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ON UNCONDITIONAL EXPONENTIAL BASES IN WEIGHTED SPACES ON REAL AXIS

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Abstract. We prove that if there exists an unconditional exponential basis in integral weighted functional space on interval $(-1, 1)$ and the generating entire function satisfies some condition, then as a normed space, space $L_2(h)$ is isomorphic to classical space L_2.

Keywords: Hilbert spaces, entire functions, unconditional exponential bases, Riesz bases.

Mathematics Subject Classification: 30D20

1. Introduction

In the paper we consider the problem on existence of unconditional exponential bases in Hilbert spaces

$$L_2(h) = \{f \in L_{\mathrm{loc}}(-1, 1) : \|f\|^2 = \int_{-1}^{1} |f(t)|^2 e^{-2h(t)} dt < \infty\},$$

where h is a convex function on $(-1, 1)$.

In the classical case $h(t) \equiv 0$ the Fourier system $\{e^{\pi ni}\}_{n\in\mathbb{Z}}$ forms an orthonormalized basis. It is obvious that in other cases there can be no exponential orthonormalized bases in spaces $L_2(h)$. The notion of Riesz basis was introduced in [1] and denotes the image of orthonormalized basis under the action of a bounded invertible basis.

The system of elements $\{e_k, \ k = 1, 2, \dots\}$ in Hilber space H is called unconditional basis ([2], [3]) if it is complete and there exist numbers $c, C > 0$ such that for each set of numbers $c_1, c_2, \dots, c_n$ the relation

$$c \sum_{j=1}^{n} |c_k|^2 \|e_k\|^2 \leqslant \| \sum_{j=1}^{n} c_k e_k\|^2 \leqslant C \sum_{j=1}^{n} |c_k|^2 \|e_k\|^2$$

holds true. It is known (see [4],[5]) that if a system $\{e_k, \ k = 1, 2, \dots\}$ is an unconditional basis, then each element in space H can be uniquely represented by the series

$$x = \sum_{k=1}^{\infty} x_k e_k,$$

and

$$c \sum_{k=1}^{\infty} |x_k|^2 \|e_k\|^2 \leqslant \|x\|^2 \leqslant C \sum_{k=1}^{\infty} |x_k|^2 \|e_k\|^2.$$

Uncoditional basis $\{e_k, \ k = 1, 2, \dots\}$ becomes Riesz basis if and only if $0 < \inf \|e_k\| \leqslant \sup \|e_k\| < \infty$.

In work [6], there was initiated the study of unconditional exponential bases in Hilbert subspaces of space $H(D)$ formed by functional analytic in a bounded convex domain $D \subset \mathbb{C}$. For

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Smirnov space $E_2(D)$ on a convex polygon there were constructed unconditional exponential bases. In work [7] the existence of exponential bases in $E_2(D)$ on a convex domain D with a smooth boundary was considered. It was proved in dissertation [8] that there exist no unconditional exponential bases in Smirnov spaces on convex domains containing a smooth arc on the boundary. It was shown in [9] that there exist no unconditional exponential bases in Bergman spaces on convex domains whose boundary contains a point with a nonzero curvature.

In dissertation [10], an analogue of this result in weighted spaces $L_2(h)$ was proved: under certain regular growth conditions of weighted function $h(t)$, if for each $k \in \mathbb{N}$

$$e^{h(t)}(1-|t|)^k \longrightarrow \infty,\ |t| \longrightarrow 1,$$

then there is no unconditional exponential bases in space $L_2(h)$.

The same essence of the problems on unconditional exponential bases in Smirnov spaces, Bergman spaces and spaces $L_2(h)$ becomes clear if by Fourier-Laplace transform we pass to an equivalent problem on unconditional bases of reproducing kernels in Hilbert spaces of entire functions.

If X is a Hilbert space of functions, in which the set of all exponentials $e^{\lambda z}$, $\lambda \in \mathbb{C}$, is complete, then Fourier-Laplace transform mapping each linear continuous functional $S \in X^*$ into the function

$$\widehat{S}(\lambda) = S(e^{\lambda z}), \quad \lambda \in \mathbb{C},$$

makes a one-to-one correspondence between the dual space X^* and some space of functions $\widehat{X}$. Under natural conditions for original space X, space $\widehat{X}$ happens to be a Hilbert space of entire functions with the structure inherited from X^*, in which point functionals $F \longrightarrow F(z)$ happen to be bounded for each $z \in \mathbb{C}$. Thus, by the self-adjointness of Hilbert space, a reproducing kernel $K(\lambda, z)$ appears [11]:

$$(F(\lambda), K(\lambda, z))_{\widehat{X}} = F(z),\ \forall F \in \widehat{X}.$$

Simple functional-analytic arguments yield that the system of exponentials $e^{\lambda_k z}$, $k \in \mathbb{Z}$, is an unconditional basis in X if and only if the system $K(\lambda, \lambda_k)$, $k \in \mathbb{Z}$, is an unconditional basis in $\widehat{X}$.

The problem on unconditional basis of reproducing kernels in weighted spaces of entire functions was studied by many authors. For instance, in works [12]–[15], the weighted spaces of entire functions

$$H(\varphi) = \{F \in H(\mathbb{C}) :\ \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-2\varphi(z)} dm(z) < \infty\}$$

where considered, where φ is a subharmonic function on the plane, $dm(z)$ is the planar Lebesgue measure. It was proven in work [13] that under a certain growth regularity of function $\varphi(z) = \varphi(|z|)$, if

$$\ln^2 t = o(\varphi(t)),\ t \longrightarrow \infty,$$

there exist no bases of reproducing kernels in space $H(\varphi)$, while in the weighted spaces

$$\varphi(t) = O(\ln^2 t),\ t \longrightarrow \infty,$$

they exist.

In work [16] there was proved a general condition for Bergman function of a weighted space of entire functions ensuring the absence of the basis of reproducing kernels in this space.

The results of work [13] suggest a some stability of existence of unconditional bases in weighted spaces under “perturbation” of the weight. The matter is that as $\varphi(\lambda) = O(\ln|\lambda|)$, $\lambda \longrightarrow \infty$, spaces $H(\varphi)$ become finite-dimensional and thus, there are unconditional bases of reproducing kernels in these spaces. Then a natural conjecture is that for weights h growing rather slowly as $|t| \longrightarrow 1$, there can be unconditional exponential spaces in space $L_2(h)$ since they exist in the classical space L_2. Theorem 1 proven in this paper provides rather an argument against conjecture.

2. Notations, preliminaries and formulation of statements

The fact that two nonnegative functions f, g satisfy the estimate $f(x) \leqslant Cg(x), \forall x \in X$, for some constant C will be denoted by the symbol $\prec$:

$$f(x) \prec g(x), \ x \in X.$$

The symbols $\succ$ and $\asymp$ have corresponding meanings.

It was shown in work [17] that a space $\widehat{L}_2(h)$ of Fourier-Laplace transforms of the functionals continuous on $L_2(h)$ is isomorphic as a normed space to the space of entire functions of exponential type with the norm

$$\|F\|^2 := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|F(x+iy)|^2}{K(x)} dy d\widetilde{h}(x), \tag{1}$$

where

$$\widetilde{h}(x) = \sup_{|t|<1}(xt - h(t))$$

is the Young dual function for function h and

$$K(x) = \|e^{(x+iy)t}\|^2 = \int_{-1}^{1} e^{2xt-2h(t)} dt.$$

If $\delta_z : F(\cdot) \longrightarrow F(z)$ is a point-wise functional on $\widehat{L}_2(h)$, then by the definition of Fourier-Laplace transform

$$\|\delta_z\|^2_{\widehat{L}_2(h)^*} = \|e^{zt}\|^2 = K(\operatorname{Re} z).$$

Let the system of exponentials $\{e^{\lambda_k t}, k \in \mathbb{Z}\}$ form an unconditional basis in space $L_2(h)$ and S_k, $k \in \mathbb{Z}$ be a biorthogonal system. We let

$$L(\lambda) := (\lambda - \lambda_0)\widehat{S}_0(\lambda).$$

Then

$$\widehat{S}_k(\lambda) = \frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)}, \quad k \in \mathbb{Z},$$

and this system forms an unconditional basis in space $\widehat{L}_2(h)$, and hence,

$$\left\|\frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)}\right\|^2 = \frac{1}{K(\lambda_k)}. \tag{2}$$

By formula (1) we have ($\lambda = x + iy$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)}\right|^2 \frac{dy d\widetilde{h}(x)}{K(x)} \asymp \frac{1}{K(\operatorname{Re}\lambda_k)}, k \in \mathbb{Z}. \tag{3}$$

In what follows, entire function L will be called a generating function of unconditional basis. In the examples of unconditional bases we know, in order to construct a basis, one takes a generating function satisfying very strict conditions for the asymptotics at infinity. To describe the system of exponentials forming an unconditional basis in the classical space L_2, B.Ya. Levin introduced the notion of sine type entire function. In what follows this notion was generalized for unconditional bases in Smirnov spaces ([6],[7]). In work [17] a more general definition of sine type function was given for a subharmonic function.

Definition 1. *Let u be a continuous subharmonic function on the plane and $\tau(u,z)$ be the radius of maximal circle centered at point z in which function u deviate from the space of harmonic functions on this circle by at most 1. A sine type function for function u is an entire function L satisfying the conditions:*

1. All the zeroes z_n, $n \in \mathbb{N}$, of function L are simple and for some $\varepsilon > 0$ the balls $B(z_n, \varepsilon\tau(u,z_n))$, $n \in \mathbb{N}$, are mutually disjoint.

2. For each $\varepsilon > 0$, outside the set of circles $B(z_n, \varepsilon\tau(u,z_n))$, $n \in \mathbb{N}$, the relation

$$|\ln|L(z)| - u(z)| \leqslant A(\varepsilon)$$

holds true.

The subharmonicity and the definition of quantity $\tau(u,z)$ implies the property:

$2'$. *For each $z \in \mathbb{C}$ the upper estimate*

$$\ln|L(z)| \leqslant |u(z)| + A_1(\varepsilon)$$

is valid.

In such general sense sine type entire functions were used for constructing unconditional bases in Bergman spaces on convex polygons [22] and in spaces $H(\varphi)$ ([13]). Since space $\widehat{L}_2(h)$ is invariant w.r.t. the shifts along the imaginary axis, it is naturally to seek the generating function of unconditional basis among the sine type functions for some subharmonic function depending only on real part of its argument, i.e., for some convex function of one variable $u(x)$. Such entire function satisfies the condition

$$|L(\lambda)| \asymp 1, \ |\mathrm{Re}\,\lambda| < q, \ \lambda \notin \bigcup_k B_k, \tag{4}$$

in each strip $\{\lambda : |\mathrm{Re}\,\lambda| < q\}$ outside the balls $B_k(\delta) := B(\lambda_k, \delta)$, where λ_k are zeroes of function L. At that, for some $\delta > 0$ balls $B_k(\delta)$ are mutually disjoint.

Theorem 1. *Suppose that the generating function L of some unconditional basis in space $L_2(h)$ satisfies condition (4). Then as a normed space, space $L_2(h)$ is isomorphic to classical space L_2.*

In the proof of this theorem we shall make use the following facts on unconditional bases and generating functions. The definition of unconditional basis implies (see, for instance, [9])

$$\frac{1}{P}K(z) \leqslant \sum_k \frac{|L(z)|^2 K(z_k)}{|L'(z_k)|^2 |z - z_k|^2} \leqslant PK(z), \ z \in \mathbb{C}, \tag{5}$$

where P is a positive constant. We introduce a more general characteristics for the functions continuous on the plane.

Definition 2. *Given a continuous in $\overline{B}(z,r)$ function f, we let*

$$\|f\|_r = \max_{w \in \overline{B}(z,r)} |f(w)|.$$

Let $d(f,z,r)$ be the distance from function f to the space of harmonic in $B(z,r)$ functions:

$$d(f,z,r) = \inf\{\|f - H\|_r, \ H \textit{is harmonic in } B(z,r)\}.$$

For a function $\boldsymbol{u}$ continuous on $\mathbb{C}$ and a positive constant we let

$$\tau(u,z,p) = \sup\{r : \ d(u,z,r) \leqslant p\}.$$

It was shown in [21, Lm. 1.1] that in the case when u is continuous subharmonic function, the quantity $\tau = \tau(u,\lambda,p)$ is completely defined by the condition: if $H(z)$ is a harmonic majorant for fucntion u in ball $B(\lambda,\tau)$, then

$$\max_{z \in \overline{B}(\lambda,\tau)} (H(z) - u(z)) = 2p.$$

We define this quantity for the function $u(\lambda) = \ln K(\lambda)$ and the number $\ln(5P)$, where P is a constant in inequalities (5). Hereafter we denote it simply by $\tau(\lambda)$.

Theorem A ([9, Thm. 1])**.** *1. Each ball $B(\lambda, 2\tau(\lambda))$ contains at least one zero λ_k of function L.*

2. For each n, k, $n \neq k$, the inequality

$$|\lambda_k - \lambda_n| \geqslant \frac{\max(\tau(\lambda_k), \tau(\lambda_n))}{10P^{\frac{3}{2}}}$$

holds true.

3. For each k, the inequalities

$$\frac{1}{5^6 P^8} K(\lambda) \leqslant \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leqslant PK(\lambda)$$

hold true in the ball $B\left(\lambda_k, \frac{\tau(\lambda_k)}{20P^{\frac{3}{2}}}\right)$.

Statement 1 of this theorem implies the estimate for the derivatives in the vicinity of the imaginary axis. Together with (5) it yields the estimate for the function in the whole plane.

Lemma 1. *If the absolute value of generating function L is bounded from above on two vertical lines, then for for each $q > 0$ the estimates*

$$\sup_{|\operatorname{Re} \lambda_k| \leqslant q} |L'(\lambda_k)| < \infty, \quad \sup_{\lambda \in \mathbb{C}} |L(\lambda)| e^{-\widetilde{h}(\operatorname{Re} \lambda)} < \infty$$

hold true.

Assuming lower bounds for generating function, by means of (3) we obtain the lower bounds for the derivatives $|L'(\lambda_k)|$. Together with (5) it gives the lower bounds for the function outside neighbourhoods of zeroes.

Lemma 2. *Suppose that generating function L of unconditional basis in space $L_2(h)$ satisfies the condition: for some $q > 0$ and each $\delta > 0$*

$$|L(\lambda)| \succ 1, \quad |\operatorname{Re} \lambda| < q, \quad \lambda \notin \bigcup_k B_k,$$

and for some δ balls $B_k(\delta)$ are mutually disjoint. Then for each k the lower estimate

$$|L'(\lambda_k)| \succ \frac{1}{|x_k| + 1} e^{\widetilde{h}(x_k)}, \quad \lambda_k = x_k + iy_k,$$

holds true and for each $\varepsilon > 0$ the lower estimate

$$|L(\lambda)| \succ e^{\widetilde{h}(x) - \frac{1}{2} \ln \ln(|x| + e)}, \quad \lambda = x + iy \notin \bigcup_k B(\lambda_k, \varepsilon(|x_k| + 1))$$

is valid.

If generating function satisfies the assumptions of Theorem 1, then both the lemmata hold true.

Lemma 3. *Suppose that generating function satisfies the assumptions of Theorem 1. Then*

$$|L'(\lambda_k)| \asymp \frac{1}{|x_k| + 1} e^{\widetilde{h}(x_k)}, \ k \in \mathbb{Z},$$

and for each $\varepsilon > 0$

$$|L(\lambda)| \succ e^{\widetilde{h}(x)}, \ \lambda = x + iy \notin \bigcup_k B(\lambda_k, \varepsilon(|x_k| + 1)).$$

In particular, generating function L should be a sine type function for function $\widetilde{h}$.

3. Proof of Theorem 1 and auxiliary statements

We prove Theorem 1 under the assumption that the weight function satisfies the condition: for some α

$$e^{h(t)} = O(1-|t|)^{\alpha}, \quad |t| \longrightarrow 1, \quad h(t) > 0, \quad t \neq 0, \quad h(0) = 0,$$

since otherwise there are no unconditional bases in space $L_2(h)$ [10]. If this condition is satisfied, then

$$\tau(z) \asymp |\operatorname{Re} z| + 1. \tag{6}$$

The nonnegativity of h implies that for each $q > 0$

$$\int_{-q}^{q} d\widetilde{h}(x) > 0. \tag{7}$$

Proof of Lemma 1. We take the vertical line $\operatorname{Re}\lambda = a$ at which $|L|$ is bounded from above. Since $K(\lambda) = \|\delta_\lambda\|^2$, by formula (2) we have

$$|L(\lambda)|^2 \prec (|\lambda| + 1)^2 K(\lambda). \tag{8}$$

As it was shown in [20],

$$K(x) \asymp \frac{e^{2\widetilde{h}(x)}}{\tau(x)}.$$

Hence, in view of (6),

$$K(x) \asymp \frac{e^{2\widetilde{h}(x)}}{|x| + 1} \prec \frac{e^{2|x|}}{|x| + 1}. \tag{9}$$

This is why it follows from (8) that on the half-plane $\operatorname{Re}\lambda > a$

$$|L(\lambda)e^{-\lambda}|^2 \prec \frac{(|\lambda| + 1)^2}{|x| + 1}.$$

The arc length γ of the semi-cirle $C := \{\lambda : |\lambda - a| = R, \operatorname{Re}\lambda > a\}$ located above the parabola $y = (x-a)^2$ is $o(R)$ as $R \longrightarrow \infty$. The projection $C \setminus \gamma$ on the real axis is the interval (x_R, R), where x_R is a positive solution to equation $x^2 + x^4 = R^2$. For our needs it is sufficient that $x_R \asymp \sqrt{R}$. Let G be a domain bounded by semi-circle C and vertical line $\operatorname{Re}\lambda = a$. By Cauchy formula,

$$g'(\lambda) = \frac{1}{2\pi i} \int_{\partial G} \frac{g(z)dz}{(z-\lambda)^2}$$

for the function $g(\lambda) = L(\lambda)e^{-\lambda}$ and each $\lambda \in G$. By the above estimates we have

$$\left| \int_{\gamma \bigcup \overline{\gamma}} \frac{g(z)dz}{(z-\lambda)^2} \right| \prec \frac{o(R)}{R} = o(1), \quad \left| \int_{C \setminus (\gamma \bigcup \overline{\gamma})} \frac{g(z)dz}{(z-\lambda)^2} \right| \prec \sup \frac{1}{\sqrt{|x| + 1}} \asymp \frac{1}{x_R} = o(1).$$

Thus,

$$g'(\lambda) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = a} \frac{g(z)dz}{(z-\lambda)^2}.$$

If $|L|$ is bounded from above on the line $\operatorname{Re} z = a$ and $\operatorname{Re}\lambda - a > \varepsilon$, then

$$|g'(\lambda)| \prec \int_{\operatorname{Re} z = a} \frac{d\operatorname{Im} z}{|z-\lambda|^2} \prec \varepsilon^{-1}.$$

For $\lambda_k :\ \operatorname{Re}\lambda_k > a + \varepsilon$ we obtain ($x_k = \operatorname{Re}\lambda_k$)

$$|L'(\lambda_k)| \prec \varepsilon^{-1} e^{|x_k|}.$$

We get the same estimate on the half-plane $\operatorname{Re} z - a < -\varepsilon$. If $\operatorname{Re} z = b$, $b \neq a$, is the other vertical line on which $|L|$ is bounded, then taking $\varepsilon < |b-a|/3$, we obtain the estimates on four half-planes covering whole the plane. Hence,

$$\sup_k |L'(\lambda_k)| < \infty.$$

The first statement of Theorem A implies that one can find $d > 0$ such that each square

$$Q_m = \{z : \ |\operatorname{Re} z| \leqslant d, \ md \leqslant \operatorname{Im} z < (m+1)d\}, \ m \in \mathbb{Z},$$

contains at least one exponent of λ_k. We denote by z_m one of the zeroes lying in square Q_m. By relation (5) we obtain

$$|L(\lambda)|^2 \sum_m \frac{1}{|\lambda - z_m|^2} \prec K(\lambda).$$

In view of the density of zeroes z_m, it is easy to show that outside the balls $B(z_m, \delta)$ the relation

$$\sum_m \frac{1}{|\lambda - z_m|^2} \asymp \frac{1}{|x|+1}$$

holds true. Thus, outside such balls by (9) we have the estimate

$$|L(\lambda) \prec e^{\widetilde{h}(\operatorname{Re}\lambda)}.$$

The proof is complete. □

Proof of Lemma 2. Under the assumptions of the lemma we choose $\delta > 0$ such that the squares

$$Q_k(2\delta) = \{|Rez - \operatorname{Re}\lambda_k| \leqslant 2\delta, \ |\operatorname{Im} z - \operatorname{Im}\lambda_k| \leqslant 2\delta\}$$

are mutually disjoint and we let

$$Q(\delta) = \bigcup_k Q_k(\delta), \ Q_x = \{z : \operatorname{Re} z = x\} \setminus Q(\delta).$$

By relation (3) for each k and by the lower-semiboundedness of $|L|$ we have

$$|L'(\lambda_k)|^2 \succ K(\lambda_k) \int_{|x|\leqslant q} \left(\int_{-\infty}^{\infty} \frac{|L(\lambda)|^2 dy}{|\lambda - \lambda_k|^2} \right) \frac{d\widetilde{h}'(x)}{K(x)} \succ K(x_k) \int_{|x|\leqslant q} \left(\int_{Q_x} \frac{dy}{|\lambda - \lambda_k|^2} \right) \frac{d\widetilde{h}'(x)}{K(x)}.$$

Taking into consideration the choice of δ, it is easy to show that

$$\int_{Q_x} \frac{dy}{|\lambda - \lambda_k|^2} \asymp \frac{1}{|x - x_k| + 1}.$$

Hence,

$$|L'(\lambda_k)|^2 \succ K(\lambda_k) \int_{|x|\leqslant q} \frac{d\widetilde{h}'(x)}{(|x - x_k| + 1)K(x)} \succ \frac{K(x_k)}{|x_k| + 1}.$$

By relation (9) we obtain the desired lower bound of the derivatives

$$|L'(\lambda_k)| \succ \frac{1}{(|x_k| + 1)} e^{\widetilde{h}(x_k)}, \ \lambda_k = x_k + iy_k.$$

If we substitute this estimate into relation (5), we obtain

$$K(\lambda) \prec |L(\lambda)|^2 \sum_k \frac{|x_k| + 1}{|\lambda - \lambda_k|^2}. \tag{10}$$

By assumption (6) and Statement 2 of Theorem A there exist numbers ε and $d \in (1,2)$ such that each of the squares

$$R_{0,n} = \{0 \leqslant \operatorname{Re} z < \varepsilon, n\varepsilon \leqslant Imz < (n+1)\varepsilon\},$$
$$R_{m,n} = \{\varepsilon d^{m-1} \leqslant \operatorname{Re} z < \varepsilon d^m, d^{m-1} n\varepsilon \leqslant Imz < d^m(n+1)\varepsilon\},\ n \in \mathbb{Z},\ m \in \mathbb{N},$$

and each square $-R_{m,n}$, $m = 0, 1, 2, \dots$, $n \in \mathbb{Z}$, contains at most one zero λ_k. We take an arbitary $\delta > 0$ and $\lambda \notin \bigcup_k B(\lambda_k, \delta(|x_k|+1))$. Let $\lambda \in R_{s,j}$. By A we denote the set of adjacent with $R_{s,j}$ squares; they are finitely many. If $\lambda_k \in R_{m,n} \in \mathsf{A}$, then

$$|\lambda - \lambda_k| \succ (|x|+1),$$

and this is why

$$\sum\nolimits' \frac{|x_k|+1}{|\lambda-\lambda_k|^2} \prec \frac{1}{|x|+1}. \tag{11}$$

The superscript prime of summation symbol means that we sum the terms (if they exist) over $\lambda_k \in R_{m,n} \in \mathsf{A}$.

Let $z_{m,n}$ be the geometric center of square $R_{m,n}$. If $\lambda_k \in R_{m,n} \notin \mathsf{A}$, then

$$|\lambda - \lambda_k| \asymp |\lambda - z_{m,n}|,\ |x_k| + 1 \asymp \operatorname{diam} R_{m,n} \asymp d^m,\ k \in \mathbb{Z}.$$

Function $|\lambda - z|^{-2}$ is subharmonic w.r.t. $z \neq \lambda$, and therefore

$$\sum\nolimits' \frac{|\operatorname{Re}\lambda_k|+1}{|\lambda-\lambda_k|^2} \prec \int_{\mathbb{C}\setminus B(\lambda,\varepsilon(|\operatorname{Re}\lambda|+1))} \frac{dm(z)}{(|Rez|+1)|\lambda-z|^2}.$$

Here the superscript prime of the summation symbol means the summation over all $\lambda_k \in R_{m,n} \notin \mathsf{A}$. It yields

$$\sum\nolimits' \frac{|\operatorname{Re}\lambda_k|+1}{|\lambda-\lambda_k|^2} \prec \frac{\ln(|x|+1)}{|x|+1}.$$

In view of (11) we obtain

$$\sum \frac{|\operatorname{Re}\lambda_k|+1}{|\lambda-\lambda_k|^2} \prec \frac{\ln(|x|+1)}{|x|+1}.$$

Together with (10), the latter relation means that

$$|L(\lambda)| \succ e^{\widetilde{h}(x)-\frac{1}{2}\ln\ln(|x|+e)},\ \lambda \notin \bigcup_k B(\lambda_k, \varepsilon(|x_k|+1)).$$

The proof is complete. □

Proof of Lemma 3. Suppose the assumptions of Theorem 1. By Lemma 1,

$$|L(\lambda)| \prec e^{\widetilde{h}(x)} \asymp \frac{1}{|\operatorname{Re}\lambda|+1} K(\lambda).$$

By the definition of function τ, there exists a harmonic function $u(z)$ in the circle $B = B(\lambda, \varepsilon(|x|+1))$ differing from $\ln K(z)$ by at most 1. Let g be a function analytic in this circle and $\operatorname{Re} g = u$. By Cauchy formula

$$(Le^{-g})'(\lambda_k) = \frac{1}{2\pi i} \int_{\partial B} \frac{L(z)e^{-g(z)}dz}{(z-\lambda_k)^2}$$

we obtain the upper bound

$$|L'(\lambda_k)| \prec \frac{e^{\widetilde{h}(x_k)}}{|x_k|+1}.$$

The corresponding lower bounds we obtained in Lemma 2.

We choose $\varepsilon > 0$ such that the circles $B(\lambda_k, \varepsilon(|\lambda_k| + 1))$ are mutually disjoint and in each of them the relation of Statement of Theorem A is satisfied. By the proven asymptotics for $L'|(\lambda_k)|$ we obtain that the lower bound

$$|L(\lambda)| \succ e^{\widetilde{h}(\operatorname{Re}\lambda)}$$

is satisfied in the annuli $S_k := B(\lambda_k, \varepsilon(|\lambda_k| + 1)) \setminus B(\lambda_k, \frac{\varepsilon}{2}(|\lambda_k| + 1))$. In view of Lemma 1

$$|L(\lambda)| \prec e^{\widetilde{h}(\operatorname{Re}\lambda)},\ \lambda \in S := \bigcup_k S_k.$$

Let

$$C = \sup_{\lambda \in S} |L(\lambda)| e^{-\widetilde{h}(\operatorname{Re}\lambda)}.$$

We denote

$$\Omega := \mathbb{C} \setminus \bigcup_k B(\lambda_k, \varepsilon(|\lambda_k| + 1)).$$

The function

$$u(\lambda) = \widetilde{h}(\operatorname{Re}\lambda) - \ln|L(\lambda)|$$

is subharmonic on $\mathbb{C} \setminus \bigcup_k \{\lambda_k\}$ and by Lemma 2 the estimate

$$u(\lambda) \leqslant \frac{1}{2} \ln\ln(|\operatorname{Re}\lambda| + e) + O(1), \quad \lambda \in \Omega$$

holds true. At that,

$$u(\lambda) \leqslant C, \quad \lambda \in \partial\Omega.$$

Function $v_1(\lambda) = \max(u(\lambda), 2C)$ is subharmonic on the plane outside zeroes λ_k

$$v_1(\lambda) \equiv 2C,\ \lambda \in \bigcup_k S_k.$$

Then the function

$$v(\lambda) = \begin{cases} v_1(\lambda), & \lambda \in \Omega, \\ 2C, & a = \lambda \notin \Omega, \end{cases}$$

is subharmonic in the whole plane and satisfies the estimate

$$v(\lambda) \leqslant \ln\ln(|\lambda| + O(1),\ \lambda \in \mathbb{C}.$$

But in this case function v must be constant [15] and thus,

$$\ln|L(\lambda)| \geqslant \widetilde{h}(\operatorname{Re}\lambda) + O(1), \lambda \in \Omega.$$

□

Completing the proof of Theorem 1.1. Relation (3) implies the estimate for each x_k

$$\frac{|L'(\lambda_k)|^2}{K(x_k)} \succ \int_0^{\frac{|x_k|}{2}} \int_{-\infty}^{\infty} \frac{|L(x+iy)|^2}{|z-\lambda_k|^2 K(x)} d\widetilde{h}'(x) dy.$$

We estimate $|L|^2$ by Lemma 3 outside mutually disjoint squares

$$Q_n := \{z : |\operatorname{Re}(z - \lambda_n)| \leqslant \varepsilon(|\operatorname{Re}\lambda_n| + 1),\ |\operatorname{Im}(z - \lambda_n)| \leqslant \varepsilon(|\operatorname{Re}\lambda_n| + 1)\}.$$

By simple lower estimates of the integral w.r.t. dy we obtain

$$\frac{|L'(\lambda_k)|^2}{K(x_k)} \succ \frac{1}{|x_k|} \int_0^{\frac{|x_k|}{2}} |x| d\widetilde{h}'(x).$$

By Lemma 3 it implies

$$\int_0^{\frac{|x_k|}{2}} |x| d\widetilde{h}'(x) \prec 1, |x_k| \longrightarrow \infty. \tag{12}$$

Integrating by parts, we have

$$\int_0^a t d\widetilde{h}'(t) = a\widetilde{h}'(a) - \widetilde{h}(a) + \widetilde{h}(0).$$

If $\widetilde{h}'$ is a strictly increasing function, then the supremum in

$$h(t) = \sup_x (xt - \widetilde{h}(x))$$

is attained at the unique point $x = x_t : \ \widetilde{h}'(x) = t$ and $|x_t| \longrightarrow \infty$ if $|t| \longrightarrow 1$. Hence,

$$h(t) = x_t t - \widetilde{h}(x_t) = x_t \widetilde{h}'(x_t) - \widetilde{h}(x_t).$$

Now by (12) we obtain that $h(t)$ is a bounded function, i.e., $L_2(h)$ is isomorphic to classic L_2. The proof is complete. □

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