{r}{\zeta_{j+1}}\right)\leqslant\Phi(b_0)$. Since

$$\frac{r}{\zeta_j}\geqslant\sqrt{\frac{\zeta_{j+1}}{\zeta_j}}\longrightarrow\infty,\qquad j\to\infty,$$

again by (59) we have

$$\max_{\sqrt{\zeta_j\zeta_{j+1}}\leqslant r\leqslant\zeta_{j+1}}\Phi\left(\frac{r}{\zeta_j}\right)\longrightarrow 0,\qquad j\to\infty.$$

In both cases we obtain

$$S(r)\leqslant\rho\,\Phi(b_0)+o(1),\qquad r\to+\infty. \tag{62}$$

Moreover, if $b_0\geqslant 1$, then letting $r=p_j=b_0\,\zeta_j$ in (61), by conditions (59) we obtain that

$$S(p_j)=\rho\left[\Phi\left(b_0\right)+\Phi\left(\frac{b_0\zeta_j}{\zeta_{j+1}}\right)\right]+o(1)=\rho\,\Phi\left(b_0\right)+o(1),\qquad j\to\infty.$$

If $b_0\leqslant 1$, then letting $r=p_j=b_0\zeta_{j+1}$ in (61), again by (59) we obtain

$$S(p_j)=\rho\left[\Phi\left(\frac{b_0\zeta_{j+1}}{\zeta_j}\right)+\Phi\left(b_0\right)\right]+o(1)=\rho\,\Phi\left(b_0\right)+o(1),\qquad j\to\infty.$$

In both case we have

$$S(p_j)=\rho\,\Phi\left(b_0\right)+o(1),\qquad j\to\infty. \tag{63}$$

It follows from (62), (63) that

$$\varlimsup_{r\to+\infty}S(r)=\rho\,\Phi(b_0).$$

In its turn, by (50) it implies

$$\varliminf_{r\to+\infty}\sigma(r)=B\frac{\pi\rho}{\sin\pi\rho}-\varlimsup_{r\to+\infty}S(r)=B\rho\left(\frac{\pi}{\sin\pi\rho}-\Phi(b_0)\right).$$

Summarizing, we conclude that in accordance with (48), (50), (57), (62), (63), function $\widetilde{f}(z)$ with constructed zero set $\widetilde{\Lambda}$ satisfy the identities

$$\underline{\sigma}(\widetilde{f})=\varliminf_{r\to+\infty}\sigma(r)=B\rho\left(\frac{\pi}{\sin\pi\rho}-\Phi(b_0)\right).$$

All the cases have been considered. Function $\widetilde{f}(z)$ is extremal since it provides the identity in (26). To complete the proof let us show that the second form of function $\Phi(b)$ is obtained from the first one by integration by parts:

$$\Phi(b)=\int\limits_{ba_2^{-1/\rho}}^{b}\frac{\tau^{-\rho}-a_2b^{-\rho}}{\tau+1}\,d\tau\;+\int\limits_{b}^{ba_1^{-1/\rho}}\frac{\tau^{-\rho}-a_1b^{-\rho}}{\tau+1}\,d\tau$$

$$=\ln(1+\tau)\left(\tau^{-\rho}-a_2b^{-\rho}\right)\Big|_{ba_2^{-1/\rho}}^{b}+\rho\int\limits_{ba_2^{-1/\rho}}^{b}\frac{\ln(1+\tau)}{\tau^{\rho+1}}\,d\tau$$

$$+\ln(1+\tau)\left(\tau^{-\rho}-a_1b^{-\rho}\right)\Big|_{b}^{ba_1^{-1/\rho}}+\rho\int\limits_{b}^{ba_1^{-1/\rho}}\frac{\ln(1+\tau)}{\tau^{\rho+1}}\,d\tau$$

$$=\rho \int_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} \frac{\ln(1+\tau)}{\tau^{\rho+1}}\, d\tau \, + \ln(1+b)(1-a_2)b^{-\rho} + \ln(1+b)(a_1-1)b^{-\rho}$$

$$=\rho \int_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} \frac{\ln(1+\tau)}{\tau^{\rho+1}}\, d\tau \, - \ln(1+b)(a_2-a_1)b^{-\rho} = \rho \int_{ba_2^{-\frac{1}{\rho}}}^{ba_1^{-\frac{1}{\rho}}} \frac{\ln(\tau+1)-\ln(b+1)}{\tau^{\rho+1}}\, d\tau.$$

The proof is complete.

4. Two-sided estimate for extremal quantity $\underline{S}^*(\alpha^*,\, \beta^*;\, \rho)$

Let us prove first relation (60) announced in the proof of Theorem 3 in the case $\alpha^* < \beta^*$. In order to do it, we estimate function $\Phi(b)$ for sufficiently large independent variable taking into consideration that the roots of equation (13) are related by strict inequalities $a_1 < 1 < a_2$. We write $\Phi(b)$ as

$$\Phi(b) = \int_{ba_2^{-1/\rho}}^{b} \frac{\tau^{-\rho}-a_2 b^{-\rho}}{\tau+1}\, d\tau \, + \int_{b}^{ba_1^{-1/\rho}} \frac{\tau^{-\rho}-a_1 b^{-\rho}}{\tau+1}\, d\tau$$

$$= \int_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\, d\tau \, - \, a_2 b^{-\rho} \ln \frac{b+1}{1+ba_2^{-1/\rho}} \, - \, a_1 b^{-\rho} \ln \frac{1+ba_1^{-1/\rho}}{1+b}.$$

We consider the first term:

$$\int_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\, d\tau > \int_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} \tau^{-\rho-1}\left(1-\frac{1}{\tau}\right)\, d\tau = \left.\frac{\tau^{-\rho}}{-\rho}\right|_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}} + \left.\frac{\tau^{-\rho-1}}{\rho+1}\right|_{ba_2^{-1/\rho}}^{ba_1^{-1/\rho}}$$

$$=\frac{b^{-\rho}}{\rho}\left(a_2-a_1\right) - \frac{b^{-\rho-1}}{\rho+1}\left(a_2^{1+1/\rho} - a_1^{1+1/\rho}\right).$$

Applying inequalities $x - \frac{x^2}{2} < \ln(1+x) < x$, we estimate remaining terms

$$\begin{aligned} a_2 b^{-\rho} \ln \frac{1+b}{1+ba_2^{-1/\rho}} &= a_2 b^{-\rho} \ln \frac{b\left(1+1/b\right)}{ba_2^{-1/\rho}\left(1+1/ba_2^{-1/\rho}\right)} \\ &< a_2 b^{-\rho}\left[\ln a_2^{1/\rho} + \left(\frac{1}{b} - \frac{a_2^{1/\rho}}{b} + \frac{1}{2}\frac{a_2^{2/\rho}}{b^2}\right)\right] \\ &= \frac{a_2 b^{-\rho}}{\rho}\ln a_2 + a_2 b^{-\rho-1}\left[\left(1-a_2^{1/\rho}\right) + \frac{a_2^{2/\rho}}{2b}\right] \end{aligned}$$

or, similarly,

$$a_1 b^{-\rho} \ln \frac{1+ba_1^{-1/\rho}}{1+b} < -\frac{a_1 b^{-\rho}}{\rho}\ln a_1 + a_1 b^{-\rho-1}\left(a_1^{1/\rho} - 1 + \frac{1}{2b}\right).$$

Gathering the obtained estimates, we can write

$$\Phi(b) > \frac{b^{-\rho}}{\rho}\left(a_2 - a_1 - a_2\ln a_2 + a_1 \ln a_1\right)$$

$$+\, b^{-\rho-1}\left[-\frac{\left(a_2^{1+1/\rho}-a_1^{1+1/\rho}\right)}{\rho+1}-\left(a_2-a_2^{1+1/\rho}+\frac{a_2^{1+2/\rho}}{2b}\right)-\left(a_1^{1+1/\rho}-a_1+\frac{a_1}{2b}\right)\right].$$

The first term disappears since the expression in the brackets is equal to

$$a_2\ln\frac{e}{a_2}-a_1\ln\frac{e}{a_1}=\frac{\alpha^*}{\beta^*}-\frac{\alpha^*}{\beta^*}=0$$

because a_1 and a_2 are roots of equation (13). Thus, we have

$$\Phi(b)>b^{-\rho-1}\left[\left(a_2^{1+1/\rho}-a_1^{1+1/\rho}\right)\frac{\rho}{\rho+1}-(a_2-a_1)-\frac{1}{2b}\left(a_1+a_2^{1+2/\rho}\right)\right],$$

or, denoting $1+1/\rho=\nu$ ($\nu>2$),

$$\Phi(b)>\frac{b^{-\rho-1}}{\nu}\left[a_2^\nu-a_1^\nu-\nu(a_2-a_1)-O\left(\frac{1}{b}\right)\right],\quad b\to+\infty.$$

Let us prove that $\psi(\nu)=a_2^\nu-a_1^\nu-\nu(a_2-a_1)>0$ for each $\nu\geqslant 1$. We have

$$\psi'(\nu)=a_2^\nu\ln a_2-a_1^\nu\ln a_1-(a_2-a_1),\qquad \psi''(\nu)=a_2^\nu\ln^2a_2-a_1^\nu\ln^2a_1.$$

Function $\psi''(\nu)$ is increasing since the first term in its definition increases w.r.t. ν, while the second term decreases. Then for $\nu\geqslant 1$ we have

$$\begin{aligned}\psi''(\nu)\geqslant\psi''(1)&=a_2\ln^2a_2-a_1\ln^2a_1=a_2(\ln^2a_2-1)+a_2-a_1(\ln^2a_1-1)-a_1\\&=(a_2\ln a_2-a_2)(\ln a_2+1)-(a_1\ln a_1-a_1)(\ln a_1+1)+a_2-a_1=-\frac{\alpha^*}{\beta^*}\ln\frac{a_2}{a_1}+a_2-a_1.\end{aligned}$$

Here we again employed the fact that a_1 and a_2 are roots of equation (13). We employ a parametric representation of the roots of this equation found in work [31]:

$$a_1=e\,s^{\frac{s}{1-s}},\quad a_2=s\,a_1=s^{\frac{1}{1-s}},\qquad s>1.$$

We obtain

$$\psi''(1)=a_2-a_1-a_1\,\ln\frac{e}{a_1}\ln\frac{a_2}{a_1}=\frac{a_1}{s-1}\left[(s-1)^2-s\ln^2s\right]>0.$$

The positivity of the expression in square brackets is implied by its monotonic increasing that can be checked by usual methods of analysis. The proven positivity of $\psi''(\nu)$ on $[1,+\infty)$ implies the increasing of $\psi'(\nu)$ on this segment. Therefore,

$$\psi'(\nu)>\psi'(1)=a_2\ln a_2-a_1\ln a_1-(a_2-a_1)=0,\qquad \nu>1.$$

In its turn, it yields the increasing of function $\psi(\nu)$ that for $\nu>1$ implies the inequality

$$\psi(\nu)>\psi(1)=a_2-a_1-a_2+a_1=0.$$

Hence, relation (60) is proven.

We proceed to studying the quantity $\underline{S}^*(\alpha^*,\beta^*;\rho)$ and we obtain first simple estimates for the integral involved in the definition of function $\Phi(b)$. Since on the integration interval we have $\tau\geqslant b$, the inequality $-b^{-\rho}\leqslant-\tau^{-\rho}$ holds true and estimating $\Phi(b)$, we can write

$$\Phi(b)\leqslant\Phi_1(b)=\int\limits_b^{ba_1^{-1/\rho}}\frac{\tau^{-\rho}-a_1b^{-\rho}}{\tau+1}\,d\tau\leqslant\int\limits_b^{ba_1^{-1/\rho}}\frac{\tau^{-\rho}(1-a_1)}{\tau+1}\,d\tau.$$

Thus, the estimate

$$\Phi(b)\leqslant(1-a_1)\int\limits_b^{ba_1^{-1/\rho}}\frac{\tau^{-\rho}}{\tau+1}\,d\tau \tag{64}$$

is true. Enlarging the integration interval leads us to the inequality

$$\Phi(b) \leqslant (1-a_1) \int\limits_0^{+\infty} \frac{\tau^{-\rho}}{\tau+1}\, d\tau = (1-a_1)\, \frac{\pi}{\sin \pi\,\rho}, \qquad b>0.$$

It follows that

$$\underline{S}^*(\alpha^*,\, \beta^*;\, \rho) = \beta^*\,\rho \left(\frac{\pi}{\sin \pi\,\rho} - \sup_{b>0} \Phi(b) \right) \geqslant \beta^*\,\rho \left(\frac{\pi}{\sin \pi\,\rho} - (1-a_1)\, \frac{\pi}{\sin \pi\,\rho} \right) = a_1\, \frac{\pi\,\rho\,\beta^*}{\sin \pi\,\rho}.$$

Employing also (60), we finally arrive at the two-sided estimate

$$a_1\, \frac{\pi\,\rho\,\beta^*}{\sin \pi\,\rho} \leqslant \underline{S}^*(\alpha^*,\, \beta^*;\, \rho) \leqslant \frac{\pi\,\rho\,\beta^*}{\sin \pi\,\rho}. \tag{65}$$

This simple estimate characterizes $\underline{S}^*(\alpha^*,\, \beta^*;\, \rho)$, when the smaller root a_1 of equation (13) is equal or close to one, i.e., when β^* coincides or differs just a little from α^*. But this estimate is completely non-informative for small values of root a_1, when β^* exceeds essentially α^*. The next result covers this problem.

Theorem 4. *For each $\rho \in (0,1)$, $\alpha^*>0$, $\beta^* \geqslant \alpha^*$ the inequalities*

$$\underline{S}^*(\alpha^*,\, \beta^*;\, \rho) \;\geqslant\; \beta^*\rho \left[\frac{\pi a_1}{\sin \pi\rho} \;+\; a_1^{1-\rho}(1-a_1)A_\rho \right], \tag{66}$$

$$\underline{S}^*(\alpha^*,\, \beta^*;\, \rho) \leqslant \beta^*\rho \left[\frac{\pi a_1}{\sin \pi\rho} + a_1^{1-\rho}(1-a_1) \left(B_\rho \ln \frac{e}{a_1} + e \right) \right] \tag{67}$$

hold true, where $A_\rho = \min\{1/2;\, \rho\}$, and $B_\rho = (\rho(1-\rho))^{-1}$.

Proof. We specify (65) by estimating the quantity

$$S := \frac{\underline{S}^*(\alpha^*,\, \beta^*;\, \rho)}{\beta^*\,\rho} \;-\; \frac{\pi\, a_1}{\sin \pi\,\rho}.$$

By means of inequality (64) we obtain

$$\begin{aligned} S =&(1-a_1)\, \frac{\pi}{\sin \pi\rho} \;-\; \max_{b>0}\; \Phi(b) \\ \geqslant&(1-a_1)\, \frac{\pi}{\sin \pi\rho} \;-\; (1-a_1) \max_{b>0} \int\limits_b^{b a_1^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\, d\tau =: (1-a_1) \min_{b>0}\; \eta(b). \end{aligned}$$

As it was shown in Theorem 4 in work [26], the function $\eta(b) = \dfrac{\pi}{\sin \pi\rho} - \displaystyle\int\limits_b^{b a_1^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\, d\tau$ satisfies the lower estimate

$$\eta(b) \;\geqslant\; A_\rho\, a_1^{1-\rho}, \qquad A_\rho = \min\{1/2; \rho\}.$$

Applying of this estimate leads us to the inequality

$$S \;\geqslant\; A_\rho\, a_1^{1-\rho}(1-a_1)$$

for each $\rho \in (0,1)$ that implies immediately (66).

More efforts are needed for an upper estimate of S. To obtain such estimate, as in work [26], we replace the maximal value of function $\Phi(b)$ by its value at point $b=a_1$:

$$\max_{b>0}\; \Phi(b) \;\geqslant\; \Phi(a_1) = \Phi_1(a_1) \;-\; \Phi_2(a_1).$$

Let us estimate each term:

$$\Phi_1(a_1) = \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho} - a_1^{1-\rho}}{\tau+1} d\tau = (1-a_1) \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau + a_1 \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau - a_1^{1-\rho} \ln \frac{1+a_1^{1-1/\rho}}{1+a_1}$$

$$\geqslant (1-a_1) \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, - \, a_1^{1-\rho}(1-a_1) \ln \frac{1+a_1^{1-1/\rho}}{1+a_1}$$

$$\geqslant (1-a_1) \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, + \, a_1^{1-\rho}(1-a_1) \ln a_1^{1/\rho}.$$

At the last step we have employed that $\dfrac{1+a_1^{1-1/\rho}}{1+a_1} \leqslant a_1^{-1/\rho}$. Thus, the inequality

$$\Phi_1(a_1) \, \geqslant (1-a_1) \int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, + \, a_1^{1-\rho}(1-a_1) \ln a_1^{1/\rho} \tag{68}$$

holds true. Then

$$\Phi_2(a_1) = \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho} - a_2\, a_1^{-\rho}}{\tau+1} d\tau$$

$$= (1-a_1) \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, + \, a_1 \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, - \, a_2 a_1^{-\rho} \ln \frac{1+a_1 a_2^{-1/\rho}}{1+a_1}$$

$$\leqslant (1-a_1) \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, + \, a_1^{-\rho}(a_1 - a_2) \ln \frac{1+a_1 a_2^{-1/\rho}}{1+a_1}$$

$$\leqslant (1-a_1) \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} d\tau \, + \, a_1^{1-\rho}(a_2 - a_1).$$

Here we have used the inequality

$$\ln \frac{1+a_1}{1+a_1 a_2^{-1/\rho}} \leqslant \ln(1+a_1) \leqslant a_1.$$

We finally have the estimate

$$\Phi_2(a_1) \leqslant (1-a_1) \int\limits_{a_1}^{a_1 a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1} \, d\tau \, + \, a_1^{1-\rho}(a_2 - a_1). \tag{69}$$

Taking into consideration both estimate (68) and (69), we obtain

$$\Phi(a_1) = \Phi_1(a_1) \, - \, \Phi_2(a_1)$$

$$\geqslant (1-a_1)\int\limits_{a_1}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\,d\tau \;+\; a_1^{1-\rho}(1-a_1)\ln a_1^{1/\rho}$$
$$-\,(1-a_1)\int\limits_{a_1}^{a_1a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\,d\tau \;-\; a_1^{1-\rho}(a_2-a_1)$$
$$=(1-a_1)\int\limits_{a_1a_2^{-1/\rho}}^{a_1^{1-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}\,d\tau \;+\; a_1^{1-\rho}(1-a_1)\ln a_1^{1/\rho} \;-\; a_1^{1-\rho}(a_2-a_1).$$

We return back to the estimate for S:

$$S=(1-a_1)\frac{\pi}{\sin\pi\rho}-\max_{b>0}\,\Phi(b)\leqslant (1-a_1)\frac{\pi}{\sin\pi\rho}\;-\;\Phi(a_1)$$
$$=(1-a_1)\left[\int\limits_0^{a_1a_2^{-1/\rho}} \frac{\tau^{-\rho}}{\tau+1}d\tau \;+\int\limits_{a_1^{1-1/\rho}}^{+\infty} \frac{\tau^{-\rho}}{\tau+1}d\tau\right] \;-\; a_1^{1-\rho}(1-a_1)\ln a_1^{1/\rho} \;+\; a_1^{1-\rho}(a_2-a_1).$$

The expression in the square brackets does not exceed

$$\int\limits_0^{a_1a_2^{-1/\rho}} \tau^{-\rho}d\tau \;+\int\limits_{a_1^{1-1/\rho}}^{+\infty} \tau^{-\rho-1}d\tau =\frac{\left(a_1a_2^{-1/\rho}\right)^{1-\rho}}{1-\rho}\;+\;\frac{\left(a_1^{1-1/\rho}\right)^{-\rho}}{\rho}$$
$$=a_1^{1-\rho}\left(\frac{a_2^{1-1/\rho}}{1-\rho}\;+\;\frac{1}{\rho}\right)\leqslant \frac{a_1^{1-\rho}}{\rho(1-\rho)}.$$

It is easy to show (geometrically it is almost evident) that roots a_1, a_2 of equation (13) ($0\leqslant a_1\leqslant 1\leqslant a_2\leqslant e$) satisfy the condition

$$a_2-a_1\leqslant e\,(1-a_1).$$

Finally we can write

$$S\leqslant a_1^{1-\rho}(1-a_1)\left(\frac{1}{\rho(1-\rho)}-\frac{1}{\rho}\ln a_1\;+\;e\right)$$
$$=a_1^{1-\rho}(1-a_1)\left(\frac{1-(1-\rho)\ln a_1}{\rho(1-\rho)}\;+\;e\right)\leqslant a_1^{1-\rho}(1-a_1)\left(\frac{\ln\frac{e}{a_1}}{\rho(1-\rho)}\;+\;e\right)$$

that is equivalent to (67). The proof is complete. □

If lower averaged ρ-density of roots α^* of an entire function is zero, as we have mentioned in the proof of Theorem 1, its lower ρ-type is independent of the upper averaged ρ-density β^* of the roots and is equal to zero. Therefore, $\underline{S}^*(0,\,\beta^*;\,\rho)=0$ for each finite β^*. The situation changes essentially if $\alpha^*>0$. In this case it is impossible to estimate from above the lower ρ-type of an entire function in terms of lower averaged ρ-density of its zeroes. It is stated by the following result being the corollary of Theorem 4.

Theorem 5. *Let $\rho\in(0,1)$ and $\alpha^*>0$. The identity*

$$\sup_{\beta^*\geqslant\alpha^*}\;\underline{S}^*(\alpha^*,\,\beta^*;\,\rho)=+\infty$$

holds true.

Proof. We fix $\alpha^* > 0$. If $\beta^* \to +\infty$, then lower root a_1 of equation

$$a \ln \frac{e}{a} = \frac{\alpha^*}{\beta^*}$$

tends to zero, while greater root a_2 tends to e. At that, in accordance with (66), we have

$$\begin{aligned} \underline{S}^*(\alpha^*, \beta^*; \rho) \geqslant \beta^* \rho\, a_1^{1-\rho}(1-a_1)\, A_\rho = \frac{\alpha^* \rho\, a_1^{1-\rho}(1-a_1) A_\rho}{\alpha^*/\beta^*} \\ = \alpha^* \rho\, A_\rho\, (1-a_1) \frac{a_1^{1-\rho}}{a_1 \ln \frac{e}{a_1}} = \alpha^* \rho\, A_\rho\, (1-a_1) \frac{1}{a_1^{\rho} \ln \frac{e}{a_1}} \to +\infty. \end{aligned}$$

□

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Georgii Genrikhovich Braichev,
Moscow State Pedagogical University,
M. Pirogovskaya str. 1,
199296, Moscow, Russia
E-mail: braichev@mail.ru