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EXACTNESS OF ESTIMATES FOR k-th ORDER OF DIRICHLET SERIES IN A SEMI-STRIP

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Dedicated to the memory of professor
Igor' Fedorovich Krasichkov-Ternovskii

Abstract. We study Dirichlet series converging only in a half-plane such that their sequence of exponents admits an extension to a "regular" sequence. We prove the exactness of two-sided estimates for k-order of the sum of the Dirichlet series in a semi-strip whose width depends on the special distribution density of the exponents.

Keywords: k-order of the Dirichlet series in a semi-strip, entire functions with a prescribed asymptotics on the positive axis.

Mathematics Subject Classification: 3010

Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\lambda_n} = H < \infty. \tag{1}$$

In studying entire functions

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad (s = \sigma + it) \tag{2}$$

defined by everywhere convergent Dirichlet series, Ritt introduced the notion of R-order [1]:

$$\rho_R = \overline{\lim_{\sigma\to+\infty}} \frac{\ln\ln M(\sigma)}{\sigma},$$

where $M(\sigma) = \sup_{|t|<\infty} |F(\sigma+it)|$. We note that by condition (1) series (2) converges absolutely in the whole plane. It is known that $\ln M(\sigma)$ is an increasing convex function of σ, $\lim_{\sigma\to+\infty} \ln M(\sigma) = +\infty$. The quantity

$$\rho_s = \overline{\lim_{\sigma\to+\infty}} \frac{\ln^+ \ln M_s(\sigma)}{\sigma} \quad (a^+ = \max(a, 0))$$

is called R-order of function F in strip $S(a, t_0) = \{s = \sigma + it : |t - t_0| \leqslant a\}$. Here $M_s(\sigma) = \max_{|t-t_0|\leqslant a} |F(\sigma + it)|$.

In [2] sufficient conditions for Λ and a ensuring $\rho_R = \rho_S$ were obtained. The most general results on relation between ρ_R and ρ_s was established by A.F. Leontiev [3].

Similar issues in the case when $H = 0$ and the convergence domain of series (2) is the half-plane $\Pi_0 = \{s = \sigma + it : \sigma < 0\}$ were studied by A.M. Gaisin in [4].

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As $H = 0$, if series (2) converges in half-plane Π_0, it converges absolutely in Π_0. Then the sum of series F is analytic in this half-plane. The class of all unbounded analytic functions represented by Dirichlet series (2) converging just in half-plane Π_0 is denoted by $D_0(\Lambda)$.

Let $S(a, t_0) = \{s = \sigma + it : |t - t_0| \leqslant a, \sigma < 0\}$ be a semi-strip. The quantities

$$\rho_R = \overline{\lim_{\sigma\to 0-}} \frac{\ln^+ \ln M(\sigma)}{|\sigma|^{-1}}, \qquad \rho_s = \overline{\lim_{\sigma\to 0-}} \frac{\ln^+ \ln M_s(\sigma)}{|\sigma|^{-1}}$$

are called Ritt orders of function F in half-plane Π_0 and semi-strip $S(a, t_0)$ [4]. In what follows we call ρ_R and ρ_s orders in the half-plane and the semi-strip. If it is necessary, instead of ρ_R and ρ_s we shall write $\rho_R(F)$ and $\rho_s(F)$.

It was shown in [4] that condition

$$\lim_{n\to\infty} \frac{\ln \lambda_n}{\lambda_n} \ln n = 0$$

is sufficient for order ρ_R of each function $F \in D_0(\Lambda)$ to be

$$\rho_R = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \ln^+ |a_n|. \tag{3}$$

Let sequence Λ have a finite upper density D. Then

$$L(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) \quad (z = x + iy)$$

is an entire function of exponential type. If $h(\varphi)$ is the growth indicatri, and τ is a type of function L, then $\tau = h(\pm\frac{\pi}{2}) \leqslant \pi D^*$ (D^* is the averaged upper density of sequence Λ) [2]. Assume that

$$|L(x)| \leqslant e^{g(x)} \quad (x \geqslant 0), \qquad \lim_{x\to+\infty} \frac{g(x)\ln x}{x} = 0, \tag{4}$$

where g is a non-negative on $\mathbb{R}_+ = [0, \infty)$ function. In this case the adjoint diagram of function L is the segment $I = [-\tau i, \tau i]$, $h(\varphi) = \tau|\sin\varphi|$.

In [4] the following theorem was proved.

Theorem I. *Suppose that function L satisfies conditions (4) and has type τ $(0 \leqslant \tau < \infty)$. We let $q = q(L)$, where*

$$q(L) = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \ln \left|\frac{1}{L'(\lambda_n)}\right|. \tag{5}$$

Then order ρ_s in the half-plane $S(a, t_0)$ as $a > \tau$ and order ρ_R of each function $F \in D_0(\Lambda)$ in half-plane Π_0 satisfy the eistimates

$$\rho_s \leqslant \rho_R \leqslant \rho_s + q. \tag{6}$$

The left estimate in (6) is exact [4]. In the general situation the right estimate is not exact, moreover, the pair of conditions (4) can fail. But there can exist an entire function Q of exponential type with simple zeroes at the points of sequence Λ, for which conditions (4) hold and $q(Q) = q^*$, where

$$q^* = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \int_0^1 \frac{n(\lambda_n; t)}{t} dt,$$

$q(Q)$ is the quantity defined in the same way as $q(L)$ in (5), and $n(\lambda_n; t)$ is the amount of points $\lambda_k \neq \lambda_n$ in the segment $\{x : |x - \lambda_n| \leqslant t\}$. Paper [5] is devoted to constructing such entire functions Q with a prescribed subset of zeroes Λ and a prescribed asymptotics at the real axis. It turns out that in terms of a special distribution density $G(R)$ of sequence Λ one can provide conditions under which the estimates

$$\rho_s \leqslant \rho_R \leqslant \rho_s + q^*$$

hold true, where ρ_s is the order in a semi-strip $S(a,t_0)$ of the width greater than $2\pi G(R)$) and these estimates are the best possible in class $D_0(\Lambda)$ [6]. Similar estimates for k-orders were obtained in [7]. The aim of the paper is prove the exactness of these estimates.

1. Definitions and needed facts

Let $\Lambda = \{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ be a sequence having a finite upper density, L be the class of positive continuous and unboundedly increasing on $[0,\infty)$ functions. By K we denote a subclass of functions h in L such that $h(0) = 0$, $h(t) = o(t)$ as $t \to \infty$, $\frac{h(t)}{t} \downarrow$ as $t \uparrow$ ($\frac{h(t)}{t}$ decreases monotonically as $t > 0$). In particular, if $h \in K$, the $h(2t) \leqslant 2h(t)$ $(t > 0)$, $h(t) \leqslant h(1)t$ as $t \geqslant 1$.

K-density of sequence Λ is the quantity

$$G(K) = \inf_{h\in K} \overline{\lim_{t\to\infty}} \frac{\mu_\Lambda(\omega(t))}{h(t)}, \tag{7}$$

where $\omega(t) = [t, t + h(t))$ is a semi-interval, $\mu_\Lambda(\omega(t))$ is the amount of points of Λ lying in semi-interval $\omega(t)$.

Let $\Omega = \{\omega\}$ be a family of semi-intervals $\omega = [a,b)$. By $|\omega|$ we denote the length of ω. Each sequence $\Lambda = \{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ generates an integer-valued counting measure μ_Λ:

$$\mu_\Lambda(\omega) = \sum_{\lambda_n\in\omega} 1, \quad \omega \in \Omega.$$

Let μ_Γ be a counting measure generated by the sequence $\Gamma = \{\mu_n\}$, $(0 < \mu_n \uparrow \infty)$. Then the inclusion $\Lambda \subset \Gamma$ means that $\mu_\Lambda(\omega) \leqslant \mu_\Gamma(\omega)$ for each $\omega \in \Omega$. In this case we shall say that measure μ_Γ majorizes measure μ_Λ.

By $D(K)$ we denote the infimum of numbers b $(0 \leqslant b < \infty)$ for which there exists measure μ_Γ majorizing μ_Λ such that

$$|M(t) - bt| \leqslant h(t) \qquad (t \geqslant 0) \tag{8}$$

for some function $h \in K$. Here $\Lambda = \{\lambda_n\}$, $\Gamma = \{\mu_n\}$, $M(t) = \sum\limits_{\mu_n\leqslant t} 1$.

It was shown in [6] that $D(K) = G(K)$.

The quantity

$$\rho_k = \overline{\lim_{\sigma\to 0_-}} \frac{\ln_k M(\sigma)}{|\sigma|^{-1}} \quad (k \geqslant 2) \tag{9}$$

is called k-order of function $F \in D_0(\Lambda)$ in half-plane $\Pi_0 = \{s : \ \sigma = \operatorname{Re} s < 0\}$ [7]. Here $\ln_0 t = t$, $\ln_k t = \underbrace{\ln\ln\ldots\ln t}_{k}$ $(k \geqslant 1)$. In view of the definition of k-order (9) we see that $\rho_2 = \rho_R$, where ρ_R is the R-order in half-plane Π_0 [4].

The following theorem was proven in [7].

Theorem II. *The condition*

$$\lim_{n\to\infty} \frac{\ln n \ln_{k-1}\lambda_n}{\lambda_n} = 0 \quad (k \geqslant 2) \tag{10}$$

is necessary and sufficient for k-order ρ_k of each function $F \in D_0(\Lambda)$ to satisfy the formula

$$\rho_k = \overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{\lambda_n} \ln_{k-1}\lambda_n \quad (k \geqslant 2;\ 0 \leqslant \rho_R \leqslant \infty). \tag{11}$$

We observe that formula (3) is a particular case of identity (11).

In the same way one introduces the notion of k-order $\rho_s^{(k)}$ in semi-strip $S(a,t_0)$. For the sake of convenience, we shall still denote it by ρ_s.

We introduce the following classes of functions:

$$L_k = \{h \in L: \ h(x)\ln_{k-1} x = o(x), \quad x\to\infty\} \quad (k \geqslant 2),$$

$$S = \left\{h \in K: \ d(h) = \varlimsup_{x\to\infty} \frac{h(x)\ln h(x)}{x\ln\frac{x}{h(x)}} < \infty\right\},$$

$$R_k = \left\{h \in S: \ h(x)\ln\frac{x}{h(x)} = o\left(\frac{x}{\ln_{k-1} x}\right), \quad x\to\infty\right\} \quad (k \geqslant 2).$$

The following theorem was proved in paper [7].

Theorem III. *Let $\Lambda = \{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ be a sequence satisfying the conditions*

1) $\Lambda(x+\rho) - \Lambda(x) \leqslant c\rho + d + \dfrac{\varphi(x)}{\ln^+\rho + 1}$, $(\rho \geqslant 0)$, where $\Lambda(x) = \sum\limits_{\lambda_n\leqslant x} 1$, φ is a function in L_k $(k \geqslant 2)$;

2) $q_k^* = \varlimsup_{n\to\infty} \dfrac{\ln_{k-1}\lambda_n}{\lambda_n}\int\limits_0^1 \dfrac{n(\lambda_n;t)}{t}dt < \infty$ $(k \geqslant 2)$, where $n(\lambda_n;t)$ is the amount of points of $\lambda_k \neq \lambda_n$ lying in the segment $\{x: \ |x-\lambda_n| \leqslant t\}$.

If density R_k of sequence Λ is equal to $G(R)$, then k-order ρ_s of each function $F \in D_0(\Lambda)$ in semi-strip $S(a,t_0)$ as $a > \pi G(R_k)$ and order ρ_R of this function in half-plane Π_0 satisfies the estimates

$$\rho_s \leqslant \rho_k \leqslant \rho_s + q_k^* \quad (k \geqslant 2). \tag{12}$$

As it is known, estimate $\rho_s \leqslant \rho_k$ in (12) is exact. In what follows we discuss the exactness of inequality $\rho_k \leqslant \rho_s + q_k^*$ $(k \geqslant 2)$.

2. Main theorem on exact estimates for k-order

The main result of the paper is the following theorem.

Theorem 1. *Let Λ be a sequence satisfying the assumptions of Theorem III. Then there exists a function $F \in D_0(\Lambda)$ such that $\rho_k(F) = \rho_s(F) + q^*$, where $\rho_k(F)$ is the order in half-plane Π_0, and $\rho_s(F)$ is order in semi-strip $S(a,t_0)$, $(a > \pi G(R))$.*

Corollary. *Suppose that Λ satisfies the assumptions of Theorem 1. Order $\rho_k(F)$ of each function $F \in D_0(\Lambda)$ is equal to order $\rho_s(F)$ in each semi-strip $S(a,t_0)$ $(a > \pi G(R))$ if and only $q^* = 0$.*

In the proof of Theorem 1 we shall make use of

Theorem IV [6]. *Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence having a finite S-density $G(S)$. Then for each $b > G(S)$ there exists a sequence $\Gamma = \{\mu_n\}$ $(0 < \mu_n \uparrow \infty)$ containing Λ and having density b such that the entire function*

$$Q(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\mu_n^2}\right) \quad (z = x+iy)$$

of exponential type πb possesses the properties:

1) $Q(\lambda_n) = 0$, $Q'(\lambda_n) \neq 0$ for each $\lambda_n \in \Lambda$;

2) there exists $H \in S$ such that

$$\ln|Q(x)| \leqslant AH(x)\ln^+\frac{x}{H(x)} + B; \tag{13}$$

3) if $\Lambda(x) = \sum\limits_{\lambda_n\leqslant x} 1$, and

$$\Lambda(x+\rho) - \Lambda(x) \leqslant a\rho + b + \frac{\varphi(x)}{\ln^+\rho + 1} \quad (\rho \geqslant 0) \tag{14}$$

(φ is an arbitrary non-negative non-decreasing function defined on the ray $[0,\infty)$, $1 \leqslant \varphi(x) \leqslant \alpha x \ln^+ x + \beta$), then there exists a sequence $\{r_n\}$, $0 < r_n \uparrow \infty$, $r_{n+1} - r_n = O(H(r_n))$ as $n \to \infty$, such that for $x = r_n$ $(n \geqslant 1)$

$$\ln|Q(x)| \geqslant -CH(x)\ln^+ \frac{x}{H(x)} - 2\varphi(x) - D; \tag{15}$$

4) if

$$\Delta = \varlimsup_{n\to\infty} \frac{1}{\lambda_n} \int\limits_0^1 \frac{n(\lambda_n;t)}{t} dt < \infty,$$

then under condition (14)

$$\left| \ln\left|\frac{1}{Q'(\lambda_n)}\right| - \int\limits_0^1 \frac{n(\lambda_n;t)}{t} dt \right| \leqslant EH(\lambda_n)\ln^+ \frac{\lambda_n}{H(\lambda_n)} + 2\varphi(\lambda_n) + F\ln\lambda_n + L \quad (n \geqslant 1), \tag{16}$$

where $n(\lambda_n;t)$ is the amount of points $\lambda_k \neq \lambda_n$ in the segment $\{x : |x-\lambda_n| \leqslant t\}$.

Here all constants are finite and positive.

Let $\Lambda = \{\lambda_n\}$ be a sequence satisfying assumptions of Theorem III. Then in accordance with Theorem IV for each $b > G(R_k)$ ($G(R_k)$ is R_k-density of sequence Λ) there exists a sequence $\Gamma = \{\mu_n\}$ $(0 < \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n \to \infty)$ contatining Λ such that

$$|M(t) - bt| \leqslant H(t) \quad (t \geqslant 0), \quad H \in R, \tag{17}$$

and entire function

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\mu_n^2}\right) \quad (z = x+iy) \tag{18}$$

of exponential type πb possesses the properties

1^0. $Q(\lambda_n) = 0,\ Q'(\lambda_n) \neq 0\,(n \geqslant 1)$;

2^0. $\ln|Q(x)| \leqslant g(x)\,(x \geqslant 0), g \in L_k$;

3^0. The estimate

$$\ln|Q(x)| \geqslant -CH(x)\ln^+ \frac{x}{H(x)} - 2\varphi(x) - D, \quad H \in R_k$$

holds true as $x = r_n$ $(n \geqslant 1)$. Estimates 2^0, 3^0 in Theorem III follows from (13), (15). But since $H \in R_k$, $\varphi \in L_k$, there exists function $V \in L_k$ such that

$$\ln|Q(z)| \geqslant \ln|Q(r)| \geqslant -V(r) \tag{19}$$

as $r = r_n$ $(r = |z|)$ $(n \geqslant 1)$.

Let$\{r_n\}$ be the sequence from Theorem IV (estimates (19) hold true as $|z| = r_n\,(n \geqslant 1)$). Let $\Delta_n = (r_{p_n}, r_{p_n+1})$ $(n \geqslant 1)$ be all the intervals each of which contains at leas one point in Λ (some of intervals (r_n, r_{n+1}) can contain no points in Λ).

By Γ_{p_n} $(n \geqslant 1)$ we denote a closed contour formed by two arcs of the circles $K_{p_n} = \{\lambda : |\lambda| = r_{p_n}\}$ and $K_{p_n+1} = \{\lambda : |\lambda| = r_{p_n+1}\}$ in the angle $\{\lambda : |\arg\lambda| \leqslant \varphi_n < \frac{\pi}{4}\}$ and by the segments of the rays $\{\lambda : |\arg\lambda| = \varphi_n\}$.

In the proof of Theorem 1 we employ the functions

$$q_n(\lambda) = \prod_{\nu_k \in \Delta_n} \left(1 - \frac{\lambda}{\nu_k}\right),$$

where $\Delta_n = (r_{p_n}, r_{p_n+1})$, $\nu = \{\nu_k\} = \Gamma\backslash\Lambda$. Sequence ν is constructed in the proof of Theorem IV and possesses the properties [6]:

a) $\inf\limits_{i\neq j} |\nu_i - \nu_j| \geqslant \tau > 0$;

b) $\inf\limits_{m\geqslant 1} |\lambda_n - \nu_m| \geqslant \frac{\gamma}{\varphi(2\lambda_n)}$ $(\gamma > 0, n \geqslant 1)$, where φ is the functions from condition (14) of Theorem IV.

Let us establish estimates for $|q_n(\lambda)|$.

Lemma 1. *There exists function $u \in L_k$ such that*

$$\max_{\lambda_j \in \Delta_n} |\ln |q_n(\lambda_j)|| \leqslant u(r_{p_n}) \quad (n \geqslant 1). \tag{20}$$

Proof. Indeed, let $\lambda_j \in \Delta_n, \nu_j'$ and ν_j'' be the closest to λ_j points of sequence ν located to the left and to right of λ_j, respectively. We have

$$\left|\frac{\nu_j' - \lambda_j}{\nu_j'}\right| \left|\frac{\nu_j'' - \lambda_j}{\nu_j''}\right| \geqslant \left[\frac{\gamma}{\varphi(2\lambda_j)}\right]^2 r_{p_n+1}^{-2} \quad (\lambda_j \in \Delta_n).$$

Since $1 \leqslant \varphi(x) \leqslant \alpha x \ln^+ x + \beta, r_{p_n}/r_{p_n+1} \to 1$ as $n \to \infty$, it implies the estimate

$$\left|1 - \frac{\lambda_j}{\nu_j'}\right| \left|1 - \frac{\lambda_j}{\nu_j''}\right| \geqslant e^{-c_1 - c_2 \ln r_{p_n}} \quad (\lambda_j \in \Delta_n), \tag{21}$$

where $0 < c_i < \infty$ $(i = 1, 2)$.

Let $\Delta_n' = \Delta_n \backslash \{\nu_j', \nu_j''\}$. Then

$$\prod_{\substack{\nu_k \in \Delta_n' \\ \nu_k < \lambda_j}} \left|\frac{\nu_k - \lambda_j}{\nu_k}\right| \geqslant \left(\frac{\tau}{r_{p_n+1}}\right)^{s_n} s_n!, \tag{22}$$

where s_n is the amount of points $\nu_k < \lambda_j$, $\nu_k \in \Delta_n'$. In the same way,

$$\prod_{\substack{\nu_k \in \Delta_n' \\ \nu_k > \lambda_j}} \left|\frac{\nu_k - \lambda_j}{\nu_k}\right| \geqslant \left(\frac{\tau}{r_{p_n+1}}\right)^{l_n} l_n!, \tag{23}$$

where l_n is the amount of points $\nu_k > \lambda_j$, $\nu_k \in \Delta_n'$. It follows from (21)–(23) that as $\lambda_j \in \Delta_n$, $(n \geqslant 1)$

$$|q_n(\lambda_j)| \geqslant e^{-c_1 - c_2 \ln r_{p_n}} \left(\frac{\delta}{r_{p_n}}\right)^{s_n + l_n} s_n! l_n! \quad (0 < \delta \leqslant 1). \tag{24}$$

If $\sup\limits_{n \geqslant 1}(s_n + l_n) < \infty$, the required lower estimate for $|q_n(\lambda_j)|$ is obvious. Otherwise we first employ the known estimate

$$s_n! l_n! \geqslant \frac{(s_n + l_n)!}{2^{s_n + l_n}},$$

and then we use Stirling's asymptotic formula

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

as $n \to \infty$.

Then by (24) we obtain

$$|q_n(\lambda_j)| \geqslant \exp\left(-c_3 - c_2 \ln r_{p_n}\right) \left[\frac{\delta(s_n + l_n)}{2e r_{p_n}}\right]^{s_n + l_n} \quad (n \geqslant 1),$$

where $0 < c_i < \infty$ $(i = 2, 3)$. Letting $s_n + l_n = m_n$, for $\lambda_j \in \Delta_n$ we have

$$|q_n(\lambda_j)| \geqslant \exp\left(-c_3 - c_2 \ln r_{p_n} - m_n \ln \frac{2e r_{p_n}}{\delta m_n}\right), \tag{25}$$

where $n \geqslant 1$, m_n is a number not exceeding the amount of points ν_k in interval Δ_n. Since $0 < r_{p_n+1} - r_{p_n} \leqslant pH(p_n)$ $(0 < p < \infty)$, taking into consideration property a) of sequence ν, we have $m_n \leqslant c_4 H(r_{p_n}), 0 < c_4 < \infty$ $(n \geqslant 1)$. Then $\frac{H(x)}{x} \downarrow 0$ as $x \uparrow \infty$ and function $\psi(x) = x \ln \frac{\Delta}{x}$ (Δ is a positive constant) is increasing as $0 < x < \frac{\Delta}{e}$. Therefore, by (25) we get that

$$\ln |q_n(\lambda_j)| \geqslant -c_5 - c_2 \ln r_{p_n} - c_6 H(r_{p_n}) \ln \frac{r_{p_n}}{H(r_{p_n})}$$

for $\lambda_j \in \Delta_n$ $(n \geqslant n_0)$, where $0 < c_i < \infty$ $(i = 2, 5, 6)$. Since $H \in R_k$, there exists $u_1 \in L_k$ such that

$$\ln |q_n(\lambda_j)| \geqslant -u_1(r_{p_n}) \quad (n \geqslant 1) \tag{26}$$

for $\lambda_j \in \Delta_n$.

Let us estimate $\ln |q_n(\lambda_j)|$ from above. In order to do it, we observe that as $n \geqslant n_1$,

$$\left|1 - \frac{\lambda_j}{\nu_k}\right| \leqslant 1 + \frac{r_{p_n+1}}{r_{p_n}} \leqslant e$$

for each $\lambda_j \in \Delta_n$. Thus,

$$\ln |q_n(\lambda_j)| \leqslant m_n + 2 \leqslant c_4 H(r_{p_n}) + 2 \quad (n \geqslant n_1)$$

for $\lambda_j \in \Delta_n$. It follows that

$$\ln |q_n(\lambda_j)| \leqslant u_2(r_{p_n}) \quad (n \geqslant 1) \tag{27}$$

for some function $u_2 \in L_k$. Thus, by (26), (27) we finally obtain that

$$\max_{\lambda_j \in \Delta_n} |\ln |q_n(\lambda_j)|| \leqslant u(r_{p_n}) \quad (n \geqslant 1),$$

where $u = u_1 + u_2$. The proof is complete. □

We let $\gamma_n = \Gamma_{p_n}$ $(n \geqslant 1)$.

Lemma 2. *For each* $n \geqslant 1$

$$M_n = \max_{\lambda \in \gamma_n} \ln |q_n(\lambda)| \leqslant u(r_{p_n}), \tag{28}$$

where u *is a function in* L_k*.*

Proof. For each $\lambda \in \gamma_n$, $\nu_k \in \Delta_n$ as $n \geqslant n_1$ we have

$$\left|1 - \frac{\lambda}{\nu_k}\right| \leqslant 1 + \frac{r_{p_n+1}}{r_{p_n}} \leqslant e.$$

Therefore, as in Lemma 1, $M_n \leqslant u_2(r_{p_n}) \leqslant u(r_{p_n})$ $(n \geqslant 1)$. Thus, estimate (28) indeed holds true. □

Now we are in position to prove Theorem 1.

Proof of Theorem 1. Let $\gamma_n = \Gamma_{p_n}$ $(n \geqslant 1)$. We let $\rho'_n = r_{p_n}$, $\rho''_n = r_{p_n+1}$. Then $\Delta_n = (\rho'_n, \rho''_n)$ $(n \geqslant 1)$.

We consider Dirichlet series

$$F(s) = \sum_{j=1}^{\infty} a_j e^{\lambda_j s} \quad (s = \sigma + it), \tag{29}$$

where

$$a_j = \exp\left((\rho - q^*) \frac{\rho'_n}{\ln_{k-1} \rho'_n}\right) \frac{q_n(\lambda_j)}{Q'(\lambda_j)} \quad (j \geqslant 1)$$

for $\lambda_j \in \Delta_n$ $(n \geqslant 1)$. Here Q is function (18), q_n is the function described in Lemmata 1, 2, $0 \leqslant \rho < \infty$, and q^* is the quantity defined in Theorem III. Since $H \in R_k$, $\varphi \in L_k$, estimate (16) in Theorem IV implies that $q^* = q(Q) \geqslant 0$, where

$$q(Q) = \overline{\lim_{n \to \infty}} \frac{\ln \lambda_n}{\lambda_n} \ln \left| \frac{1}{Q'(\lambda_n)} \right|.$$

Since $\rho''_n/\rho'_n \to 1$ as $n \to \infty$, $q(Q) < \infty$, by (20) we obtain that

$$\overline{\lim_{j\to\infty}} \frac{\ln|a_j|}{\lambda_j} = 0.$$

Hence, $F \in D_0(\Lambda)$. Taking into consideration (20) once again and employing formula (11) for calculating k-order ρ_k, we have

$$\rho_k(F) = \overline{\lim_{j\to\infty}} \frac{\ln_{k-1}\lambda_j}{\lambda_j} \ln_{k-1}\left|\frac{1}{Q'(\lambda_j)}\right| + \lim_{j\to\infty} \frac{\ln_{k-1}\lambda_j}{\lambda_j} \ln|q_n(\lambda_j)|$$
$$+ \lim_{j\to\infty} \frac{\ln\lambda_j}{\lambda_j}(\rho - q^*)\frac{\rho'_n}{\ln_{k-1}\rho'_n} = q(Q) + \rho - q^* = \rho.$$

Let us estimate order $\rho_s(F)$ in semi-strip $S(a,t_0)$ $(a > \pi G(R_k))$. Sequence $\Gamma = \{\mu_n\}$ of zeroes of function Q has density b that follows from (17) and $G(R) < b$. Given $G(R_k)$ and a, we choose parameter b in Theorem IV so that $G(R_k) < b < \frac{a}{\pi}$.

Then we observe that

$$A_n \overset{def}{\equiv} \sum_{\lambda_j\in\Delta_n} a_j e^{\lambda_j s} = e^{(\rho-q^*)\frac{\rho'_n}{\ln_{k-1}\rho'_n}} \frac{1}{2\pi i}\int_{\gamma_n} \frac{q_n(\xi)}{Q(\xi)} e^{s\xi} d\xi, \tag{30}$$

where γ_n is a closed contour formed by the arcs of circles $K_{\rho'_n}$ and $K_{\rho''_n}$ in the angle $\{\lambda : |\arg\lambda| \leqslant \varphi_n < \frac{\pi}{4}\}$ and by the segments of rays $\{\lambda : |\arg\lambda| = \varphi_n\}$. We take $\varphi_n = \varepsilon_0\frac{H(\rho'_n)}{\rho'_n}$ $(0 < \varepsilon_0 < 1)$. Since $H \in R_k$, then $\varphi_n \downarrow 0$ as $n \to \infty$. We choose number ε_0 so that $0 < \varphi_n < \frac{\pi}{4}$ $(n \geqslant 1)$.

Let us estimate function $\left|\frac{q_n(\xi)}{Q(\xi)}\right|$ on contour γ_n. In order to do it, we employ (17) and estimate [5]

$$-\ln|Q(re^{\pm i\varphi_n})| \leqslant 6H(r)\ln\frac{r}{H(r)} + \frac{8\pi}{|\varphi_n|}\frac{H^2(r)}{r} + 3\mu_1 b, \quad r \geqslant \rho'_{n_0}.$$

We note that this "effective" estimate of Weierstrass product is valid under the only restriction, which is condition (17).

Let $\rho'_n \leqslant r \leqslant \rho''_n$, $n \geqslant n_0$. Since $\frac{H(r)}{r} \downarrow$ as $r \uparrow$, then $H(r) \leqslant \frac{r}{\rho'_n}H(\rho'_n) \leqslant \frac{\rho''_n}{\rho'_n}H(\rho'_n)$. Hence,

$$-\ln|Q(re^{\pm i\varphi_n})| \leqslant 12H(\rho'_n)\ln\frac{\rho'_n}{H(\rho'_n)} + \frac{32\pi}{\varepsilon_0}H(\rho'_n) + 3\mu_1 b \tag{31}$$

as $\lambda_j \in \Delta_n$ $(n \geqslant 1)$. Estimates (19) hold true on the arcs of circles $K_{\rho'_n}$ and $K_{\rho''_n}$ in contour γ_n. Since $H \in R_k$, in view of $\rho''_n/\rho'_n \to 1$ as $n \to \infty$, by (19), (31) we obtain that

$$-\ln|Q(\xi)| \leqslant w(\rho'_n), \quad \xi \in \gamma_n \quad (n \geqslant n_1)$$

for some function $w \in L_k$. Therefore, employing Lemma 2, we obtain the estimate

$$\max_{\xi\in\gamma_n}\left|\frac{q_n(\xi)}{Q(\xi)}\right| \leqslant e^{u(\rho'_n)+w(\rho'_n)} \quad (n \geqslant n_1),$$

where u, w are functions in L_k. But then it follows from (30) that

$$|A_n| \leqslant 2\rho''_n e^{(\rho-q^*)\frac{\rho'_n}{\ln\rho'_n}+u(\rho'_n)+w(\rho'_n)} e^{\max\limits_{\xi\in\gamma_n}\mathrm{Re}\,(s\xi)} \tag{32}$$

as $n \geqslant n_1$.

Let $s \in S(a,t_0)$, $\xi \in \gamma_n$, $s = \sigma + it$, $\xi = \xi_1 + i\xi_2$. Then

$$\left|\sum_{\lambda_j<\rho'_{n_1}} a_j e^{\lambda_j s}\right| \leqslant \sum_{\lambda_j<\rho'_{n_1}} |a_j| e^{\lambda_j\sigma} \leqslant \sum_{\lambda_j<\rho'_{n_1}} |a_j| = M, \tag{33}$$

$\operatorname{Re}(s\xi) = \sigma\xi_1 - t\xi_2 \leqslant \sigma\rho_n' + (|t_0| + a)|\operatorname{Im}\xi|$. Since $|\operatorname{Im}\xi| \leqslant \rho_n''|\sin\varphi_n| \leqslant \rho_n'|\varphi_n| = \varepsilon_0\frac{\rho_n''}{\rho_n'}H(\rho_n')$ as $\xi \in \gamma_n$, then there exists $d(0 < d < \infty)$ such that

$$\max_{\xi\in\gamma_n}(s\xi) \leqslant \sigma\rho_n' + dH(\rho_n'), \quad (n \geqslant 1) \tag{34}$$

as $s \in S(a, t_0)$. Therefore, by (32)–(34) we obtain

$$M_s(\sigma) = \max_{|t-t_0|\leqslant a}|F(\sigma + it)| \leqslant M + \sum_{n=n_1}^{\infty}\gamma_n e^{\sigma\rho_n'} \quad (\sigma < 0),$$

where

$$\gamma_n = \exp\left[\ln(2\rho_n'') + (\rho - q^*)\frac{\rho_n'}{\ln\rho_n'} + dH(\rho_n') + u(\rho_n') + w(\rho_n')\right].$$

We introduce an auxiliary series

$$\Phi(s) = \sum_{n=1}^{\infty}\gamma_n e^{s\rho_n'} \quad (s = \sigma + it).$$

Since H, u, w belong to L_k, $\rho_n''/\rho_n' \to 1$ as $n \to \infty$, then in accordance with formula (11), the order of function Φ in half-plane Π_0 is equal to $\rho_k(\Phi) = \rho - q^*$. But $M_s(\sigma) \leqslant \Phi(\sigma) + M$. Hence, $\rho_s(F) \leqslant \rho - q^*$. It follows from Theorem III that $\rho_k(F) \leqslant \rho_s(F) + q^*$. Since $\rho_k(F) = \rho$, then $\rho_k(F) = \rho_s(F) + q^*$, and the proof of Theorem 1 is complete. □

BIBLIOGRAPHY

1. J.F. Ritt. *On certain points in the theory of Dirichlet series* // Amer. J. Math. **50**:1, 73–86 (1928).
2. S. Mandelbrojt. *Séries adhérentes. Régularisation des suites. Applications.* Gauthier-Villars, Paris (1952).
3. A.F. Leontiev. *Exponential series.* Nauka, Moscow (1976). (in Russian).
4. A.M. Gaĭsin. *A bound for the growth in a half-strip of a function represented by a Dirichlet series* // Matem. Sbornik. **117(159)**:3, 412–424 (1982). [Math. USSR-Sbornik. **43**:3, 411–422 (1983).]
5. A.M. Gaisin, D.I. Sergeeva. *Entire functions with a given sequence of zeros and of regular behavior on the real axis. I* // Sibir. Matem. Zhurn. **48**:5, 995–1007 (2007). [Siber. Math. J. **48**:5, 798–808 (2007).]
6. A.M. Gaisin, D.I. Sergeeva. *An estimate for the Dirichlet series in a half-strip in the case of the irregular distribution of exponents. II* // Sibir. Matem. Zhurn. **49**:2, 280–298 (2008). [Siber. Math. J. **49**:2, 222–238 (2008).]
7. N.N. Aitkuzhina, A.M. Gaisin. *Two-sided k-order estimate for Dirichlet series in a half-strip* // Ufimskij Matem. Zhurn. **6**:4, 19–31 (2014). [Ufa Math. J. **6**:4, 18–30 (2014).]

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