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BOUNDARY VALUE PROBLEM FOR PARTIAL DIFFERENTIAL EQUATION WITH FRACTIONAL RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. For a differential equation involving a fractional order diffusion equations, we study a non-local problem in an unbounded domain, where the boundary condition involves a linear combination of generalized operators of a fractional integro-differentiation.

For various values of the parameters of these operators by Tricomi method we prove the uniqueness of solution to the considered problem. The existence of solution is obtained in the closed form as a solution to the appropriate equation with fractional derivative of various order.

Keywords: boundary value problem, generalized operator of fractional integro-differentiation, Wright's function, fractional order differential equation.

Mathematics Subject Classification: 35M10

1. Formulation of problem

We consider the second order partial differential equation

$$\begin{cases} u_{xx} - D^{\alpha}_{0+,y}u = 0, & (y > 0,\, 0 < \alpha < 1), \\ (-y)^m u_{xx} - u_{yy} = 0, & (m > 0,\, y < 0), \end{cases} \tag{1}$$

where $D^{\alpha}_{0+,y}$ is the Riemann-Liouville fractional derivative of order α of function $u(x,y)$ w.r.t. the second variable [1]

$$\left(D^{\alpha}_{0+,y}u\right)(x,y) = \frac{\partial}{\partial y}\frac{1}{\Gamma(1-\alpha)}\int\limits_0^y \frac{u(x,t)dt}{(y-t)^{\alpha}} \quad (0 < \alpha < 1,\, y > 0)$$

in a domain Ω being the union of the upper half-plane $\Omega^+ = \{(x,y) : -\infty < x < +\infty, y > 0\}$ and domain Ω^- lying in the lower half-plane $(y < 0)$ and bounded by the characteristics

$$AC:\, \xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0, \quad BC:\, \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$$

and the segment $[0,1]$ in the axis $y = 0$.

We introduce the notation. Let $I = (0,1)$ be the unit interval in the axis $y = 0$, $\Theta_0(x) = \frac{x}{2} - i\left[\frac{(m+2)x}{4}\right]^{\frac{2}{m+2}}$ is the intersection of the characteristics of equation (1) leaving points $(x,0)$ $(x \in I)$ with characteristics AC.

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By $I_{0+}^{\alpha,\beta,\eta}$ we denote the operator of generalized fractional integro-differentiating with Gauss hypergeometric function $F(a,b;c;z)$ introduced in [2] (see also [1], [3], [4], [5]). For real α, β, η and $x>0$ it reads as

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)=\begin{cases}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int\limits_0^x(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)\,dt & (\alpha>0),\\ \frac{d^n}{dx^n}\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) & (\alpha\leqslant 0,\ n=[-\alpha]+1).\end{cases} \tag{2}$$

In particular,

$$\left(I_{0+}^{0,0,\eta}f\right)(x)=f(x),\ \left(I_{0+}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{0+}^{\alpha}f\right)(x),\ \left(I_{0+}^{-\alpha,\alpha,\eta}f\right)(x)=\left(D_{0+}^{\alpha}f\right)(x), \tag{3}$$

where I_{0+}^{α} and D_{0+}^{α} are the operators of fractional integration and differentiation of order $\alpha>0$ [1].

We study a boundary value problem for equation (1): find a solution $u(x,y)$ to equation (1) in domain Ω satisfying the conditions:

$$y^{1-\alpha}u|_{y=0}=0\quad(-\infty<x\leqslant 0,\quad 1\leqslant x<\infty), \tag{4}$$

$$A\left(I_{0+}^{a,b,\beta-1-a}u[\Theta_0(t)]\right)(x)+B\left(I_{0+}^{a+1,b-1-\beta,\beta-1-a}u_y(t,0)\right)(x)=g(x),\quad(x\in I) \tag{5}$$

and the conjugation conditions

$$\lim_{y\to 0+}y^{1-\alpha}u(x,y)=\lim_{y\to 0-}u(x,y),\quad(x\in\overline{I}), \tag{6}$$

$$\lim_{y\to 0+}y^{1-\alpha}(y^{1-\alpha}u(x,y))_y=\lim_{y\to 0-}u_y(x,y),\quad(x\in I). \tag{7}$$

Here $\beta=\frac{m}{2m+4}$, a and b are real numbers, at that, $a>\max\{-\beta,\beta-1\}$, A and B are real constants of opposite signs, $g(x)$ is a given function such that $g(x)\in C^1(\overline{I})\cap C^2(I)$.

We shall seek solution $u(x,y)$ of the formulated problem in the class of twice differentiable functions in domain Ω such that

$$\begin{aligned}&u(x,y)\quad\text{tends to zero as}\quad(x^2+y^2)\to\infty,\\&y^{1-\alpha}u(x,y)\in C(\overline{\Omega^+}),\quad u(x,y)\in C(\overline{\Omega^-}),\\&y^{1-\alpha}(y^{1-\alpha}u)_y\in C(\Omega^+\cup\{(x,y):\ 0<x<1,y=0\}),\\&u_{xx}\in C(\Omega^+\cup\Omega^-),\quad u_{yy}\in C(\Omega^-).\end{aligned}$$

We note that in papers [6], [7] we studied nonlocal boundary value problems for equation (1). This work is a continuation of the mentioned study and is its generalization.

2. Uniqueness of solution to problem

Suppose that there exists a solution to the formulated problem.

We introduce the notations

$$\lim_{y\to 0+}y^{1-\alpha}u(x,y)=\tau_1(x),\qquad\lim_{y\to 0-}u(x,y)=\tau_2(x), \tag{8}$$

$$\lim_{y\to 0+}y^{1-\alpha}(y^{1-\alpha}u(x,y))_y=\nu_1(x),\qquad\lim_{y\to 0-}u_y(x,y)=\nu_2(x),\quad(x\in I). \tag{9}$$

It is known (see, for instance, [6]) that solutions to equation (1) in the half-plane $y>0$ satisfying condition (4) and

$$\lim_{y\to 0+}y^{1-\alpha}u(x,y)=\tau_1(x),\quad(x\in\overline{I}), \tag{10}$$

are given by the formula

$$u(x,y)=\int_0^1 G(x,y,t)\tau_1(t)dt, \tag{11}$$

where

$$G(x,y,t)=\frac{\Gamma(\alpha)}{2}\,y^{\frac{\alpha}{2}-1}e^{1,\frac{\alpha}{2}}_{1,\frac{\alpha}{2}}\left(-|x-t|y^{-\frac{\alpha}{2}}\right)$$

and

$$e^{\mu,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\mu)\Gamma(\delta-\beta n)},\quad \alpha>\beta,\alpha>0$$

is a Wright type function [8]. It is also known [9] that a functional relation between $\tau_1(x)$ and $\nu_1(x)$ moved from domain Ω^+ on the line $y=0$ reads as

$$\nu_1(x)=\frac{1}{\Gamma(1+\alpha)}\tau_1''(x). \tag{12}$$

Let us find a functional relation between $\tau_2(x)$ and $\nu_2(x)$ moved on the line $y=0$ from hyperbolic part Ω^- of domain Ω.

Employing the solution to the Cauchy problem for equation (1) as $y<0$, in work [10] there was found $u[\Theta_0(x)]$. It reads as

$$u[\Theta_0(x)]=\gamma_1\Gamma(\beta)\left(I^{\beta,0,\beta-1}_{0+}\tau_2(t)\right)(x)-\gamma_2\Gamma(1-\beta)\left(I^{1-\beta,2\beta-1,\beta-1}_{0+}\nu_2(t)\right)(x), \tag{13}$$

where

$$\gamma_1=\frac{\Gamma(2\beta)}{\Gamma^2(\beta)},\quad \gamma_2=\frac{1}{2}\left(\frac{4}{m+2}\right)^{2\beta}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}.$$

Substituting (13) into (5), taking into consideration (8)–(9) and applying the relation [1]

$$\left(I^{\alpha,\beta,\eta}_{0+}I^{\gamma,\delta,\alpha+\eta}_{0+}f\right)(x)=\left(I^{\alpha+\gamma,\beta+\delta,\eta}_{0+}f\right)(x)\quad(\gamma>0), \tag{14}$$

we obtain

$$\begin{aligned}k_1\left(I^{a+\beta,b,\beta-1-a}_{0+}\tau_2(t)\right)(x)+k_2\left(I^{a+1-\beta,b+2\beta-1,\beta-a-1}_{0+}\nu_2(t)\right)(x)\\+B\left(I^{a+1,b-1+\beta,\beta-a-1}_{0+}\nu_2(t)\right)(x)=g(x),\end{aligned} \tag{15}$$

where

$$k_1=A\gamma_1\Gamma(\beta),\quad k_2=-A\gamma_2\Gamma(1-\beta).$$

We apply the operator $I^{-a-\beta,-b,2\beta-1}_{0+}$ to the both sides of relation (15).

Straightforward calculating with employing formulae (14) and (3) show that

$$\tau_2(x)=-k_3\left(I^{1-2\beta}_{0+}\nu_2(t)\right)(x)-k_4\left(I^{1-\beta}_{0+}\nu_2(t)\right)(x)+\frac{1}{k_1}\left(I^{-a-\beta,-b,2\beta-1}_{0+}g(t)\right)(x), \tag{16}$$

where

$$k_3=-\frac{\gamma_2(1-\beta)}{\gamma_1\Gamma(\beta)},\quad k_4=\frac{B}{A\gamma_1\Gamma(\beta)}. \tag{17}$$

Let us estimate the integral

$$I=\int_0^1\tau_2(x)\nu_2(x)dx.$$

By conjugation conditions (6), (7) and relation (12), we have

$$I=\frac{1}{\Gamma(1+\alpha)}\int_0^1\tau_1(x)\tau_1''(x)dx.$$

Integrating by parts and taking into account that $\tau_1(0) = \tau_1(1) = 0$, we obtain

$$I = -\frac{1}{\Gamma(1+\alpha)} \int_0^1 [\tau_1'(x)]^2 dx \leqslant 0. \tag{18}$$

Let us find the lower estimate for integral I.

As $g(x) = 0$, identity (16) becomes

$$\begin{aligned}\tau_2(x) =& - k_3 \left(I_{0+}^{1-2\beta} \nu_2(t) \right)(x) - k_4 \left(I_{0+}^{1-\beta} \nu_2(t) \right)(x) \\ =& - \frac{k_3}{\Gamma(1-2\beta)} \int_0^x \nu_2(t)(x-t)^{-2\beta} dt - \frac{k_4}{\Gamma(1-\beta)} \int_0^x \nu_2(t)(x-t)^{-\beta} dt,\end{aligned}$$

and therefore,

$$I = -\frac{k_3}{\Gamma(1-2\beta)} \int_0^1 \nu_2(x)dx \int_0^x (x-t)^{-2\beta} \nu_2(t)dt - \frac{k_4}{\Gamma(1-\beta)} \int_0^1 \nu_2(x)dx \int_0^x (x-t)^{-\beta} \nu_2(t)dt.$$

We employ the known formula for Gamma-function $\Gamma(\mu)$ [11]

$$\int_0^\infty s^{\mu-1} \cos(ks)ds = \frac{\Gamma(\mu)}{k^\mu} \cos\left(\frac{\mu\pi}{2}\right) \quad (k > 0, 0 < \mu < 1).$$

Letting $k = |x-t|$, $\mu = 2\beta$, we obtain

$$|x-t|^{-2\beta} = \frac{1}{\Gamma(2\beta)\cos(\pi\beta)} \int_0^\infty s^{2\beta-1} \cos(s|x-t|)ds, \left(0 < \beta < \frac{1}{2}\right).$$

As $k = |x-t|$, $\mu = \beta$, we have

$$|x-t|^{-\beta} = \frac{1}{\Gamma(\beta)\cos\left(\frac{\pi\beta}{2}\right)} \int_0^\infty s^{\beta-1} \cos(s|x-t|)ds.$$

Applying these formulae and Dirichlet formula for switching the order of integration in an iterated integral, we arrive at the identity

$$\begin{aligned}I =& - \frac{k_3 \sin(\pi\beta)}{\pi} \int_0^\infty s^{2\beta-1} \left[\left(\int_0^1 \nu_2(x)\cos(sx)dx \right)^2 + \left(\int_0^1 \nu_2(x)\sin(sx)dx \right)^2 \right] ds \\ &- \frac{k_4 \sin\left(\frac{\pi\beta}{2}\right)}{\pi} \int_0^\infty s^{\beta-1} \left[\left(\int_0^1 \nu_2(x)\cos(sx)dx \right)^2 + \left(\int_0^1 \nu_2(x)\sin(sx)dx \right)^2 \right] ds \geqslant 0.\end{aligned} \tag{19}$$

It follows from (18) and (19) that $I = 0$ and hence, in accordance with (18),

$$\int_0^1 [\tau_1(x)]^2 dx = 0.$$

By the identities $\tau_1(0) = \tau_1(1) = 0$ it implies $\tau_1(x) = 0$ for each $x \in \overline{I}$.

In accordance with formula (11),

$$u(x,y) = \int_0^1 G(x,y,t)\tau_1(t)dt,$$

and it allows to state that $u(x,y) \equiv 0$ in domain $\overline{\Omega^+}$.

By conjugation conditions (8), $\tau_2(x) = \tau_1(x)$, and hence, $\tau_2(x) = 0$. By (9) and (12) it yields $\nu_2(x) = 0$. And since $u(x,y) \equiv 0$ also in domain Ω^- as the solution to the Cauchy problem with zero data, it proves the uniqueness of a solution to the original problem.

3. Existence of solution to problem

According to (11), to prove the existence of solution to the studied problem it is sufficient to find $\nu_1(x)$.

We differentiate twice (16) w.r.t. x:

$$\frac{d^2}{dx^2}\tau_2(x) = -k_3\frac{d^2}{dx^2}\left(I_{0+}^{1-2\beta}\nu_2(t)\right)(x) - k_4\frac{d^2}{dx^2}\left(I_{0+}^{1-\beta}\nu_2(t)\right)(x) + \\ +\frac{1}{k_1}\frac{d^2}{dx^2}\left(I_{0+}^{-a-\beta,-b,2\beta-1}g(t)\right)(x)$$

or, letting $\tau_1(x) = \tau_2(x) = \tau(x)$, $\nu_1(x) = \nu_2(x) = \nu(x)$,

$$\left(D_{0+}^{1+2\beta}\nu\right)(x) - \lambda\left(D_{0+}^{1+\beta}\nu\right)(x) - \mu\nu(x) = g_1(x), \tag{20}$$

where

$$\lambda = -\frac{B}{k_2} = \frac{B}{A\gamma_2\Gamma(1-\beta)}, \quad \mu = -\frac{\Gamma(1+\alpha)}{k_3} = \frac{\gamma_1\Gamma(1+\alpha)\Gamma(\beta)}{\gamma_2\Gamma(1-\beta)},$$
$$g_1(x) = -\frac{1}{A\gamma_2\Gamma(1-\beta)}\left(I_{0+}^{-a-\beta-2,2-b,1+2\beta}g(t)\right)(x).$$

In monograph [12] there was considered the equation with fractional derivatives

$$\left(D_{0+}^{\alpha}y\right)(x) - \lambda\left(D_{0+}^{\beta}y\right)(x) - \mu y(x) = f(x), \tag{21}$$

where $x > 0$, $\alpha > \beta > 0$, $\lambda, \mu \in R$, $f(x)$ is defined on $R_+ = [0,\infty)$. It solution was written as follows:

$$y(x) = \int_0^x (x-t)^{\alpha-1}G_{\alpha,\beta,\lambda,\mu}(x-t)f(t)dt.$$

Here

$$G_{\alpha,\beta,\lambda,\mu}(z) = \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\,z^{\alpha n}\,{}_1\Psi_1\left[\begin{array}{c}(n+1,1)\\(\alpha n+\alpha,\alpha-\beta)\end{array}\middle|\lambda z^{\alpha-\beta}\right],$$
$${}_p\Psi_q(z) = \sum_{n=0}^{\infty}c_k\,z^k, \quad c_k = \frac{\Pi_{i=1}^p\Gamma(a_i+\alpha_i k)}{\Pi_{j=1}^q\Gamma(b_j+\beta_j k)}\frac{1}{k!} \quad (k \in N_0 = \{0,1,\ldots\}),$$

$z, a_i, b_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$, $i = \overline{1,p}$, $j = \overline{1,q}$.

Famous mathematicians Fox C. [13], [14] and Wright E.M. [15], [16], [17] devoted their works to these functions.

Equation (20) is a particular case of equation (21), and this is why its solution is given by the formula

$$\nu(x) = \int_0^x (x-t)^{2\beta} G_{1+2\beta,1+\beta,\lambda,\mu}(x-t) g_1(t)dt,$$

$$G_{1+2\beta,1+\beta,\lambda,\mu}(x-t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} (x-t)^{(1+2\beta)n} {}_1\Psi_1 \left[\begin{array}{c} (n+1,1) \\ ((1+2\beta)n+1+2\beta,\beta) \end{array} \middle| \lambda(x-t)^{\beta} \right],$$

that completes the proof of the existence of solution to the original problem.

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