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ON SPECTRAL PROPERTIES OF STURM-LIOUVILLE OPERATOR WITH MATRIX POTENTIAL

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Abstract. In the work we obtain asymptotic estimates for the eigenvalues, eigenvectors and spectral projectors of a Sturm-Liouville operator with a matrix potential subject to quasi-periodic boundary conditions. The matrix potential is formed by functions square summable on the segment $[0,1]$ and the matrix of the means of the functions have simple eigenvalues. We consider also the case when the matrix of the means has a simple structure.

Keywords: Similar operator method, spectrum, linear operators, spectral projectors.

Mathematics Subject Classification: 35L75, 35Q53, 37K10, 37K35

1. Introduction

We consider Hilbert space $L_2[0,1]$ of functions measurable and square integrable on $[0,1]$. Let $\mathcal{H} = L_2([0,1],\mathbb{C}^m) = L_2^m[0,1] = \underbrace{L_2[0,1]\times\cdots\times L_2[0,1]}_{m \text{ times}}$ be the Hilbert space of functions with values in $\mathbb{C}^m$ measurable and square integrable on $[0,1]$. The scalar product in $L_2^m[0,1]$ is defined by the formula

$$(f,g)=\sum_{i=1}^{m}(f_i,g_i),\quad f=(f_1,f_2,\dots,f_m)\in L_2^m[0,1],\quad g=(g_1,g_2,\dots,g_m)\in L_2^m[0,1],$$

$(f_i,g_i)=\int_0^1 f_i(x)\overline{g_i(x)}\,dx$ is the scalar product in space $L_2[0,1]$ of complex-valued functions and the norm

$$\|f\|^2=\sum_{i=1}^{m}\int_0^1 |f_k(x)|^2\,dx$$

is generated by this scalar product.

In space $L_2^m[0,1]$ we consider the differential operator L generated by the differential expression

$$(Ly)(t) = -y''(t) + Q(t)y(t), \tag{1}$$

and the quasiperiodic boundary condtions

$$y'(1) = y'(0)e^{i\theta},\quad y(1) = y(0)e^{i\theta}, \tag{2}$$

$\theta\in(0,2\pi)$, $\theta\neq\pi$. It is a Sturm-Liouville operator with matrix potential $Q(t)=\{b_{ij}(t)\}$, $i,j=1,\dots,m$, and $b_{ij}\in L_2[0,1]$ are complex-valued functions. By the symbol $Q_0=(b_{0ij})$, $i,j=1,2,\dots,m$, we denote the matrix formed by the mean values of functions b_{ij}, i.e., $b_{0ij}=\int_0^1 b_{ij}(t)\,dt$. In what follows we study the spectral characteristics of operator L with square integrable functions b_{ij} in the case when Q_0 is a normal matrix of simple structure.

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We note that in this case matrix Q_0 is similar to the diagonal matrix having eigenvalues $\mu_1, \mu_2, \dots, \mu_m$, on the diagonal and the basis in space $\mathbb{C}^m$ is formed by the associated (orthogonal) eigenvectors of matrix Q_0. This is why without loss of generality we can assume that matrix Q_0 is diagonal and the corresponding basis is formed by its orthonormalized eigenvectors $f_1, f_2, \dots, f_m$. We also assume that numbers $\mu_1, \mu_2, \dots, \mu_m$ are taken in the ascending order, i.e., $\mu_1 \leqslant \mu_2 \leqslant \dots \leqslant \mu_m$.

We consider operator L_0 such that $L_0 y = -y''$ with the domain coinciding with the domain of operator L. In what follows operator L_0 plays the role of an unperturbed one. It is known that its eigenvalues are the numbers

$$\lambda_n = \lambda_{n,j} = (2\pi n + \theta)^2, \quad n \in \mathbb{Z}, \quad j = 1, 2, \dots, m,$$

and the associated eigenvectors are the functions $e_{n,j}(t) = e^{i(2\pi n+\theta)t} f_j$, $n \in \mathbb{Z}$, $j = 1, 2, \dots, m$. The multiplicity of such eigenvalue is equal to m.

Operators of such class were considered by O.A. Veliev in [1] under the condition $b_{ij} \in L_1[0,1]$, $i = 1, 2, \dots, m$. In that work and under the same assumptions for θ the Riesz basis property was proven for the root functions of operator L in the case of simple eigenvalues of matrix Q_0. Estimates for the eigenvalues and eigenvectors of operator L were obtained (see [1, Thm. 2]):

$$\widetilde{\lambda}_{k,j} = (2\pi k + \theta)^2 + \mu_j + \mathcal{O}\left(\frac{\ln|k|}{k}\right), \quad \widetilde{e}_{k,j}(x) = \mathrm{e}^{i(2\pi k+\theta)x} f_j + \mathcal{O}\left(\frac{\ln|k|}{k}\right).$$

In the considered case ($b_{ij} \in L_2[0,1]$), it is possible to improve the asymptotics of the eigenvalues and eigenvectors of operator L and to obtain also the estimates of equiconvergence of the corresponding spectral resolutions. We note that in work [1], there were not considered the estimates for the weighted mean in the case of multiple eigenvalues of matrix Q_0, deviation of spectral projectors and the equiconvergence. The main result of the work is the following theorems.

Theorem 1. *There exists a natural number l such that the spectrum $\sigma(L)$ of operator L is represented as*

$$\sigma(L) = \widetilde{\sigma}_{(l)} \bigcup \Big(\bigcup_{|k|>l} \widetilde{\sigma}_k \Big), \tag{3}$$

where $\widetilde{\sigma}_{(l)}$ is a finite set of at most $2lm + m$ eigenvalues, $\widetilde{\sigma}_k = \{\widetilde{\lambda}_{k,1}, \ \widetilde{\lambda}_{k,2}, \dots, \widetilde{\lambda}_{k,m}\}$. In the case of simple eigenvalues μ_j of matrix Q_0 the estimates ($|k| > l$)

$$\widetilde{\lambda}_{k,j} = (2\pi k + \theta)^2 + \mu_j + \mathcal{O}(|k|^{-1}), \quad j = 1, 2, \dots, m, \tag{4}$$

$$\widetilde{e}_{k,j} = \mathrm{e}^{i(2\pi k+\theta)x} f_j + \mathcal{O}(|k|^{-1}) \tag{5}$$

hold true. If eigenvalues μ_j are semi-simple, the formula

$$\frac{1}{m}\sum_{j=1}^{m} \widetilde{\lambda}_{k,j} = (2\pi k + \theta)^2 + \frac{1}{m}\sum_{j=1}^{m} \mu_j + \mathcal{O}(|k|^{-1}) \tag{6}$$

holds true. The spectral projectors $P_k = P(\sigma_k, L_0)$ and $\widetilde{P}_k = P(\widetilde{\sigma}_k, L)$ satisfy the asymptotic representation ($|k| > l$)

$$\|\widetilde{P}_k - P_k\|_2 = \mathcal{O}(|k|^{-1}),$$

and

$$\sum_{|k|>l} \|\widetilde{P}_k - P_k\|_2^2 < \infty. \tag{7}$$

Corollary 1. *Operator L is spectral in the sense of Dunford* [2] *w.r.t. resolution (3).*

Theorem 2. *The estimate for uniform absolute equiconvergence of spectral resolutions*

$$\|\widetilde{P}(\Omega) - P(\Omega)\|_2 \leqslant \mathrm{const}\, \mathcal{O}(k_0^{-\frac{1}{2}}),$$

where $P(\Omega) = \sum\limits_{k\in\Omega} P_k$, $\widetilde{P}(\Omega) = \sum\limits_{k\in\Omega} \widetilde{P}_k$, $\Omega = \{i \in \mathbb{Z}, |i| > l\}$, $k_0 = \min\limits_{k\in\Omega} |k|$.

We note that since projectors $P(\Omega)$ and $\widetilde{P}(\Omega)$ are similar, the series $\sum\limits_{k\in\Omega} \widetilde{P}_k$ converges absolutely.

2. On similar operators methods

We introduce first the following spaces of operators. By $\operatorname{End}\mathcal{H}$ we denote the Banach algebra of operators acting in $\mathcal{H}$ with the norm $\|X\|_\infty$. Hereafter by A we denote a closed linear operator acting in $\mathcal{H}$ with a domain $D(A)$, a spectrum $\sigma(A)$ and a resolvent set $\rho(A)$. Operator A plays a role of an unperturbed operator whose spectral properties are well-studied. By the symbol $\mathfrak{L}_A(\mathcal{H})$ we denote the space of operators acting in $\mathcal{H}$ and relatively bounded w.r.t. operator A, i.e., $B \in \mathfrak{L}_A(\mathcal{H})$ if $D(B) \supset D(A)$ and there exists a constant $C > 0$ such that $\|Bx\| \leqslant C(\|x\| + \|Ax\|), x \in D(A)$. The norm in $\mathfrak{L}_A(\mathcal{H})$ is introduced by the formula $\|B\|_A = \inf\{C > 0 : \|Bx\| \leqslant C(\|x\| + \|Ax\|), x \in D(A)\}$. We observe that $B \in \mathfrak{L}_A(\mathcal{H})$ means that there exists $\lambda_0 \in \rho(A)$ such that $\|B(A - \lambda_0 I)^{-1}\|$ is finite. For the considered operator, without loss of generality we can assume that $D(A) = D(B)$.

Definition 1 ([6])**.** *Two linear operators* $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1, 2$, *are called similar if there exists a continuously invertible operator* $U \in \operatorname{End}\mathcal{H}$, *such that* $UD(A_2) = D(A_1)$ *and* $A_1Ux = UA_2x$, $x \in D(A_2)$. *Operator* U *is called operator of transforming operator* A_1 *into* A_2.

As a method of studying operator (1), (2), the similar operators method will serve. The main ideas of the similar operators method are presented in the chronological order in works [3]–[7]. We shall follow work [6]. We note that by means of the similar operators method, the spectral properties of various differential operators were studied, for instance, in works [8]–[11], [16]. To operator L with a matrix potential and quasiperiodic boundary conditions this method was not applied before.

In accordance with the terminology of M.G. Krein, an operator acting in the space of operators will be called transformer.

One of important notions in the similar operators method is that of admissible triple which is to satisfy certain condition to make the method applicable.

Definition 2 ([6])**.** *Let* $\mathcal{M} \subset \mathfrak{L}_A(\mathcal{H})$ *be a linear subspace of operators and* $J : \mathcal{M} \to \mathcal{M}$, $\Gamma : \mathcal{M} \to \operatorname{End}\mathcal{H}$ *are transformers. The triple* $(\mathcal{M}, J, \Gamma)$ *is called admissible triple for operator* A *and* $\mathcal{M}$ *is an admissible perturbation space if the following conditions hold true:*

1) $\mathcal{M}$ *is a Banach space with a norm* $\|\cdot\|_\mathcal{M}$ *continuously embedded in* $\mathfrak{L}_A(\mathcal{H})$, *i.e., there exists a constant* $C > 0$ *such that* $\|X\|_A \leqslant C\|X\|_\mathcal{M}$ *for each* $X \in \mathfrak{L}_A(\mathcal{H})$*;*

2) J *and* Γ *are continuous transformers and* J *is a projector, i.e.,* $J^2 = J$*;*

3) $(\Gamma X)D(A) \subset D(A)$, $A\Gamma X - (\Gamma X)A = X - JX$, $\forall X \in \mathcal{M}$ *and* $Y = \Gamma X$ *is the unique solution to the equation* $AY - YA = X - JX$ *satisfying the condition* $JY = 0$*;*

4) $X(\Gamma Y)$, $(\Gamma X)Y \in \mathcal{M}$, $\forall X, Y \in \mathcal{M}$, *and there exists a constant* $\gamma > 0$ *such that*

$$\|\Gamma\| \leqslant \gamma, \quad \text{and} \quad \max\{\|X\Gamma Y\|_\mathcal{M}, \|\Gamma XY\|_\mathcal{M}\} \leqslant \gamma\|X\|_\mathcal{M}\|Y\|_\mathcal{M};$$

5) for each $X \in \mathcal{M}$ *and each* $\varepsilon > 0$ *there exists a number* $\lambda_\varepsilon \in \rho(A)$ *such that* $\|X(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.

Let $(\mathcal{M}, J, \Gamma)$ be an admissible triple for operator A. We perturbed operator A by some operator B in the space of admissible perturbations $\mathcal{M}$.

Theorem 3 ([6])**.** *Suppose that the condition* $4\|B\|_{\mathcal{M}}\gamma < 1$ *is satisfied. Then operators* $A - B$ *and* $A - JX$ *are similar, i.e.,*

$$(A - B)(I + \Gamma X) = (I + \Gamma X)(A - JX),$$

where operator $X \in \mathcal{M}$ *solves the nonlinear operator equation*

$$X = B\Gamma X - (\Gamma X)(JB) - (\Gamma X)J(B\Gamma X) + B, \tag{8}$$

and it can be found by method of successive approximations using operator B *as the first apporximation.*

Sometimes it is difficult to choose a space of admissible perturbations $\mathcal{M}$ such that operator B belongs to $\mathcal{M}$ and this space is convenient for further studies. This is why sometimes it is convenient to make a preliminary similarity transformation of operator $A - B$ into operator $A - \widetilde{B}$, where $\widetilde{B} \in \mathcal{M}$ ($B \notin \mathcal{M}$).

Continuations of transformers J and Γ on space $\mathfrak{L}_A(\mathcal{H})$ are denoted by the same symbols and is made by as follows (see [6]). Let $\lambda_0 \in \rho(A)$ and

$$JX = J(X(A - \lambda_0 I)^{-1})(A - \lambda_0 I), \quad X \in \mathfrak{L}_A(\mathcal{H}), \tag{9}$$

$$\Gamma X = \Gamma(X(A - \lambda_0 I)^{-1})(A - \lambda_0 I), \quad X \in \mathfrak{L}_A(\mathcal{H}). \tag{10}$$

These continuations are well-defined, i.e., they are independent of choice of number $\lambda_0 \in \rho(A)$. If $x \in D(A)$, it follows from (9) and (10) that the action of operators JX and ΓX on a vector x is introduced in the same way as in Definition 2 and they are used in Theorem 3.

Such transformation is possible under the validity of the following assumption [6], [12].

Assumption 1. *Operators* ΓB, JB, B *satisfy the following conditions:*
1) $\Gamma B \in \operatorname{End}\mathcal{H}$ *and* $\|\Gamma B\| < 1$*;*
2) $(\Gamma B)D(A) \subset D(A)$ *and* $(A\Gamma B)x - (\Gamma BA)x = Bx - (JB)x$, $\forall x \in D(A)$*;*
3) $B\Gamma B$, $(\Gamma B)JB \in \mathcal{M}$*;*
4) for each number $\varepsilon > 0$ *there exists a number* $\lambda_\varepsilon \in \rho(A)$ *such that* $\|B(A - \lambda_\varepsilon I)^{-1}\|_\infty < \varepsilon$.

Theorem 4 ([6], [12])**.** *Suppose Assumption 1, then operator* $A-B$ *is similar to an operator* $A - \widetilde{B}$ *being*

$$A - JB - (I + \Gamma B)^{-1}(B\Gamma B - (\Gamma B)JB) = A - \widetilde{B},$$

and operator $I + \Gamma B$ *serves as the operator of transforming* $A - B$ *into operator* $A - \widetilde{B}$.

3. Construction of admissible triple for an operator close to L_0

We begin with the following remark. As $\theta \neq 0$ and $\theta \neq \pi$, the eigenvalue λ_n of operator L_0 has multiplicity m, but as $\theta \to 0$ and $\theta \to \pi$, the distance between the pairs of corresponding eigenvalues λ_n and λ_{-n} ($\theta \to 0$) or λ_n and $\lambda_{-(n+1)}$ ($\theta \to \pi$) tends to zero. This is why, if we consider only the case of simple eigenvalues of operator L, then $\theta \in [\varepsilon_1, \pi - \varepsilon_2] \cup [\pi + \varepsilon_2, 2\pi - \varepsilon_3]$, where ε_1, ε_2, ε_3 are non-zero small quantities. Under this condition all the estimated obtained in what follows are uniform in θ in the considered segment. In what follows we deal with such θ only.

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a normal linear unbounded operator having semi-simple eigenvalues

$$\lambda_{n,i} = (an + \theta)^2 + \mu_i, \quad i = 1, \dots, m, \quad n \in \mathbb{Z},$$

where a, θ are constants, $\theta \in [\varepsilon_1, \pi - \varepsilon_2] \cup [\pi + \varepsilon_2, 2\pi - \varepsilon_3]$, $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_m$, and the associated eigenvectors $e_{n,i}$ form an orthonormalized basis in $\mathcal{H}$. Let $P_n = P(\sigma_n, A)$ be the Riesz projectors constructed by spectral set $\sigma_n = \{\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nm}\}$ of operator A. To each operator $X \in \operatorname{End}\mathcal{H}$, we associate two matrices: operator one $X = (X_{ij})$, where $X_{ij} = P_i X P_j$, $i, j \in \mathbb{Z}$, and a scalar one $X = (x_{n_i k_j})$, where $x_{n_i k_j} = (Xe_{k_j}, e_{n_i})$, $n, k \in \mathbb{Z}$, $1 \leqslant i, j \leqslant m$.

By the symbol $\mathfrak{S}_2(\mathcal{H})$ we denote Hilbert-Schmidt ideal of operators in $\operatorname{End}\mathcal{H}$. Since projectors P_n, $n\in\mathbb{Z}$, are orthoprojectors, then the norm in $\mathfrak{S}_2(\mathcal{H})$ can be defined by the formula

$$\|X\|_2^2=\sum_{i,j}\|P_iXP_j\|_2^2,\quad \forall X\in\mathfrak{S}_2(\mathcal{H}),\quad i,j\in\mathbb{Z}.$$

In what follows we shall also employ the notation X_p for p-th diagonal of operator matrix of operator X, i.e., $X_p=\sum\limits_{i-j=p}X_{ij}$. It is obvious that $\|X\|_2^2=\sum\limits_p\|X_p\|_2^2$, $X\in\mathfrak{S}_2(\mathcal{H})$.

As the space of admissible perturbation we choose space $\mathfrak{S}_2(\mathcal{H})$ and we perturb operator A by an operator $B\in\mathfrak{S}_2(\mathcal{H})$. We define a family of transformers by the formula

$$J_kX=Q_kXQ_k+\sum_{|i|>k}P_iXP_i,\quad X\in\mathfrak{S}_2(\mathcal{H}),$$

where $Q_k=\sum\limits_{|j|\leqslant k}P_j$ and $J_0X=\sum\limits_{i\in\mathbb{Z}}P_iXP_i$. It is obvious that $\|J_k\|=1$ since

$$\|J_kX\|_2^2\leqslant\|Q_kXQ_k\|_2^2+\sum_{i\geqslant k}\|P_iXP_i\|_2^2\leqslant\|X\|_2^2.$$

On the other hand, if $X=\sum\limits_iP_iXP_i$, i.e., matrix of operator X is diagonal, then $J_kX=X$ and $\|J_kX\|_2=\|X\|_2$.

We proceed to constructing of transformer $\Gamma_0:\mathfrak{S}_2(\mathcal{H})\to\operatorname{End}\mathcal{H}$. First we define it on operator blocks $X_{ij}=P_iXP_j$, where $X\in\mathfrak{S}_2(\mathcal{H})$ (and $X_{ij}\in\mathfrak{S}_2(\mathcal{H})$). For each X_{ij}, $i\neq j$, we let $\Gamma_0X_{ij}=Y_{ij}$, where Y_{ij} is a solution to the equation

$$AY_{ij}-Y_{ij}A=X_{ij},\quad i\neq j,\quad i,j\in\mathbb{Z},$$

and $Y_{ii}=0$. We observe that the latter equation can be rewritten as

$$A_iY_{ij}-Y_{ij}A_j=X_{ij},\tag{11}$$

$A_i=A\big|_{\mathcal{H}_i}$, $\mathcal{H}_i=\operatorname{Ran}P_i$. Since $\sigma(A_i)\cap\sigma(A_j)=\varnothing$, equations (11) are solvable [13], [14] and

$$\|Y_{ij}\|_2\leqslant\frac{\text{const}\|X_{ij}\|_2}{\operatorname{dist}(\sigma_i,\sigma_j)}.$$

It follows from [15] that const can be assumed to be one.

We recover operator Γ_0X by operator blocks $Y_{ij}=(\Gamma X)_{ij}$: $\Gamma_0X=\sum\limits_{i,j}(\Gamma X)_{ij}$, and

$$\|\Gamma_0X\|_2^2=\sum_{i,j}\|(\Gamma X)_{ij}\|_2^2\leqslant\sum_{i\neq j}\frac{\|X_{ij}\|_2^2}{\operatorname{dist}^2(\sigma_i,\sigma_j)}\leqslant d_0^{-2}\sum_{i\neq j}\|X_{ij}\|_2^2\leqslant d_0^{-2}\|X\|_2^2.$$

Thus, operator Γ_0X belongs to the space $\mathfrak{S}_2(\mathcal{H})$ and $\|\Gamma_0\|\leqslant d_0^{-1}$, where the symbol d_0 stands for $\min\limits_{i\neq j}\operatorname{dist}(\sigma_i,\sigma_j)$.

We define the family of transformers Γ_kX by the formula

$$\Gamma_kX=\Gamma_0X-\Gamma_0(Q_kXQ_k)=\Gamma_0X-Q_k\Gamma_0XQ_k.\tag{12}$$

It follows from this identity that

$$\begin{aligned}d_k=&\operatorname{dist}(\sigma_k,\sigma_{k+1})=|(a(k+1)+\theta)^2+\mu_m-(ak+\theta)^2-\mu_1|\\=&|2a^2k+2a\theta+\mu_m-\mu_1|=\mathcal{O}(k),\end{aligned}$$

i.e.,

$$\|\Gamma_kX\|_2^2\leqslant\frac{\text{const}}{k^2}\|X\|_2^2.\tag{13}$$

It implies that Condition 4 of Definition 2 holds true with γ of order k^{-1}.

Let us chech other conditions in Definition 2 concerning transformer Γ. We consider operator $AQ_n\Gamma XA^{-1}$ and represent it as ($x \in \mathcal{H}$)

$$AQ_n\Gamma XA^{-1}x = Q_n\Gamma Xx + Q_n(X - JX)A^{-1}x, \quad x \in \mathcal{H}.$$

Since $Q_n\Gamma X \to \Gamma Xx$, $Q_n(X - JX)A^{-1}x \to (X - JX)A^{-1}x$ as $n \to \infty$, then $AQ_n\Gamma XA^{-1}x \to y_0 \in \mathcal{H}$. Let $Q_n\Gamma XA^{-1}x \to x_0 = \Gamma XA^{-1}x$, then due to the closedness of operator A we have $x_0 \in D(A)$ and $Ax_0 = y_0$, where $y_0 = \lim\limits_{n\to\infty} y_n$.

Condition 5) is obvious since

$$\|X(A - \lambda_\varepsilon I)^{-1}\|_2^2 \leqslant \|X\|_2^2 \cdot \|(A - \lambda_\varepsilon I)^{-1}\|_\infty,$$

at that, the first factor is finite and the other can be chosen arbitrarily small.

Thus, we have proven

Theorem 5. *For each $k \geqslant 0$ the triple $(\mathfrak{S}_2(\mathcal{H}), J_k, \Gamma_k)$ is admissible for operator A.*

Theorem 5 and estimate (13) imply

Theorem 6. *There exists a number $l \geqslant 0$ such that operator $A - B$ is similar to the block-diagonal operator*

$$A - Q_lXQ_l - \sum_{|i|>l} P_iXP_i,$$

where X solves nonlinear operator equation (8) with Γ_l and J_l. The operator of transforming is the operator $I + \Gamma_lX$ an the first approximation for solution X in the iteration method is operator B.

Lemma 1. *Let $X \in \mathfrak{S}_2(\mathcal{H})$. Then operator $\Gamma_kX \in \mathfrak{S}_2(\mathcal{H})$ can be represented as $\Gamma_kX = YA^{-\frac{1}{2}} = A^{-\frac{1}{2}}Z$, where $Z, Y \in \mathfrak{S}_2(\mathcal{H})$.*

The proof follows from the matrix representation of operator Γ_kX, $k \geqslant 0$.

The operator $A^{-\frac{1}{2}}$ in Lemma 1 is defined on basis vectors as follows: if $x = \sum\limits_{\substack{i\in\mathbb{Z}\\k=1,\dots,m}} (x, e_{i,k})e_{i,k}$ and $Ae_{i,k} = \lambda_{i,k}e_{i,k}$, then $A^{-\frac{1}{2}}x = \sum\limits_{\substack{i\in\mathbb{Z}\\k=1,\dots,m}} \lambda_{i,k}^{-\frac{1}{2}}(x, e_{i,k})e_{i,k}$.

Lemma 2. *Under the assumptions of Theorem 6, we have$X - B \in \mathfrak{S}_1(\mathcal{H})$ and*

$$\|P_i(X - B)P_i\| \leqslant \alpha_i i^{-1}, \quad i > l, \tag{14}$$

where $\{\alpha_i\} \in l_1$.

Proof. By (8) we have

$$X - B = B\Gamma_lX - \Gamma_lXJ_lB - \Gamma_lXJ_l(B\Gamma_lX); \quad J_l(X - B) = J_l(B\Gamma_lX) = J_l(BYA^{-\frac{1}{2}})$$

and $J_l(X - B) \in \mathfrak{S}_1(\mathcal{H})$, since the product of two Hilbert-Schmidt operators is a nuclear operator. Then $P_i(X - B)P_i = P_iBYA^{-\frac{1}{2}}P_i = Z_{ii}\lambda_i^{-\frac{1}{2}}$, where $Z = BY \in \mathfrak{S}_1$ and $Z_{ii} = P_iZP_i$. The proof is complete. □

Assume that Theorem 6 holds true. Then the similarity of operators $A - B$ and $A - J_lX$ yields $\sigma(A - B) = \sigma(A - J_lX) = \sigma(A_{(l)}) \cup \big(\cup_{|i|>l} A_i\big)$, where $A_{(l)} = (A - Q_lXQ_l)|\mathcal{H}_{(l)}$, $\mathcal{H}_{(l)} = \operatorname{Ran} Q_l$ and $A_i = (P_iA - P_iX)|\mathcal{H}_i$. Since operator X is unknown and we know only the first approximation (the second approximation is in fact not used because it is too bulky), then $A_i = (P_iA - P_iB - P_i(X - B))|\mathcal{H}_i$ and the first two operators are known, while for the third operator we known only estimate (14) from Lemma 2.

The similarity of operators $A - B$ and $A - J_lX$ implies also the following representations of the projectors ($U = I + \Gamma_lX$)

$$\widetilde{Q}_l = U^{-1}Q_lU, \quad \widetilde{P}_i = U^{-1}P_iU,$$

where $\widetilde{Q}_l$ is the projector on the space $U^{-1}\mathcal{H}_{(l)}$, and $\widetilde{P}_i$ is the projector on $U^{-1}\mathcal{H}_i$. It implies immediately the formulae

$$\widetilde{Q}_l - Q_l = (\Gamma_l X Q_l - Q_l \Gamma_l X)U^{-1}; \quad \widetilde{P}_i - P_i = (\Gamma_l X P_i - P_i \Gamma_l X)U^{-1}.$$

By Lemmata 1 and 2 for $|i| > l$ we obtain

$$\|\widetilde{P}_i - P_i\|_2 \leqslant \operatorname{const} \delta_i i^{-1},$$

where $\delta_i \in l_2$ and

$$\|P_i - \widetilde{P}_i - \Gamma_l B P_i - P_i \Gamma_l B\|_2 \leqslant \operatorname{const} \delta_i' i^{-2},$$

where $\delta_i' \in l_2$.

4. Similarity transformation of operator L

We return back to operator (1), (2). We recall that as the unperturbed operator we choose operator $(Ay)(t) = -y''(t)$, while the perturbation is the operator of multiplication by matrix potential $(By)(t) = -Q(t)y(t)$. It is obvious that operator matrix consists of the entries $P_i B P_j = B_{i-j} = (b_{lk\,i-j})$, $1 \leqslant l,k \leqslant m$, $i,j \in \mathbb{Z}$ (respectively, scalar matrix reads as $(b_{lk\,i-j})$). The operators on the diagonals of matrix B_{ij} parallel to the main one are same and on the main diagonal we have blocks B_0 consisting of matrices $B_0 = (b_{lk0})$, $1 \leqslant l,k \leqslant m$, where $b_{lk0} = \int_0^1 b_{lk}(t)\,\mathrm{d}t$. Since all the functions b_{lk} belong to $L_2[0,1]$, they are represented by the Fourier series $b_{lk}(t) = \sum\limits_m b_{lkm} e^{i2\pi mt}$ and $\sum\limits_m |b_{lkm}|^2 < \infty$. We note that the convergence of Fourier series for the perturbation does not ensure the condition$B \in \mathfrak{S}_2(\mathcal{H})$. This is why we first have to make the preliminary similarity transformation of operator $A - B$ into the operator $\widetilde{A} - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{S}_2(\mathcal{H})$.

We define transformer $J_0 B$ by the formula

$$J_0 B = \sum_{n\in\mathbb{Z}} P_n B P_n,$$

where the series $\sum\limits_{n\in\mathbb{Z}} P_n B P_n$ converges and its sum is equal to B_0, i.e., transformer J_0 is well-defined. We note that $J_0 B \notin \mathfrak{S}_2(\mathcal{H})$.

We proceed to operator $\Gamma_0 B$. Employing Condition 1 of Assumption 1, first we introduce operator $\Gamma_0 B_{ij} = \Gamma_0 P_i B P_j = P_i \Gamma_0 B P_j$, $i,j \in \mathbb{Z}$, as a solution $Y_{ij} \in \operatorname{End}\mathcal{H}$ of the operator equation

$$AY_{ij} - Y_{ij}A = B_{ij} - J_0 B_{ij} \tag{15}$$

satisfying the condition $Y_{ii} = 0$. Equations (15) are solvable and each of them has solution [13], [14]. Since this equation is rewritten as

$$A_i Y_{ij} - Y_{ij} A_j = B_{ij} - J_0 B_{ij},$$

where $A_i = A|\mathcal{H}_i$, $\mathcal{H}_i = \operatorname{Ran} P_i$, $A_i P_i = \lambda_i P_i$, then ([13]–[15])

$$\|Y_{ij}\| \leqslant \frac{\operatorname{const}\|B_{ij}\|}{|\lambda_i - \lambda_j|}.$$

We recover operator $\Gamma_0 B$ by operator blocks $\Gamma_0 B_{ij}$, $i \neq j$, and let us show that $\Gamma_0 B \in \mathfrak{S}_2(\mathcal{H})$.

Indeed,

$$\|\Gamma_0 B_p\|_2^2 = \sum_{\substack{i-j=p\\ p\neq 0}} \|\Gamma_0 B_{ij}\|_2^2 = \sum_j \frac{\|B_p\|_2^2}{4\pi^2 p^2 (2\pi(p+2j)+2\theta)^2} \leqslant \frac{\|B_p\|_2^2}{4\pi^2 p^2} \sum_j \frac{1}{(2\pi(p+2j)+2\theta)^2},$$

the latter series converges and $\|\Gamma_0 B_p\|_2^2 \leqslant \text{const}\frac{\|B_p\|_2^2}{p^2}$;

$$\|\Gamma_0 B\|_2^2 = \sum_p \|\Gamma_0 B_p\|_2^2 = \text{const}\sum_p \frac{\|B_p\|_2^2}{p^2} < \infty.$$

Thus, $\Gamma_0 B \in \mathfrak{S}_2(\mathcal{H})$. It is obvious that $\Gamma_0 B \in \operatorname{End}\mathcal{H}$ and $\Gamma_0 B$ can be represented as $C_0 A^{-\beta}$, where $\beta < \frac{1}{2}$, since $\Gamma_0 B = \Gamma_0 B A^{\beta} A^{-\beta}$ and for the operator $C_0 = \Gamma_0 B A^{\beta}$, $\beta < \frac{1}{2}$ we have

$$\|\Gamma_0 B A^{\beta}\|_2^2 = \sum_{p\neq 0} \frac{\|B_p\|_2^2}{p^2} \sum_j \frac{j^{2\beta}}{(p+2j)^2}.$$

Here the internal series converges if $2\beta < 1$.

We consider operator $B\Gamma_0 B$:

$$\|B\Gamma_0 B\|_2^2 = \sum_p \Big\| \sum_k B_k (\Gamma_0 B)_{p-k} \Big\|_2^2 \leqslant \sum_p \Big(\sum_k \|B_k\|_\infty \|\Gamma_0 B_{p-k}\|_2 \Big)^2$$

and two sequences: the sequence $\xi_{1k} = \|B_k\|_\infty$, by the assumption, $\xi_{1k} \in l_2$, and the sequence $\xi_{2k} = \|\Gamma_0 B_p\|_2 \leqslant \frac{\|B_p\|_2}{p} (\sum_j \frac{1}{(p+2j)^2})^{\frac{1}{2}} \in l_1$.

The convolution of two sequences ξ_{1k} in l_2 and ξ_{2k} in l_1 is an element l_2, i.e., $\sum_p (\sum_k \xi_{1k}\xi_{2(p-k)})^2 < \infty$. Thus, $B\Gamma_0 B \in \mathfrak{S}_2(\mathcal{H})$.

We consider operator $(\Gamma_0 B) J_0 B$. By similar arguments:

$$\begin{aligned}\sum_p \|(\Gamma_0 B) J_0 B\|_2^2 \leqslant & \sum_p \Big\| \sum_{k\neq 0} (\Gamma_0 B)_k (J_0 B)_{p-k} \Big\|_2^2 = \sum_{p\neq 0} \|(\Gamma_0 B)_p B_0\|_2^2 \\ \leqslant & \|B_0\|_\infty^2 \sum_p \|(\Gamma_0 B)_p\|_2^2 < \infty,\end{aligned}$$

i.e., $(\Gamma_0 B) J_0 B \in \mathfrak{S}_2(\mathcal{H})$.

Together with operators Γ_0 and J_0 we consider also operators Γ_n and J_n defined by the identities:

$$\begin{aligned} J_n B = & J_0 B - J_0(Q_n B Q_n) + Q_n B Q_n = J_0 B - Q_n J_0 B Q_n + Q_n B Q_n \\ = & Q_n B Q_n + \sum_{|k|>n} P_k B P_k; \end{aligned} \tag{16}$$

$$\Gamma_n B = \Gamma_0 B - \Gamma_0(Q_n B Q_n) = \Gamma_0 B - Q_n \Gamma_0 B Q_n. \tag{17}$$

These identities imply that $\Gamma_n B \in \mathfrak{S}_2(\mathcal{H})$ for each n. Moreover,

$$\lim_{n\to\infty} \|\Gamma_n B\|_2^2 = \lim_{n\to\infty} \|\Gamma_0 B - \Gamma_0(Q_n B Q_n)\|_2^2 = 0,$$

and hence, we can choose sufficiently large n such that $\|\Gamma_n B\|_2 < 1$.

Lemma 3. *Operators $\Gamma_n B$, $J_n B$, B satisfy Assumption 1.*

Proof. The validity of Conditions 1 and 3 was proven above. Consider Condition 4. Since $B \in \mathfrak{L}_A(\mathcal{H})$ and

$$\|B(A-\lambda_\varepsilon I)^{-1}\| = \|BA^{-\frac{1}{2}}\| \cdot \|A^{\frac{1}{2}}(A-\lambda_\varepsilon I)^{-1}\|,$$

and the last fact can be made arbitrarily small by an appropriate choice of number λ_ε.

The inclusion $(\Gamma B)D(A) \subset D(A)$ is proven in the same way as in Theorem 5. The proof is complete. □

Theorem 7. *There exists a number n such that the operator $A-B$ is similar to the operator*
$$A - J_nB - (I + \Gamma_nB)^{-1}(B\Gamma_nB - \Gamma_nBJ_nB) = \widetilde{A} - \widetilde{B},$$
where $\widetilde{A} = A - J_0B$ and $\widetilde{B} = (I + \Gamma_nB)^{-1}(B\Gamma_nB - \Gamma_nBJ_nB) - J_0B + J_nB \in \mathfrak{S}_2(\mathcal{H})$. The operator of transforming operator $A - B$ into operator $\widetilde{A} - \widetilde{B}$ is the operator $I + \Gamma_nB$, where $\Gamma_nB \in \mathfrak{S}_2(\mathcal{H})$.

Now it is convenient to choose operator $\widetilde{A}$ as the unperturbed operator, and operator $\widetilde{B} \in \mathfrak{S}_2(\mathcal{H})$ as the perturbation. We note that J_0B, $J_nB \notin \mathfrak{S}_2(\mathcal{H})$ and $J_0B = \sum\limits_{k\in\mathbb{Z}} B_0I_k$, where the symbol I_k stands for an operator such that $P_kI_k = I_kP_k = P_k$, $I_kP_l = P_lI_k = 0$ as $l \neq k$, i.e., operator J_0B is formed by the blocks B_0 on the main diagonal, while all other elements of its operator matrix are zero. Operator $J_nB - J_0B$ is finite-dimensional and hence, it belongs also to $\mathfrak{S}_2(\mathcal{H})$, and $\mathfrak{S}_1(\mathcal{H})$. Moreover, the main diagonal of its operator matrix is zero.

We apply Theorem 6 to operator $\widetilde{A} - \widetilde{B}$ and arrive at

Theorem 8. *There exists a number $l \in \mathbb{Z}$ $(l \geqslant n)$ such that the operator $\widetilde{A} - \widetilde{B}$ is similar to the block-diagonal operator $\widetilde{A} - J_lX$, i.e.,*
$$(\widetilde{A} - \widetilde{B})(I + \Gamma_lX) = (I + \Gamma_lX)(\widetilde{A} - J_lX),$$
where $X \in \mathfrak{S}_2(\mathcal{H})$ is a solution in the space $\mathfrak{S}_2(\mathcal{H})$ of nonlinear operator equation (8) with the perturbing operator $\widetilde{B}$ and operators Γ_l and J_l defined by formulae (16) and (17).

Theorem 8 implies

Theorem 9. *There exist natural numbers n and l $(l > n)$ such that operator L defined by identities (1), (2) is similar to an operator block-diagonal w.r.t. the system of projectors P_k*
$$-\frac{d^2}{dt^2} - Q_nBQ_n - \sum_{|k|>n} P_kBP_k - Q_lXQ_l - \sum_{|k|>l} P_kXP_k, \tag{18}$$
where $X \in \mathfrak{S}_2(\mathcal{H})$ is a solution to nonlinear equation (8) with J_l and Γ_l, $X - \widetilde{B} \in \mathfrak{S}_1(\mathcal{H})$, operator $\widetilde{B}$ is from Theorem 7 and $P_kBP_k = Q_0I_k$. Operator of transforming is
$$V = I + \Gamma_nB + \Gamma_lX + \Gamma_nB\Gamma_lX = I + Y^{nl}, \tag{19}$$
where $Y^{nl} \in \mathfrak{S}_2(\mathcal{H})$.

Proof of Theorem 1. Theorem 9 yields that
$$\sigma(L) = \sigma(A - J_nB - J_lX) = \widetilde{\sigma}_{(l)} \bigcup \Big(\bigcup_{|k|>l} \widetilde{\sigma}_k \Big),$$
where $\widetilde{\sigma}_k = \sigma(AP_k - P_kB|_{\mathcal{H}_k} - P_kX|_{\mathcal{H}_k})$, $AP_k = \lambda_kP_k$ and $\sigma(P_kBP_k) = \sigma(B_0)$ are known.

We represent operator $\widetilde{B}$ as
$$\widetilde{B} = B\Gamma_nB - \Gamma_nBJ_nB + \sum_{k=1}^{\infty}(-1)^k\Gamma_nB(B\Gamma_nB - \Gamma_nBJ_nB) = B\Gamma_nB - \Gamma_nBJ_nB + T_1,$$
where $T_1 \in \mathfrak{S}_1(\mathcal{H})$. Hence,
$$J_l\widetilde{B} = \widetilde{J}_l(B\Gamma_nB) + J_lT_1 = \sum_{|k|>l} P_kB\Gamma_nBP_k + Q_lB\Gamma_nBQ_l + J_lT_1,$$
$$\|P_kB\Gamma_nBP_k\|_2 = \Big\| \sum_{j\neq k} \frac{B_{kj}B_{jk}}{\lambda_k - \lambda_j} \Big\|_2 \leqslant \Big(\sum_{j\neq k} \|B_{kj}\|_2^2 \Big)^{\frac{1}{2}} \Big(\sum_{j\neq k} \frac{\|B_{jk}\|_2^2}{(\lambda_k - \lambda_j)^2} \Big)^{\frac{1}{2}}$$
$$\leqslant \text{const}\,|k|^{-1}, \quad |k| > l.$$

Thus, if eigenvalues μ_i, $i = 1, 2, \dots, m$, of matrix Q_0 are simple, formula (4) holds true, while if they are semi-simple, formula (6) is valid.

The similarity of operators L and $A - \widetilde{B}$ imply that spectral projectors P_k of operator A and $\widetilde{P}_k$ of operator L are similar and $\widetilde{P}_k$ can be represented as

$$\widetilde{P}_k = (I + Y^{nl})P_k(I + Y^{nl})^{-1},$$

or

$$\widetilde{P}_k - P_k = (Y^{nl}P_k - P_kY^{nl})(I + Y^{nl})^{-1}.$$

It is obvious that $\widetilde{P}_k - P_k \in \mathfrak{S}_2(\mathcal{H})$. We estimate quantites $Y^{nl}P_k$, P_kY^{nl} in the norm of space $\mathfrak{S}_2(\mathcal{H})$. Taking into account that $\Gamma_l X = YA^{-\frac{1}{2}}$, $\Gamma_n B\Gamma_l X = \Gamma_n BYA^{-\frac{1}{2}} = ZA^{-\frac{1}{2}}$, where $Y \in \mathfrak{S}_2(\mathcal{H})$, $Z \in \mathfrak{S}_1(\mathcal{H})$, we have

$$\|(\Gamma_n B + \Gamma_l X + \Gamma_n B\Gamma_l X)P_k\|_2 \leqslant \left(\sum_{|l| \neq k} \frac{\|B_{lk}\|_2^2}{(\lambda_l - \lambda_k)^2}\right)^{\frac{1}{2}} + \mathcal{O}\left(\frac{1}{|k|}\right). \tag{20}$$

Since $B = (b_{ij})$, $i = 1, 2, \dots, m$, and $b_{ij} \in L_2[0, 1]$, then $\sup \sum_{l} \|B_{lk}\|_2^2 < \infty$ is bounded in k and therefore, $\|Y^{nl}P_k\|_2 = \mathcal{O}(\frac{1}{|k|})$. Then

$$\|\widetilde{P}_k - P_k\|_2 = \mathcal{O}\left(\frac{1}{|k|}\right),$$

i.e., formula (7) is true.

The similarity of operator $L = A - B$ defined by formulae (1), (2) to operator (18) implies the identity (in the case of simple eigenvalues of matrix Q_0)

$$\widetilde{e}_{k,j} = Ve_{k,j} = e_{k,j} + Y^{nl}e_{k,j}, \quad |k| > l,$$

where operator V is defined by formula (19). In order to prove identity (5), it remains to estimate quantity $Y^{nl}e_{k,j}$, and thus, estimate (5) is implied by the estimate

$$\|\Gamma_n Be_{k,j}\|_2 = \left(\sum_{i \neq k} \frac{\|B_{ik}\|_2^2}{(\lambda_i - \lambda_k)^2}\right)^{\frac{1}{2}} \leqslant \frac{1}{|k|}\left(\sum_{i \neq k} \|B_{ik}\|_2^2\right)^{\frac{1}{2}} = \text{const}\, |k|^{-1},$$

and the representation of $\Gamma_l X$ and $\Gamma_n B\Gamma_l X$, $X \in \mathfrak{S}_2(\mathcal{H})$ as $YA^{-\frac{1}{2}}$, $ZA^{-\frac{1}{2}}$, $Y \in \mathfrak{S}_2(\mathcal{H})$, $Z \in \mathfrak{S}_1(\mathcal{H})$. The proof is complete. □

We recall

Definition 3 ([2]). *Let $C : D(C) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator whose spectrum can be represented as the union*

$$\sigma(C) = \bigcup_{k \in \mathbb{J}} \sigma_k, \quad \mathbb{J} = \{\mathbb{N}, \mathbb{Z}\}, \tag{21}$$

of mutually disjoint sets σ_k, $k \in \mathbb{J}$. Let P_k be the Riesz projector constructed by set σ_k. Operator C is called spectral w.r.t. the resolution (21) (or generalized spectral) if the series $\sum_{k \in \mathbb{J}} P_k x$ converges absolutely for each vector x in $\mathcal{H}$.

Theorem 1 implies immediately Corollary 1.

Proof of Theorem 2. Under the assumptions of Theorem 1 the formula

$$\widetilde{P}(\Omega) - P(\Omega) = (Y^{nl}P(\Omega) - P(\Omega)Y^{nl})(I + Y^{nl})^{-1}$$

holds true, and hence,

$$\|\widetilde{P}(\Omega) - P(\Omega)\|_2 \leqslant \text{const}\,(\|Y^{nl}P(\Omega)\|_2 + \|P(\Omega)Y^{nl}\|_2).$$

Consider the former of the norms. Since

$$\|Y^{nl}P(\Omega)\|_2 = \|\Gamma_n BP(\Omega) + \Gamma_l XP(\Omega) + \Gamma_n B\Gamma_l XP(\Omega)\|_2 \leqslant \|\Gamma_n BP(\Omega)\|_2 + \|YA^{-\frac{1}{2}}P(\Omega) + ZA^{-\frac{1}{2}}P(\Omega)\|_2 \leqslant \|\Gamma_n BP(\Omega)\|_2 + \mathcal{O}(|k_0|^{-\frac{1}{2}}),$$

we need to estimate only the quantity $\|\Gamma_n BP(\Omega)\|$.

Reproducing the arguments for (20), we obtain $\|\Gamma_n BP(\Omega)\|_2 \leqslant \mathcal{O}(k_0^{-\frac{1}{2}})$. The estimate for the second norm is similar. The proof is complete. □

In conclusion we mention that in work [1] the estimates for the spectral projectors (Theorems 1, 2) were not considered.

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