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ASYMPTOTIC APPROACH TO THE PERFECT CUBOID PROBLEM

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Abstract. The problem of perfect cuboids is one of the old unsolved problems in number theory. By means of various methods it can be reduced to finding a solution of some single Diophantine equation of high degree obeying certain restrictions in the form of inequalities. Each such Diophantine equation is called a characteristic equation of a perfect cuboid. In this paper we present the results obtained by applying asymptotic metods to one of the characteristic equations of a perfect cuboid in the case of the second cuboid conjecture. This results shrink the domain of the integer parameters of the considered characteristic equation and thus make more effective the computer search of perfect cuboids based on this equation.

Keywords: perfect cuboid, Diophantine equations, asymptotic methods.

1. Introduction

A perfect cuboid is a rectangular parallelepiped whose edges, the diagonals on the faces, as well as the spatial diagonal are of integer length. The search of perfect cuboids has a long history [1–50]. The problem was first mentioned in 1719, but none of cuboids is found yet. The number 1719 by itself is probably not so impressive. It is interesting to compare it with some prominent historical dates: Foundation of Saint-Petersburg (1703), Battle of Poltava (1709), Rebellions of Bashkirs (1735–1740 and 1755), the Celsius scale (1742), births of Mozart (1756) and Beethoven (1770), Pugachev's Rebellion (1773), United States Declaration of Independence (1776), World First Steamship (1783), Great French Revolution (1789), World First Steamship Locomotive (1804), Napoleon Invasion of Russia (1812), Decembrist revolt (1825), opening of the first railway in Russia (1837). In mathematics these are the births of Euler (1707), Lagrange (1736), Fourier (1768), Gauss (1777), Cauchy (1789), Lobachevskii (1792), Abel (1802), Galois (1811). As we see, a lot of events happened in the world which changed essentially our life , but the problem of perfect cuboid was and still remains unsolved. Here we consider one of the approaches to this difficult problem that probably will change the situation.

In paper [51] the problem of constructing a perfect cuboid was reduced to a Diophantine equation of 12-th degree in variables a, b, u and t:

$$\begin{gathered}
u^4\,a^4\,b^4+6\,a^4\,u^2\,b^4\,t^2-2\,u^4\,a^4\,b^2\,t^2-2\,u^4\,a^2\,b^4\,t^2+4\,u^2\,b^4\,a^2\,t^4+\\
+4\,a^4\,u^2\,b^2\,t^4-12\,u^4\,a^2\,b^2\,t^4+u^4\,a^4\,t^4+u^4\,b^4\,t^4+a^4\,b^4\,t^4+\\
+6\,a^4\,u^2\,t^6+6\,u^2\,b^4\,t^6-8\,a^2\,b^2\,u^2\,t^6-2\,u^4\,a^2\,t^6-2\,u^4\,b^2\,t^6-\\
-2\,a^4\,b^2\,t^6-2\,b^4\,a^2\,t^6+u^4\,t^8+b^4\,t^8+a^4\,t^8+4\,a^2\,u^2\,t^8+\\
+4\,b^2\,u^2\,t^8-12\,b^2\,a^2\,t^8+6\,u^2\,t^{10}-2\,a^2\,t^{10}-2\,b^2\,t^{10}+t^{12}=0.
\end{gathered}\tag{1.1}$$

The precise result of paper [51] is formulated in the next theorem.

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Theorem 1.1. *A perfect cuboid does exist if and only if Diophantine equation* (1.1) *has a solution such that* a, b, u, t *are positive numbers satisfying the inequalities* $a < t$, $b < t$, $u < t$, $(a+t)\,(b+t) > 2\,t^2$.

The analysis of equation (1.1) in [52] showed that there are two cases where the polynomial in the left hand side of equation (1.1) is reducible and we can remove inessential factors producing no perfect cuboids. In these cases the degree of equation (1.1) reduces to 8-th and 10-th, respectively. For these cases in [52] the first and the second cuboid conjectures were formulated. For the general case, where no reduction of equation (1.1) does happen, the third cuboid conjecture was formulated.

The first cuboid conjecture $a = b \neq u$. This case turned out to be the simplest one. Despite the first cuboid conjecture remained unproved, in paper [53] it was established that in this case there are no perfect cuboids.

The cases of the second and third cuboid conjectures turned out to be more complicated. For these cases in [54] and [55] only certain structural theorems on integer solutions of equation (1.1) were obtained while the question on whether such solutions do exist and produce perfect cuboids remained open. The second and third conjectures on cuboid also remain neither proved no disproved.

In the present work we consider only the case of the second cuboid conjecture and we summarize the results obtained by the author for this case in eletronic publications [56–59]. On May 19, 2015, the most part of these results was presented at the city seminar on differential equations of mathematical physics named after A.M. Il'in. The author is grateful to the head of the seminar L.A. Kalyakin for the opportuinity to present a talk. The author is grateful to the other head of the seminr V.Yu. Novokshenov and to all participants for the attention and valuable remarks during the talk. The author is also greatefull to B.I. Suleimanov who mentioned the works by A.D. Bruno on Newton polygons and their application in asymptotic analysis.

Papers [60–72] made in the time period between papers [51–55] and [56–59] develop an absolutely different approach based on the discrete S_3-symmetry in the perfect cuboid equations and employing the technique of multi-symmetric polynomials. This approach is planned to be reviewed in a separated publication.

2. The case of second conjecture of cuboid

The case of the second cuboid conjecture arises when parameters a, b, u in equation (1.1) are related to each other by one of the following two identities:

$$b\,u = a^2. \tag{2.1}$$

The first of identities (2.1) is resolved by substituting

$$a = p\,q, \quad b = p^2, \quad u = q^2. \tag{2.2}$$

The second identity in (2.1) is resolved by substituting

$$a = p^2, \quad b = p\,q, \quad u = q^2. \tag{2.3}$$

Applying each of substitutions (2.2) and (2.3) to equation (1.1), we remove an inessential factor and the degree of the equation reduces from 12-th to 10-th. As a result of each of these substitutions the same equation of 10-th degree w.r.t. p, q and t arises:

$$\begin{aligned} &t^{10} + (2\,q^2 + p^2)\,(3\,q^2 - 2\,p^2)\,t^8 + (q^8 + 10\,p^2\,q^6 + 4\,p^4\,q^4 \\ &-\ 14\,p^6\,q^2 + p^8)\,t^6 - p^2\,q^2\,(q^8 - 14\,p^2\,q^6 + 4\,p^4\,q^4 + 10\,p^6\,q^2 \\ &+ p^8)\,t^4 - p^6\,q^6\,(q^2 + 2\,p^2)\,(3\,p^2 - 2\,q^2)\,t^2 - q^{10}\,p^{10} = 0. \end{aligned} \tag{2.4}$$

Theorem 1.1 implies the following theorem.

Theorem 2.1. *In the case of second cuboid conjecture, a perfect cuboid does exist if and only if Diophantine equation* (2.4) *has a solution such that* p, q, t *are positive numbers satisfying the inequalities* $p\,q < t$, $p^2 < t$, $q^2 < t$, $(p\,q + t)\,(p^2 + t) > 2\,t^2$.

The second conjecture itself is formulated as follows (see [52]).

Conjecture 2.1 (Second cuboid conjecture)**.** *For each two coprime positive integers* $p \neq q$, *the polynomial of* 10*-th degree in the left hand side of equation* (2.4) *is irreducible over the field of raional numbers* $\mathbb{Q}$.

Conjecture 2.1 is provided here just to inform the reader and it is not considered in what follows. The main efforts are aimed at the study of equation (2.4).

3. Motivation of the asymptotic approach

A computer search of perfect cuboids via equation (2.4) is performed iteratively. For instance, like this: for each fixed p we iterate over q and for each q we solve equation (2.4) w.r.t. t. Theoretically, for the complete search, for each p we should iterate over all integer values of q from 1 to $+\infty$. In practice, one has to replace $+\infty$ by a fixed large number $q_{\max}(p)$. Equation (2.4) is nice since the choice of the upper bound $q_{\max}(p)$ can be made reasonable and equivalent to the infinite iteration. Roots of polynomial equations of two variables w.r.t. one of the variables demonstrate a rather simple and easily calculable asymptotics as the second variable tend to infinity. Using such asymptotics, in certain cases one can show that for q greater than a certain value $q_{\max}(p)$ equation (2.4) has no integer roots or its integer roots do not satisfy inequalities in Theorem 2.1.

4. Asymptotics of roots t_i as $q \to +\infty$

We denote by $Q_{pq}(t)$ the polynomial in the left hand side of equation (2.4). By this notation we choose variable t as the main one, while variables p and q are regarded as parameters. Polynomial $Q_{pq}(t)$ is even w.r.t. t. This is why together with each root t, it has the opposite root $-t$. The condition

$$\begin{cases} t > 0, & \text{if } t \text{ is a real root,} \\ \operatorname{Im}(t) > 0, & \text{if } t \text{ is a complex root,} \end{cases} \tag{4.1}$$

selects a group of five roots t_1, t_2, t_3, t_4, t_5 of polynomial $Q_{pq}(t)$. The other five roots are obtained by the changing their signs:

$$t_6 = -t_1, \quad t_7 = -t_2, \quad t_8 = -t_3, \quad t_9 = -t_4, \quad t_{10} = -t_5.$$

The asymptotics of roots of polynomial equations are power [73]. For a fixed $p = \text{const}$ they are written as the series

$$t_i(p, q) = C_i\,q^{\alpha_i}\left(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\right) \quad \text{при} \quad q \to +\infty. \tag{4.2}$$

Coefficients C_i in (4.2) must be non-zero $C_i \neq 0$. These coefficients and the exponents α_i can be calculated graphically by means of Newton polygon related with the polynomial $Q_{pq}(t)$ (Fig. 4.1). We write polynomial $Q_{pq}(t)$ formally as

$$Q_{pq}(t) = \sum_{m=0}^{10}\sum_{r=0}^{10} A_{m\,r}(p)\,q^r\,t^m. \tag{4.3}$$

The number of terms in formal sum (4.3) is 121. The actual number of terms in polynomial $Q_{pq}(t)$ is much less. Drawing the Newton polygon, one chooses non-zero terms in (4.3) and they are marked as the points with coordinates (m, r) on the coordinate plane. Since coordinates (m, r) are integer, these points are the sites of the integer lattice (Fig. 4.1).

Definition 4.1. *For each polynomial of two variables $P(t, q)$, the convex hull of all sites of the integer lattice associated with its monomials is called the Newton polygon of this polynomial $P(t, q)$.*

The boundary of the Newton polygon shown in Figure 4.1 consists of two parts, the upper one and the lower one. The upper part of the boundary contains the sites $A_{0\,10}$, $A_{2\,10}$ $A_{4\,10}$, $A_{6\,8}$ $A_{8\,4}$ and $A_{10\,0}$. The associated coefficients of polynomial $Q_{pq}(t)$ are given by the formulae

$$\begin{aligned} &A_{0\,10} = -p^{10}, && A_{2\,10} = 2\,p^6, && A_{4\,10} = -p^2,\\ &A_{6\,8} = 1, && A_{8\,4} = 6, && A_{10\,0} = 1. \end{aligned} \tag{4.4}$$

The following theorem expresses a well-known fact. Its proof can be found in [56].

Theorem 4.1. *Exponents α_i in asymptotic expansions* (4.2) *are calculated by the formula $\alpha_i = -k$, where k is the slope of the segments of the polyline being the upper boundary of the Newton polygon in Figure 4.1.*

In our case Theorem 4.1 gives the following possible values for exponents α_i in asymptotic expansions (4.2):

$$\alpha_i = 0, \quad \alpha_i = 1, \quad \alpha_i = 2. \tag{4.5}$$

The number of the roots satisfying condition (4.1) exceeds three. This is why some of roots t_1, t_2, t_3, t_4, t_5 have the same growth rate as $q \to +\infty$.

Besides α_i, each of roots t_1, t_2, t_3, t_4, t_5 is characterized by its coefficient C_i in expansions (4.2). These coefficients are also calculated by means of the Newton polygon.

The case $\alpha_i = 0$ in (4.5) corresponds to the horizontal segment on the upper boundary of the Newton polygon. On this segment there three sites $A_{4\,10}$, $A_{2\,10}$, and $A_{0\,10}$. This is why the corresponding equation for C_i reads as

$$A_{4\,10}\,C_i^{\,4} + A_{2\,10}\,C_i^{\,2} + A_{0\,10} = 0. \tag{4.6}$$

In view of (4.4), equation (4.6) has two real roots

$$C_i = p^2, \quad C_i = -p^2, \tag{4.7}$$

each being of multiplicity 2. Condition (4.1) excludes the root $C_i = -p^2$ in (4.7), keeping only one double root. The associated expansion is

$$t_i(p, q) = p^2\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr). \tag{4.8}$$

Case $\alpha_i = 1$ in (4.5) corresponds to the short sloping segment on the uppoer boundary of Newton polygon. This segment contains two sites $A_{4\,10}$ and $A_{6\,8}$. The associated coefficient C_i in (4.2) is determined by the equation

$$A_{6\,8}\,C_i^{\,6} + A_{4\,10}\,C_i^{\,4} = 0. \tag{4.9}$$

In view of (4.4) and $C_i \neq 0$, equation (4.9) has two simple real roots:

$$C_i = p, \quad C_i = -p. \tag{4.10}$$

Condition (4.1) excludes the root $C_i = -p$ in (4.10) keeping only one root $C_i = p$. The expansion associated with this root reads as

$$t_i(p, q) = p\,q\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr). \tag{4.11}$$

Case $\alpha_i = 2$ in (4.5) corresponds to a long sloping segment on the upper boundary of Newton polygon. This segment contains three sites $A_{6\,8}$, $A_{8\,4}$ and $A_{10\,0}$. This is why the corresponding equation for C_i reads as

$$A_{10\,0}\,C_i^{\,10} + A_{8\,4}\,C_i^{\,8} + A_{6\,8}\,C_i^{\,6} = 0. \tag{4.12}$$

In view of (4.4) and $C_i \neq 0$, equation (4.12) has four complex roots:

$$C_i = (\sqrt{2} + 1)\,\mathrm{i}, \qquad C_i = (\sqrt{2} - 1)\,\mathrm{i}, \tag{4.13}$$

$$C_i = -(\sqrt{2} + 1)\,\mathrm{i}, \quad C_i = -(\sqrt{2} - 1)\,\mathrm{i}. \tag{4.14}$$

Here $\mathrm{i} = \sqrt{-1}$. Roots (4.14) are excluded by condition (4.1). Only two roots (4.13) are kept and they give the asymptotic expansions:

$$t_i(p, q) = (\sqrt{2} + 1)\,\mathrm{i}\,q^2\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr), \quad t_i(p, q) = (\sqrt{2} - 1)\,\mathrm{i}\,q^2\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr). \tag{4.15}$$

Formulae (4.8), (4.11), (4.15) are summarized by the following theorem.

Theorem 4.2. *For sufficiently large q, i.e., as $q > q_{\,min}$, equation* (2.4) *of* 10*-th degree has five simple roots satisfying condition* (4.1). *Three of them t_1, t_2, t_3 are real. Their asymptotics as $q \to +\infty$ are given by the formulae*

$$t_1 \sim p^2, \quad t_2 \sim p^2, \quad t_3 \sim p\,q. \tag{4.16}$$

Other two roots of equation (2.4) *are complex-valued. The asymptotics of these roots as $q \to +\infty$ are described by the formulae*

$$t_4 \sim (\sqrt{2} + 1)\,\mathrm{i}\,q^2, \quad t_5 \sim (\sqrt{2} - 1)\,\mathrm{i}\,q^2. \tag{4.17}$$

5. Asymptotic estimate for real roots

Asymptotic expansions (4.8), (4.11), (4.15) and formulae (4.16), (4.17) describe the limiting behavior of the roots to equation (2.4) as $q \to +\infty$. They give no precise information on location of roots for finite q. In order to have such information, we need asymptotic estimates, namely, the estimates for the error terms for finite sums of asymptotic series.

In accordance with formulae (4.16), roots t_1 and t_2 do not grow as $q \to +\infty$. This is why their asymptotic series (4.8) can be truncated right after the leading term and we can write formula (4.8) as the following sums:

$$t_1 = p^2 + R_1(p,q), \quad t_2 = p^2 + R_2(p,q). \tag{5.1}$$

Our next aim is to obtain the estimates

$$|R_i(p,q)| < \frac{C(p)}{q}, \text{ where } i = 1,\,2. \tag{5.2}$$

To achieve this aim, we substitute

$$t = p^2 + \frac{c}{q} \tag{5.3}$$

into equation (2.4). In the obtained equation we make one more substitution

$$q = \frac{1}{z}. \tag{5.4}$$

After two substitutions (5.3) and (5.4) and after removing the denominators equation (2.4) becomes a new polynomial equation in new variables c and z. A feature of this equation is that it is written as

$$16\,p^{12} + f(c,p,z) = 4\,p^6\,c^2. \tag{5.5}$$

Here $f(c,p,z)$ is a polynomial given by an explicit formula. But the formula for $f(c,p,z)$ is bulky. It is written in a machine-readable formate and is provided in the ancillary file `strategy_formulas.txt` attached to electronic publication [56]. The link for downloading this file is provided on the site http://arXiv.org/abs/1504.07161.

Polynomial $f(c,p,z)$ vanishes at $z = 0$. This is why its values are sufficiently small for small z. Let $q \geqslant 59\,p$ and parameter c range the interval

$$-\,5\,p^3 < c < 0. \tag{5.6}$$

By $q \geqslant 59\,p$ and (5.4) we obtain the estimate $|z| \leqslant 1/59\,p^{-1}$. Employing this estimate and inequalities (5.6), by straightforward calculations we can get the following estimate:

$$|f(c,p,z)| < 15\,p^{12}. \tag{5.7}$$

For fixed p and z estimate (5.7) means that the left hand side of equation (5.5) is a continuous function of c ranging from p^{12} to $31\,p^{12}$ as c ranges in interval (5.6). The right hand side of equation (5.5) is also a continuous function of c. It decreases monotonically from $100\,p^{12}$ to 0 on interval (5.6). Hence, interval (5.6) contains at least one root of equation (5.5).

Parameter c is related with variable t by formula (5.3). Inequalities (5.5) for c imply the following inequalities for t:

$$p^2 - \frac{5\,p^3}{q} < t < p^2. \tag{5.8}$$

Inequalities (5.8) and the above arguments give the following theorem.

Theorem 5.1. *For each* $q \geqslant 59\,p$ *equation* (2.4) *has at least one real root satisfying inequalities* (5.8).

The above arguments can be reproduced for the case when parameter c ranges the interval mirror-symmetric to interval (5.6):

$$0 < c < 5\,p^3. \tag{5.9}$$

In this case (5.9) and (5.3) imply the inequalities

$$p^2 < t < p^2 + \frac{5\,p^3}{q} \tag{5.10}$$

for variable t and we obtain the following theorem.

Theorem 5.2. *For each $q \geqslant 59\,p$ equation* (2.4) *has at least one real root satisfying inequalities* (5.10)*.*

Theorems 5.1 and 5.2 provide asymptotic estimates (5.2) with $C(p) = 5\,p^3$ for error terms $R_1(p,q)$ and $R_2(p,q)$ in formulae (5.1).

In accordance with formulae (4.16), root t_3 grows as $q \to +\infty$. We write the asymptotic formula with an error term for this root:

$$t_3 = p\,q - \frac{16\,p^3}{q} + R_3(p,q) \tag{5.11}$$

and seek an estimate for error term $R_3(p,q)$ as

$$|R_3(p,q)| < \frac{C(p)}{q^2}. \tag{5.12}$$

We then write inequalities

$$p\,q - \frac{16\,p^3}{q} - \frac{5\,p^4}{q^2} < t < p\,q - \frac{16\,p^3}{q} + \frac{5\,p^4}{q^2} \tag{5.13}$$

and formulate the following theorem.

Theorem 5.3. *For each $q \geqslant 59\,p$ equation* (2.4) *has at least one real root satisfying inequalities* (5.13)*.*

The proof of Theorem 5.3 can be found in [56]. It is almost the same as the proof of Theorems 5.1 and 5.2 given above.

Inequalities (5.13) prove estimate (5.12) with $C(p) = 5\,p^4$ for error term $R_3(p,q)$ in asymptotic formula (5.11).

6. Asymptotic estimates for complex roots

The complexity of roots t_4 and t_5 is implied by formulae (4.15). For complex root t_4 we write the asymptotic formula

$$t_4 = (\sqrt{2}+1)\,\mathrm{i}\,q^2 + (\sqrt{2}-2)\,\mathrm{i}\,p^2 + R_4(p,q), \quad \text{where} \quad \mathrm{i} = \sqrt{-1}. \tag{6.1}$$

For error term $R_4(p,q)$ we seek an estimate

$$|R_4(p,q)| < \frac{C(p)}{q}. \tag{6.2}$$

We write the inequalities

$$(\sqrt{2}+1)\,q^2 + (\sqrt{2}-2)\,p^2 - \frac{5\,p^3}{q} < \operatorname{Im} t < (\sqrt{2}+1)\,q^2 + (\sqrt{2}-2)\,p^2 + \frac{5\,p^3}{q} \tag{6.3}$$

and formulate the following theorem.

Theorem 6.1. *For each $q \geqslant 59\,p$ equation* (2.4) *has at least one pure imaginary root satisfying inequalities* (6.3)*.*

Inequalities (6.3) prove estimate (6.2) with $C(p) = 5\,p^3$ for the error term in asymptotic formula (6.1).

Theorem 6.1 is similar to Theorems 5.1, 5.2, 5.3. Its proof is provided in [56].

Complex root t_5 of equation (2.4) is similar to root t_4. For complex root t_5 we write asymptotic formula

$$t_4 = (\sqrt{2} - 1)\,\mathrm{i}\,q^2 + (\sqrt{2} + 2)\,\mathrm{i}\,p^2 + R_5(p, q), \quad \text{where} \quad \mathrm{i} = \sqrt{-1}. \tag{6.4}$$

For error term $R_5(p, q)$ we seek estimate

$$|R_5(p, q)| < \frac{C(p)}{q}. \tag{6.5}$$

We write the inequalities

$$(\sqrt{2} - 1)\,q^2 + (\sqrt{2} + 2)\,p^2 - \frac{5\,p^3}{q} < \operatorname{Im} t < (\sqrt{2} - 1)\,q^2 + (\sqrt{2} + 2)\,p^2 + \frac{5\,p^3}{q} \tag{6.6}$$

and formulate the following theorem.

Theorem 6.2. *For each $q \geqslant 59\,p$ equation* (2.4) *has at least one pure imaginary root satisfying inequalities* (6.6).

Inequalities (6.6) prove estimate (6.5) with $C(p) = 5\,p^3$ for error term in asymptotic formula (6.4). The proofs of inequality (6.6) and Theorem 6.2 can be found in work [56].

7. Asymptotic intervals

Inequalities (5.8), (5.10), (5.13), (6.3), (6.6) define five open intervals. The first three of them are on the real axis. The other two are on the imaginary axis. Several pure technical results on asymptotic intervals (5.8), (5.10), (5.13), (6.3), (6.6) were proven in [56]. The essence of these technical results is that as $q \geqslant 59\,p$, the asymptotic intervals are mutually disjoint. The first three of them are on the positive part of the real axis, and the other two are on the positive part of the imaginary axis. Along with Theorems 5.1, 5.2, 5.3, 6.1, 6.2 it leads us to the following result.

Theorem 7.1. *For $q \geqslant 59\,p$, five roots t_1, t_2, t_3, t_4, t_5 of equation* (2.4) *satisfying conditions* (4.1) *are simple. They are located in five mutually disjoint intervals* (5.8), (5.10), (5.13), (6.3), (6.6) *so that each interval contains exactly one root.*

8. Integer points of asymptotic intervals

In view of formulae (5.8), (5.10), (5.13), (6.3), (6.6) it is easy to see that the lengths of asymptotic intervals tend to zero as $q \to +\infty$. It naturally lowers the chance of integer points to be in these intervals. The following theorems were proven in work [56].

Theorem 8.1. *For $q \geqslant 59\,p$ and $q > 5\,p^3$ asymptotic intervals* (5.8) *and* (5.10) *contain no integer points.*

Theorem 8.2. *For $q \geqslant 59\,p$ and $q^2 > 10\,p^4$ asymptotic interval* (5.13) *contains at most one integer point.*

Theorem 8.3. *For $q \geqslant 59\,p$ and $q \geqslant 16\,p^3 + 5\,p/16$ asymptotic interval* (5.13) *contains no integer points.*

We note that in accordance with Theorem 2.1, not each integer solution to equation (2.4) produces a perfect cuboid. In addition, the inequalities

$$p\,q < t, \quad p^2 < t, \quad q^2 < t, \quad (p\,q + t)\,(p^2 + t) > 2\,t^2. \tag{8.1}$$

should be satisfied.

Belonging of the roots to equation (2.4) to asymptotic intervals (5.8), (5.10), (5.13), (6.3), (6.6) as $q \geqslant 59\,p$ produces new inequalities for them. By comparing inequalities (8.1) with inequalities (5.8), (5.10), (5.13), it was shown in [56] that for $q \geqslant 59\,p$ the integer points in the real asymptotic intervals did not satisfy inequalities (8.1). It implied the main result of work [56].

Theorem 8.4. *For $q \geqslant 59\,p$ Diophantine equation* (2.4) *has no solutions producing perfect cuboids.*

For each fixed p Theorem 8.4 provides the upper bound $q_{\max}(p) = 59\,p$, up to which one should perform a numerical search of perfect cuboids in the case of second cuboid conjecture.

9. Reverse asymptotics

In [57], parameters p and q of equation (2.4) switch their roles. In this paper parameter q is kept fixed, while parameter p tends to infinity. The asymptotics obtained under these conditions were called reverse. A substantial part of the results of [56] was extendedto the case of reverse asymptotics. Namely, the asymptotic expansions with asymptotic estimates were found, as well as asymptotic intervals for roots t_1, t_2, t_3, t_4, t_5 of equation (2.4) satisfying conditions (4.1). These asymptotic intervals are defined by the following inequalities:

$$p\,q+\frac{16\,q^3}{p}-\frac{5\,q^4}{p^2}<t<p\,q+\frac{16\,q^3}{p}+\frac{5\,q^4}{p^2}, \tag{9.1}$$

$$p^2-2\,q\,p-2\,q^2-\frac{9\,q^3}{p}<t<p^2-2\,q\,p-2\,q^2, \tag{9.2}$$

$$p^2+2\,q\,p-2\,q^2<t<p^2+2\,q\,p-2\,q^2+\frac{9\,q^3}{p}, \tag{9.3}$$

$$(\sqrt{2}+1)\,q^2-\frac{5\,q^3}{p^2}<\operatorname{Im} t<(\sqrt{2}+1)\,q^2+\frac{5\,q^3}{p^2}, \tag{9.4}$$

$$(\sqrt{2}-1)\,q^2-\frac{5\,q^3}{p^2}<\operatorname{Im} t<(\sqrt{2}-1)\,q^2+\frac{5\,q^3}{p^2}. \tag{9.5}$$

An analogue of Theorem 7.1 is the following theorem proven in [57].

Theorem 9.1. *For $p \geqslant 59\,q$ five roots t_1, t_2, t_3, t_4, t_5 to equation* (2.4) *satisfying conditions* (4.1) *are simple. They are located in five mutually disjoint intervals* (9.1), (9.2), (9.3), (9.4), (9.5) *such that each interval contains one root.*

It is easy to see that the lengths of asymptotic intervals tend to zero as $p \to +\infty$. Employing this fact, in [57] the following three theorems were formulated.

Theorem 9.2. *For $p \geqslant 59\,q$ and $p > 9\,q^3$ asymptotic intervals* (9.2) *and* (9.3) *contain no integer points.*

Theorem 9.3. *For $p \geqslant 59\,q$ and $p^2 > 10\,q^4$ asymptotic interval* (9.1) *contains at most one integer point.*

Theorem 9.4. *For $p \geqslant 59\,q$ and $p \geqslant 16\,q^3 + 5\,q/16$ asymptotic interval* (9.1) *contains no integer points.*

Theorems 9.2, 9.3, 9.4 are analogues of Theorems 8.1, 8.2, 8.3. Their proofs reproduce almost literally the proof of Theorems 8.1, 8.2, 8.3 provided in [56]. This is why it was not given in [57] and is not provided here.

Despite of essential likeness, the parallel between papers [56] and [57] is not complete. We did not succeed to prove an analogue of Theorem 8.4 in [57]. Instead of this, a weaker result is obtained. It is provided in the following theorem.

Theorem 9.5. *Under the condition* $p \geqslant 59\,q$ *and the condition* $p > 9\,q^3$ *Diophanite equation* (2.4) *has no solutions producing perfect cuboids.*

Summarizing the results of Theorems 8.4 and 9.5, in [57] three regions in the positive quadrant of the coordinate $p\,q$-plane were defined:

1. **Linear region** defined by linear inequalities
$$\frac{q}{59} < p, \quad p < 59\,q; \tag{9.6}$$
2. **Nonlinear region** defined by nonlinear inequalities
$$59\,q \leqslant p, \quad p \leqslant 9\,q^3; \tag{9.7}$$
3. **No cuboids region**, which includes all points outside the first two regions.

10. Asymptotics as both p and q simultaneously tend to infinity

In [58] and [59], the asymptotics for the roots to equation (2.4) were considered as both parameters p and q simultaneously tend to infinity. The main result of [58] is based on the asymptotics where

$$p - q = \text{const}\,.$$

This result can be formulated as the following theorem.

Theorem 10.1. *If positive integer parameters* p *and* q *of equation* (2.4) *satisfy the inequalities*

$$q - \frac{q}{97} \leqslant p, \quad p \leqslant q + \min\Bigl(\frac{q}{97}, \sqrt[3]{\frac{q}{74}}\Bigr), \tag{10.1}$$

then equation (2.4) *has no solutions producing perfect cuboids.*

The main result of [59] is based on the asymptotics as

$$p - B\,q^3 = \text{const} \quad \text{and} \quad B = 1,\,2,\,\ldots,\,9.$$

Here, instead of (10.1), the inequalities

$$B\,q^3 - \frac{q^3}{3600^3} \leqslant p \leqslant B\,q^3 + \frac{q^3}{3600^3}, \tag{10.2}$$
$$B\,q^3 - 2\,q < p < B\,q^3 + 2\,q, \tag{10.3}$$

arise and the following theorems are formulated.

Theorem 10.2. *For each* $B = 1,\,2,\,\ldots,\,9$ *except* $B = 5$*, if inequalities* (10.2) *are satisfied, then equation* (2.4) *has no solutions producing perfect cuboids.*

Theorem 10.3. *As* $B = 5$*, if both inequalities* (10.2) *and* (10.3) *are satisfied, then Dipohantine equation* (2.4) *has no solutions producing perfect cuboids.*

The results stated in Theorem 10.1 and Theorems 10.2, 10.3 are of a restricted nature. By means of (10.1), (10.2) and (10.3) they specify small subdomains in the linear and nonlinear regions defined by inequalities (9.6) and (9.7) above and ensure the absence of perfect cuboids in the specified subdomains.

11. Further perspectives

Until the problem of perfect cuboids is solved, nobody can say in which way its solution will be found. Concerning the asymptotic approach to the problem in the case of the second cuboid conjecture, the author believes that it is promising to consider invertible transformations of parameters

$$\tilde{p} = \tilde{p}(p, q), \quad \tilde{q} = \tilde{q}(p, q),$$

see [74], and then to study the asymptotics as $\tilde{q} \to +\infty$ and $\tilde{p} = \text{const}$. It is also possible to study the asymptotics as p and q tend independently to infinity by employing the techniques of Newton polyhedra developed in numerous works by Bruno, Arnold, Bernstein, Varchenko, Volevich, Gindikin, Kushnerenko, Khovansky, Soleev and other. This approach is more complicated since it will take some time to learn the above mentioned technique.

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