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ON AN INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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Abstract. In this paper, the direct and inverse problems for Sturm-Liouville operator with discontinuous coefficient are studied. The spectral properties of the Sturm-Liouville problem with discontinuous coefficient such as the orthogonality of its eigenfunctions and simplicity of its eigenvalues are investigated. Asymptotic formulas for eigenvalues and eigenfunctions of this problem are examined. The resolvent operator is constructed and the expansion formula with respect to eigenfunctions is obtained. It is shown that eigenfunctions of this problem are in the form of a complete system. The Weyl solution and Weyl function are defined. Uniqueness theorems for the solution of the inverse problem according to Weyl function and spectral date are proved.

Keywords: Sturm-Liouville operator, expansion formula, inverse problem, Weyl function.

Mathematics Subject Classification: 34A55, 34B24, 47E05

1. Introduction

In non-homogeneous environment, vibration, diffusion and other physical problems are described by differential equations with discontinuous coefficient [1]-[8]. While solving these problems, the spectral properties of Sturm - Liouville problem with discontinuous coefficient should be analyzed [7]-[21].

We consider the differential equation

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \leqslant x \leqslant \pi \tag{1.1}$$

with the boundary conditions

$$y'(0) = 0, \quad y(\pi) = 0 \tag{1.2}$$

where λ is a complex spectral parameter. We suppose that $q \in L_2(0, \pi)$ is a real-valued function, ρ is a piecewise continuous function:

$$\rho(x) = \begin{cases} 1 & 0 \leqslant x < a \\ \alpha^2 & a \leqslant x \leqslant \pi. \end{cases} \tag{1.3}$$

As $\rho(x) \equiv 1$, the same problems were studied in [4], [22]-[24]. In general case, problem (1.1), (1.2) is solved by considering it in the intervals $[0, a)$ and $[a, \pi]$. But in this study, problem (1.1), (1.2) was reduced to an equivalent integral equation in $[0, \pi]$ interval. For the special solution of problem, the integral representation in this interval was used. There were shown the simplicity and reality of eigenvalues and the orthogonality of the eigenfunctions associated

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with different eigenvalues and the asymptotic formulas for the eigenvalues and eigenfunctions were obtained. The definition of the system of eigenfunctions is given in the weighted space $L_{2,\rho}(0,\pi)$ with the weight ρ. The expansion formula for eigenfunctions and Parseval identity were obtained. In the last part of the study, the inverse problem boundary value problem for (1.1), (1.2) was studied for Weyl function, the uniqueness of solution of inverse problem for the spectral data formed by the eigenvalues and normalizing numbers was shown.

Here we deal with boundary value problem (1.1), (1.2). Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of boundary value problem (1.1), (1.2) satisfying the initial conditions

$$\varphi(0,\lambda)=1, \varphi'(0,\lambda)=0 \tag{1.4}$$

and

$$\psi(\pi,\lambda)=0, \psi'(\pi,\lambda)=1. \tag{1.5}$$

Denote

$$\Delta(\lambda)=W[\varphi(x,\lambda),\psi(x,\lambda)]=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda). \tag{1.6}$$

Function $\Delta(\lambda)$ is called the characteristic function of problem (1.1), (1.2). Substituting $x=0$ and $x=\pi$ into (1.6), we get

$$\Delta(\lambda)=\varphi(\pi,\lambda)=\psi'(0,\lambda). \tag{1.7}$$

Lemma 1. *The eigenfunctions $y_1(x,\lambda_1)$ and $y_2(x,\lambda_2)$ corresponding to different eigenvalues $\lambda_1\neq\lambda_2$ are orthogonal.*

Proof. Since $y_1(x,\lambda_1)$ and $y_2(x,\lambda_2)$ are eigenfunctions of problem (1.1), (1.2), we get

$$-y_1''(x,\lambda_1)+q(x)y_1(x,\lambda_1)=\lambda_1^2\rho(x)y_1(x,\lambda_1),$$

$$-y_2''(x,\lambda_2)+q(x)y_2(x,\lambda_2)=\lambda_2^2\rho(x)y_2(x,\lambda_2).$$

Multiplying these identities by $y_2(x,\lambda_2\ )$ and $-y_1(x,\lambda_1)$, respectively, and summing up, we obtain

$$\frac{d}{dx}\{<y_2(x,\lambda_2),y_1(x,\lambda_1)>\}=(\lambda_1^2-\lambda_2^2)\rho(x)y_1(x,\lambda_1)y_2(x,\lambda_2).$$

Integrating from 0 to π and using the condition (1.2), we have

$$(\lambda_1^2-\lambda_2^2)\int_0^\pi \rho(x)y_1(x,\lambda_1)y_2(x,\lambda_2)dx=0.$$

Since $\lambda_1\neq\lambda_2$,

$$\int_0^\pi \rho(x)y_1(x,\lambda_1)y_2(x,\lambda_2)dx=0.$$

□

Corollary 1. *The eigenvalues of boundary value problem (1.1),(1.2) are real.*

Lemma 2. *The zeros λ_n of characteristic function coincide with the eigenvalues of boundary value problem (1.1), (1.2). The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence β_n such that*

$$\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n),\ \beta_n\neq 0. \tag{1.8}$$

Proof. 1) Let λ_0 be zero of $\Delta(\lambda)$. Then, because of (1.6), $\psi(x,\lambda_0)=\beta_0\varphi(x,\lambda_0)$ and the function $\psi(x,\lambda_0)$ and $\varphi(x,\lambda_0)$ satisfy boundary condition (1.2). Thus, λ_0 is an eigenvalue and $\psi(x,\lambda_0)$, $\varphi(x,\lambda_0)$ are associated eigenfunctions.

2) Let λ_0 be an eigenvalue of problem (1.1), (1.2) and let $y_0(x)$ be an associated eigenfunction. Then, $y_0(x)$ satisfies boundary condition (1.2). Clearly, $y_0(x)\neq 0$. Without loss of generality, we let $y_0(0)=1$. Then $y_0'(0)=0$ and therefore, $y_0(x)\equiv\varphi(x,\lambda)$. Hence, by (1.7), $\Delta_0(\lambda)=0$. We have proved that for each eigenvalue there exists only one eigenfunction. □

Definition 1. *The normalizing number of boundary value problem (1.1), (1.2) is described as*

$$\alpha_n := \int_0^\pi \rho(x)\varphi^2(x,\lambda_n)dx.$$

Lemma 3. *The eigenvalues of boundary value problem (1.1), (1.2) are simple and*

$$\dot{\Delta}(\lambda)=2\lambda_n\alpha_n\beta_n. \tag{1.9}$$

Proof. Since $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ are the solutions of this problem, the identities

$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n^2\rho(x)\varphi(x,\lambda_n),$$
$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda^2\rho(x)\psi(x,\lambda)$$

hold true. Multiplying them by $\psi(x,\lambda\ )$ and $-\varphi(x,\lambda_n)$, respectively, and summing up, we get

$$\frac{d}{dx}\{<\varphi(x,\lambda_n),\psi(x,\lambda)>\}=(\lambda_n^2-\lambda^2)\rho(x)\varphi(x,\lambda_n)\psi(x,\lambda).$$

Integrating from 0 to π and using the condition (1.2), we obtain:

$$\int_0^\pi \rho(x)\varphi(x,\lambda_n)\psi(x,\lambda)=\frac{\Delta(\lambda_n)-\Delta(\lambda)}{\lambda_n^2-\lambda^2}.$$

Since $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$ as $\lambda\to\lambda_n$, by Lemma 3, we obtain

$$\dot{\Delta}(\lambda_n)=2\lambda_n\alpha_n\beta_n$$

where $\beta_n=\psi(0,\lambda_n)$. Thus, it follows that $\dot{\Delta}(\lambda_n)\neq 0$. □

2. On the Eigenvalues of Problem (1.1), (1.2) as $q(x)\equiv 0$

We denote by $\varphi_0(x,\lambda)$ the solution to the equation $-y''=\lambda^2\rho(x)y$ satisfying condition (1.4). It reads as:

$$\varphi_0(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^+(x)+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^-(x) \tag{2.1}$$

where $\mu^\pm(x)=\pm x\sqrt{\rho(x)}+a(1\mp\sqrt{\rho(x)})$.

We see that if $(\lambda_n^0)^2$ are the eigenvalues of problem (1.1), (1.2) as $q(x)\equiv 0$, then λ_n^0 can be found via the equation $\varphi_0(\pi,\lambda)=0$, that is, by the equation

$$\Delta_0(\lambda)=\frac{1}{2}(1+\frac{1}{\alpha})\cos\lambda\mu^+(\pi)+\frac{1}{2}(1-\frac{1}{\alpha})\cos\lambda\mu^-(\pi)=0$$

$$\cos\lambda\mu^+(\pi)+\frac{\alpha-1}{\alpha+1}\cos\lambda\mu^-(\pi)=0. \tag{2.2}$$

It follows from (2.2) that

$$\lambda_n^0=\frac{\pi}{\mu^+(\pi)}\left(n-\frac{1}{2}\right)+h_n \tag{2.3}$$

where

$$\sup_n |h_n| < \infty.$$

Lemma 4. *The roots of function $\Delta_0(\lambda)$ are isolated, i.e.,*

$$\inf_{n\neq k} \left|\lambda_n^0 - \lambda_k^0\right| = \gamma > 0.$$

Proof. We argue by the contradiction. Assume the contrary, then there are two sequences $\left\{\lambda_{k,1}^0\right\}$, $\left\{\lambda_{k,2}^0\right\}$ of zeros of function $\Delta_0\left(\lambda\right)$ such that

$$\lambda_{k,1}^0 \neq \lambda_{k,2}^0, \quad \lambda_{k,1}^0 \to \infty, \quad \lambda_{k,2}^0 \to \infty, \quad \lim_{k\to\infty}\left(\lambda_{k,1}^0 - \lambda_{k,2}^0\right) = 0.$$

By Lemma 1, functions $\varphi_0\left(x, \lambda_{k,1}^0\right)$ and $\varphi_0\left(x, \lambda_{k,2}^0\right)$ are orthogonal:

$$\begin{aligned}
0 = &\int_0^\pi \rho\left(x\right)\varphi_0\left(x,\lambda_{k,1}^0\right)\varphi_0\left(x,\lambda_{k,2}^0\right)dx = \\
= &\int_0^\pi \rho\left(x\right)\varphi_0\left(x,\lambda_{k,1}^0\right)\left[\varphi_0\left(x,\lambda_{k,2}^0\right) - \varphi_0\left(x,\lambda_{k,1}^0\right)\right]dx + \\
&+\int_0^\pi \rho\left(x\right)\varphi_0^2\left(x,\lambda_{k,1}^0\right)dx = \\
= &I_k + \int_0^\pi \rho\left(x\right)\varphi_0^2\left(x,\lambda_{k,1}^0\right)dx \geqslant I_k + \int_0^a \rho\left(x\right)\varphi_0^2\left(x,\lambda_{k,1}^0\right)dx = \\
= &I_k + \int_0^a \cos^2\left(\lambda_{k,1}^0 x\right)dx = I_k + \frac{a}{2} - \frac{\sin\left(2a\lambda_{k,1}^0\right)}{4\lambda_{k,1}^0},
\end{aligned}$$

where

$$I_k = \int_0^\pi \rho\left(x\right)\varphi_0\left(x,\lambda_{k,1}^0\right)\left[\varphi_0\left(x,\lambda_{k,2}^0\right) - \varphi_0\left(x,\lambda_{k,1}^0\right)\right]dx.$$

Let us show that

$$\lim_{k\to\infty} I_k = 0.$$

Indeed, by (2.1) and the estimate

$$\left|\cos\lambda_{k,1}^0 x - \cos\lambda_{k,2}^0 x\right| \leqslant C\left|\lambda_{k,1}^0 - \lambda_{k,2}^0\right| \quad (C>0)$$

we conclude that

$$\left|\varphi_0\left(x,\lambda_{k,1}^0\right) - \varphi_0\left(x,\lambda_{k,2}^0\right)\right| \leqslant C\left|\lambda_{k,1}^0 - \lambda_{k,2}^0\right| \quad (C>0).$$

Thus, $\lim_{k\to\infty}\left(\varphi_0\left(x,\lambda_{k,1}^0\right) - \varphi_0\left(x,\lambda_{k,2}^0\right)\right) = 0$ is valid uniformly for $x \in [0,\pi]$. Passing to the limit in the inequality $0 \geqslant I_k + \frac{a}{2} - \frac{\sin\left(2a\lambda_{k,1}^0\right)}{4\lambda_{k,1}^0}$ as $k \to \infty$, we have $0 \geqslant \frac{a}{2}$. We arrive at the contradiction. □

3. Asymptotic Formulas of Eigenfunctions and Eigenvalues

Using representation for solution $e(x,\lambda)$ to equation (1.1) satisfying the initial conditions $e(0,\lambda) = 1$, $e'(0,\lambda) = i\lambda$, we obtain the following integral representation for solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^{\mu^+(x)} A(x,t)\cos\lambda t dt$$

where $K(x,.) \in L_1(-\mu^+(x),\mu^+(x))$ and $A(x,t) = K(x,t) - K(x,-t)$. Kernel $A(x,t)$ processes the following properties:

i) $A(\pi,\mu^+(\pi)) = \frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\rho(t)}}\left(1 + \frac{1}{\sqrt{\rho(t)}}\right)q(t)dt,$

ii) $A(\pi, \mu^-(\pi)+0) - A(\pi, \mu^-(\pi)-0) = \frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\rho(t)}}\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)dt.$

Lemma 5. *As $|\lambda|\to\infty$, the asymptotic formulas*

$$\begin{aligned}\varphi(x,\lambda) &= \varphi_0(x,\lambda) + O\left(\frac{e^{|\mathrm{Im}\lambda|\mu^+(x)}}{|\lambda|}\right) = O\left(e^{|\mathrm{Im}\lambda|\mu^+(x)}\right)\\ \varphi'(x,\lambda) &= \varphi_0'(x,\lambda) + O\left(e^{|\mathrm{Im}\lambda|\mu^+(x)}\right) = O\left(|\lambda|\, e^{|\mathrm{Im}\lambda|\mu^+(x)}\right)\end{aligned} \tag{3.1}$$

$$\begin{aligned}\psi(x,\lambda) &= \psi_0(x,\lambda) + O\left(\frac{e^{|\mathrm{Im}\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}}{|\lambda|^2}\right) = O\left(\frac{e^{|\mathrm{Im}\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}}{|\lambda|}\right)\\ \psi'(x,\lambda) &= \psi_0'(x,\lambda) + O\left(\frac{e^{|\mathrm{Im}\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}}{|\lambda|}\right) = O\left(e^{|\mathrm{Im}\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}\right)\end{aligned} \tag{3.2}$$

hold true uniformly with respect to $x\in[0,\pi]$.

Proof. The standard method of variations of an arbitrary constants leads us to the following integral equation for the solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x g(x,t;\lambda)q(t)\varphi(t,\lambda)dt \tag{3.3}$$

where

$$\begin{aligned}g(x,t;\lambda) =& \frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\sqrt{\rho(t)}}\right)\frac{\sin\lambda(\mu^+(x)-\mu^+(t))}{\lambda}+\\ &+\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\sqrt{\rho(t)}}\right)\frac{\sin\lambda(\mu^-(x)-\mu^+(t))}{\lambda}\end{aligned} \tag{3.4}$$

and $\varphi_0(x,\lambda)$ is the solution of equation (1.1) as $q(x)\equiv 0$ satisfying the conditions (1.4).

Denote

$$\sigma(\lambda) = \max_{0\leqslant x\leqslant\pi}\left(|\varphi(x,\lambda)|\, e^{-|\mathrm{Im}\lambda|\mu^+(\pi)}\right).$$

Since

$$\left|\sin\lambda\mu^+(x)\right| \leqslant e^{|\mathrm{Im}\lambda|\mu^+(x)}, \quad \left|\cos\lambda\mu^+(x)\right| \leqslant e^{|\mathrm{Im}\lambda|\mu^+(x)}$$

and

$$|g(x,t;\lambda)| \leqslant \frac{C}{|\lambda|}e^{|\mathrm{Im}\lambda|\left(\mu^+(x)-\mu^+(t)\right)},$$

by (3.3) we get that for $|\lambda|\geqslant 1$ and $x\in[0,\pi]$,

$$|\varphi(x,\lambda)|\, e^{-|\mathrm{Im}\lambda|\mu^+(\pi)} \leqslant C_1 + \frac{C_2}{|\lambda|}\sigma(\lambda)\int_0^x |q(t)|\, dt,$$

and therefore

$$\sigma(\lambda) \leqslant C_1 + \frac{\widetilde{C}_2}{|\lambda|}\sigma(\lambda).$$

For sufficiently large $|\lambda|$ it yields $\sigma(\lambda)=O(1)$, i.e. $\varphi(x,\lambda)=O\left(e^{|\mathrm{Im}\lambda|\mu^+(x)}\right)$. By this identity and $|g(x,t;\lambda)|\leqslant\frac{C}{|\lambda|}e^{|\mathrm{Im}\lambda|\left(\mu^+(x)-\mu^+(t)\right)}$ we conclude that

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + O\left(\frac{e^{|\mathrm{Im}\lambda|\mu^+(x)}}{|\lambda|}\right).$$

Differentiating (3.3), we calculate:

$$\varphi'(x,\lambda)=\varphi_0'(x,\lambda)+g(x,x;\lambda)q(x)\varphi(x,\lambda)+\int_0^x g_x'(x,t;\lambda)q(t)\varphi(t,\lambda)dt \tag{3.5}$$

where by (3.4), $g(x,x;\lambda)=0$ and

$$\begin{aligned} g_x'(x,t;\lambda)=&\sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\sqrt{\rho(t)}}\right)\cos\lambda(\mu^+(x)-\mu^+(t))+\\ &+\sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\sqrt{\rho(t)}}\right)\cos\lambda(\mu^-(x)-\mu^+(t)). \end{aligned} \tag{3.6}$$

Substituting the identity $\varphi(x,\lambda)=O\left(e^{|\mathrm{Im}\lambda|\mu^+(x)}\right)$ into the right hand side of (3.5), we arrive at (3.1). In the same way one can get (3.2). □

Theorem 1. *Boundary value problem (1.1),(1.2) has a countable set of simple eigenvalues* $\{\lambda_n^2\}_{n\geqslant1}$*:*

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{k_n}{n},\ (\lambda_n\geqslant 0) \tag{3.7}$$

where λ_n^0 *are the zeros of the function*

$$\Delta_0(\lambda)=\frac{1}{2}\left(1+\frac{1}{\alpha}\right)\cos\lambda\mu^+(\pi)+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\cos\lambda\mu^-(\pi), \tag{3.8}$$

$\{\lambda_n^0\}^2$ *are the eigenvalues of problem (1.1),(1.2) as* $q(x)\equiv 0$,

$$d_n=\frac{h^+\sin\lambda_n^0\mu^+(\pi)+h^-\sin\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(1+\frac{1}{\alpha})\mu^+(\pi)\sin\lambda_n^0\mu^+(\pi)+\frac{1}{2}(1-\frac{1}{\alpha})\mu^-(\pi)\sin\lambda_n^0\mu^-(\pi)} \tag{3.9}$$

is a bounded sequence and $k_n\in l_2$.

Proof. Since $\Delta(\lambda)=\varphi(\pi,\lambda)$ is the characteristic function of boundary value problem (1.1), (1.2), we have

$$\Delta(\lambda)=\Delta_0(\lambda)+\int_0^\pi g(\pi,t;\lambda)q(t)\varphi(t,\lambda)dt. \tag{3.10}$$

By (3.1) we obtain

$$\Delta(\lambda)=\Delta_0(\lambda)+h^+\frac{\sin\lambda\mu^+(\pi)}{\lambda}+h^-\frac{\sin\lambda\mu^-(\pi)}{\lambda}+K_0(\lambda), \tag{3.11}$$

where

$$h^\pm=\pm\frac{1}{4}\left(1\pm\frac{1}{\alpha}\right)\int_0^a q(t)dt+\frac{1}{4}(\alpha\pm1)\int_a^\pi q(t)dt, \tag{3.12}$$

and

$$\begin{aligned} K_0(\lambda)=&\frac{1}{4\lambda}(1+\alpha)\int_0^a\cos\lambda(2\mu^+(t)-\mu^+(\pi))q(t)dt+\\ &+\frac{1}{4\lambda}(1-\alpha)\int_0^a\cos\lambda(2\mu^+(t)-\mu^-(\pi))q(t)dt+\\ &+\frac{1}{4\lambda}\left(1+\frac{1}{\alpha}\right)\int_a^\pi\cos\lambda(2\mu^+(t)-\mu^+(\pi))q(t)dt+\\ &+\frac{1}{4\lambda}\left(1-\frac{1}{\alpha}\right)\int_a^\pi\cos\lambda(\mu^+(\pi)+\mu^-(t)-\mu^+(t))q(t)dt+O\left(\frac{e^{|\mathrm{Im}\lambda|\mu^+(\pi)}}{|\lambda|^2}\right). \end{aligned} \tag{3.13}$$

We denote $G_\delta = \{\lambda : |\lambda - \lambda_n^0| \geqslant \delta\}$, where δ is a sufficiently small positive number $\delta < \frac{\gamma}{2}$ (see lemma 5). Let us show that

$$|\Delta_0(\lambda)| \geqslant C_\delta e^{|\mathrm{Im}\lambda|\mu^+(\pi)}, \lambda \in G_\delta, \quad C_\delta > 0. \tag{3.14}$$

We have $|\cos\lambda\mu^+(\pi)| \geqslant C_\delta e^{|\mathrm{Im}\lambda|\mu^+(\pi)}$, $\lambda \in G_\delta$. Then, using (3.8), we get (3.14). Further, by (3.11) we arrive at

$$|\Delta(\lambda)| \geqslant \widetilde{C}_\delta e^{|\mathrm{Im}\lambda|\mu^+(\pi)}, \lambda \in G_\delta, \quad \widetilde{C}_\delta > 0. \tag{3.15}$$

On the other hand, by (3.11) we obtain

$$\Delta(\lambda) - \Delta_0(\lambda) = O\left(\frac{e^{|\mathrm{Im}\lambda|\mu^+(\pi)}}{|\lambda|}\right), \quad |\lambda| \to \infty. \tag{3.16}$$

Consider the contour $\Gamma_n = \{\lambda : |\lambda| = |\lambda_n^0| + \frac{\gamma}{2}\}$ $(n = 1, 2, \ldots)$. Enlarging unboundedly contour Γ_n, for sufficiently large n by (3.14) and (3.16) we have

$$|\Delta(\lambda) - \Delta_0(\lambda)| \leqslant |\Delta_0(\lambda)|, \quad \lambda \in \Gamma_n . \tag{3.17}$$

Applying now Rouche's theorem, we see that the number of zeros of $\Delta_0(\lambda)$ inside Γ_n coincides with the number of zeros of $\Delta(\lambda) = \{\Delta(\lambda) - \Delta_0(\lambda)\} + \Delta_0(\lambda)$. We apply the Rouche's theorem to the circle $\gamma_n(\delta) = \{\lambda : |\lambda - \lambda_n^0| \leqslant \delta\}$ and conclude that for sufficiently large n, there exist only one zero λ_n of the function $\Delta(\lambda)$ in $\gamma_n(\delta)$. Since $\delta > 0$ is arbitrary, then

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty. \tag{3.18}$$

Substituting (3.18) into (3.11) and taking into consideration the relations

$$\Delta_0(\lambda_n^0) = \frac{1}{2}(1 + \frac{1}{\alpha}) \cos\lambda_n^0\mu^+(\pi) + \frac{1}{2}(1 - \frac{1}{\alpha}) \cos\lambda_n^0\mu^-(\pi) = 0,$$

$$\sin\varepsilon_n\mu^+(\pi) \sim \varepsilon_n\mu^+(\pi), \quad \cos\varepsilon_n\mu^+(\pi) \sim 1, \quad n \to \infty,$$

we get

$$\varepsilon_n = \frac{d_n}{\lambda_n^0 + \varepsilon_n} + \frac{\varepsilon_n}{\lambda_n^0 + \varepsilon_n}\tilde{d}_n + \frac{\tilde{k}_n}{\lambda_n^0 + \varepsilon_n} \tag{3.19}$$

where

$$d_n = \frac{h^+ \sin\lambda_n^0\mu^+(\pi) + h^- \sin\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(1 + \frac{1}{\alpha})\mu^+(\pi) \sin\lambda_n^0\mu^+(\pi) + \frac{1}{2}(1 - \frac{1}{\alpha})\mu^-(\pi) \sin\lambda_n^0\mu^-(\pi)},$$

$\tilde{k}_n = k_0(\lambda_n^0 + \varepsilon_n)$ and

$$\tilde{d}_n = \frac{h^+\mu^+(\pi) \cos\lambda_n^0\mu^+(\pi) + h^-\mu^-(\pi) \cos\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(1 + \frac{1}{\alpha})\mu^+(\pi) \sin\lambda_n^0\mu^+(\pi) + \frac{1}{2}(1 - \frac{1}{\alpha})\mu^-(\pi) \sin\lambda_n^0\mu^-(\pi)}.$$

Since $\frac{1}{\lambda_n^0+\varepsilon_n} = O(\frac{1}{n})$, $\frac{\varepsilon_n}{\lambda_n^0+\varepsilon_n} = o(\frac{1}{n})$, $n \to \infty$, we have that d_n, $\tilde{d}_n$ are bounded, $\tilde{k}_n \in l_2$ and (3.19) implies

$$\varepsilon_n = O(\frac{1}{n}), \quad n \to \infty.$$

Using (3.19) once more, we can obtain more precisely that

$$\varepsilon_n = \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad k_n \in l_2, \quad n \to +\infty, \tag{3.20}$$

where $k_n = \frac{\mu^+(\pi)}{\pi}\tilde{k}_n + O(\frac{1}{n})$, $n \to \infty$. The prove is complete. $\square$

4. Spectral Expansion Formula

Theorem 2. *1) The system of eigenfunftions* $\{\varphi(x,\lambda_n)\}_{n\geqslant 1}$ *of boundary value problem (1.1), (1.2) is complete in* $L_{2,\rho}(0,\pi)$*;*

2) If $f(x)$ *is an absolutely continuous function on the segment* $[0,\pi]$*, and* $f'(0)=f(\pi)=0$*, then*

$$f(x)=\sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n), \tag{4.1}$$

where

$$a_n=\frac{1}{\alpha_n}\int_0^{\pi} f(t)\varphi(t,\lambda_n)\rho(t)dt, \tag{4.2}$$

and series (4.1) converges uniformly on $[0,\pi]$*;*

3) For $f\in L_{2,\rho}(0,\pi)$ *series (4.1) converges in* $L_{2,\rho}(0,\pi)$*, moreover, the Parseval identity*

$$\int_0^{\pi}|f(x)|^2\rho(x)dx=\sum_{n=1}^{\infty}\alpha_n|a_n|^2 \tag{4.3}$$

holds.

Proof. Let $\psi(x,\lambda)$ be a solution of equation (1.1) under the initial conditions (1.5). Denote

$$G(x,t;\lambda)=-\frac{1}{\Delta(\lambda)}\begin{cases}\psi(x,\lambda)\varphi(t,\lambda), & x<t\\ \varphi(x,\lambda)\psi(t,\lambda), & t<x\end{cases} \tag{4.4}$$

and let us consider the function

$$Y(x,\lambda)=\int_0^{\pi}\rho(t)f(t)G(x,t;\lambda)dt \tag{4.5}$$

which is a solution to the boundary value problem

$$-Y''(x,\lambda)+\rho(x)Y(x,\lambda)=\lambda^2\rho(x)Y(x,\lambda)-f(x)\rho(x), \tag{4.6}$$

$$Y'(0,\lambda)=0, Y(\pi,\lambda)=0.$$

Using (1.9), we obtain

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=-\frac{1}{2\alpha_n\lambda_n}\varphi(x,\lambda_n)\int_0^{\pi}\rho(t)f(t)\varphi(t,\lambda_n)dt. \tag{4.7}$$

Let $f(x)\in L_{2,\rho}(0,\pi)$ be such that

$$\int_0^{\pi}\rho(t)f(t)\varphi(t,\lambda_n)dt=0 \quad n=1,2,3,\ldots.$$

Then, from (4.7), we have $\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=0$. Hence, for fixed $x\in[0,\pi]$, function $Y(x,\lambda)$ is entire with respect to λ. On the other hand, substituting (3.1), (3.2) and (3.15) into (4.5), we see that that for a fixed $\delta>0$ and a sufficiently large $\lambda^*>0$:

$$|Y(x,\lambda)|\leqslant\frac{C_\delta}{|\lambda|},\qquad \lambda\in G_\delta,\quad |\lambda|\geq\lambda^*.$$

Using the maximum principle for module of analytic functions and Liouville theorem, we conclude that $Y(x,\lambda)\equiv 0$. This fact and (4.6) imply that $f(x)=0$ a.e. on $[0,\pi]$. Thus, statement 1) of the theorem is proved.

Let $f \in AC[0,\pi]$. We rewrite function $Y(x,\lambda)$ as

$$Y(x,\lambda) = -\frac{1}{\lambda^2\Delta(\lambda)}\left\{\psi(x,\lambda)\int_0^x \left(-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda)\right)f(t)dt+ \right.$$
$$\left. +\varphi(x,\lambda)\int_x^\pi \left(-\psi''(t,\lambda)+q(t)\psi(t,\lambda)\right)f(t)dt\right\}.$$

Integrating by parts the term with the second-order derivatives and taking into consideration the conditions $f'(0)=0,\ f(\pi)=0$, we obtain

$$Y(x,\lambda)=\frac{f(x)}{\lambda^2}-\frac{1}{\lambda^2}\left(Z_1(x,\lambda)+Z_2(x,\lambda)\right), \tag{4.8}$$

where

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x g(t)\varphi'(t,\lambda)dt+\varphi(x,\lambda)\int_x^\pi g(t)\psi'(t,\lambda)dt\right].$$

Here $g(t)=f'(t)$,

$$Z_2(x,\lambda)=\frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x \varphi(t,\lambda))f(t)q(t)dt+\varphi(x,\lambda)\int_x^\pi \psi(t,\lambda))f(t)q(t)dt-\varphi(x,\lambda)f(\pi)\right].$$

We consider the contour integral

$$I_N(x)=\frac{1}{2\pi i}\int_{\Gamma_n}\lambda Y(x,\lambda)d\lambda,$$

where $\Gamma_n=\{\lambda:|\lambda|=|\lambda_N^0|+\frac{\gamma}{2}\}$ is a counter-clockwise oriented contour.

By means of the residue theorem we have

$$I_N(x)=2\sum_{n=1}^N \operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda)=\sum_{n=1}^N a_n\varphi(x,\lambda_n) \tag{4.9}$$

where

$$a_n=\frac{1}{\alpha_n}\int_0^\pi \rho(t)f(t)\varphi(t,\lambda_n)dt.$$

On the other hand, taking into consideration (4.8), we have

$$I_N(x)=f(x)-\frac{1}{2\pi i}\int_{\Gamma_N}\frac{1}{\lambda}(Z_1(x,\lambda)+Z_2(x,\lambda))d\lambda. \tag{4.10}$$

Comparing (4.9) and (4.10), we obtain

$$f(x)=\sum_{n=1}^N a_n\varphi(x,\lambda_n)+\xi_N(x),$$

where

$$\xi_N(x)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{1}{\lambda}(Z_1(x,\lambda)+Z_2(x,\lambda))d\lambda.$$

Therefore, in order to prove the item 2) of the theorem, it suffices to show that

$$\lim_{N\to\infty}\max_{0\leqslant x\leqslant\pi}|\xi_N(x)|=0 \tag{4.11}$$

From (3.1), (3.2) and (3.15) it follows that for fixed $\delta>0$ and sufficiently large $\lambda^*>0$

$$\max_{0\leqslant x\leqslant\pi}|Z_2(x,\lambda)|\leqslant\frac{C_2}{|\lambda|},\quad \lambda\in G_\delta,\quad |\lambda|\geqslant\lambda^*, C_2>0. \tag{4.12}$$

Let us show that

$$\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}}\max_{0\leqslant x\leqslant\pi}|Z_1(x,\lambda)|=0. \tag{4.13}$$

First we suppose that $g(t)$ is absolutely continuous on $[0,\pi]$. In this case, integration by parts gives

$$Z_1(x,\lambda) = -\frac{1}{\Delta(\lambda)}\left\{\psi(x,\lambda)\int_0^x \varphi(t,\lambda)g'(t)dt + \varphi(x,\lambda)\int_x^\pi \psi(t,\lambda)g'(t)dt\right\}.$$

Therefore, similarly to $Z_2(x,\lambda)$, we have

$$\max_{0\leqslant x\leqslant\pi}|Z_1(x,\lambda)| \leqslant \frac{C_1}{|\lambda|}, \quad \lambda\in G_\delta, \quad |\lambda|\geqslant\lambda^*, C_1>0.$$

In the general case, we fix $\varepsilon>0$ and choose an absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi |g_\varepsilon(t)-g(t)|\,dt<\varepsilon.$$

Then, using estimates (3.1), (3.2) and (3.15), one can find $\lambda^{**}>0$ such that for $\lambda\in G_\delta$, $|\lambda|\geqslant\lambda^{**}$ the relation

$$Z_1(x,\lambda) = \frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x \varphi'(t,\lambda)(g(t)-g_\varepsilon(t))dt + \varphi(x,\lambda)\int_x^\pi \psi'(t,\lambda)(g(t)-g_\varepsilon(t))dt\right] + \\ + \frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x \varphi(t,\lambda)g_\varepsilon'(t)dt - \varphi(x,\lambda)\int_x^\pi \psi(t,\lambda)g_\varepsilon'(t)dt\right]$$

yields

$$\max_{0\leqslant x\leqslant\pi}|Z_1(x,\lambda)| \leqslant C\int_0^\pi |g_\varepsilon(t)-g(t)|\,dt + \frac{\widetilde{C}(\epsilon)}{|\lambda|} < C_\epsilon + \frac{\widetilde{C}(\epsilon)}{|\lambda|}, \quad \lambda\in G_\delta, \quad |\lambda|\geqslant\lambda^{**}.$$

Therefore,

$$\overline{\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}}}\max_{0\leqslant x\leqslant\pi}|Z_1(x,\lambda)| \leqslant C_\epsilon.$$

Since ε is an arbitrary positive number, we arrive at identity (4.13). Relations (4.12), (4.13) immediately imply (4.11), and thus, statement 2) of the theorem is proved.

System of eigenfunction $\{\varphi(x,\lambda_n)\}_{n\geqslant 1}$ is complete and orthogonal in $L_{2,\rho}(0,\pi)$. Therefore, it forms the orthogonal basis in $L_{2,\rho}(0,\pi)$ and Parseval identity (4.3) is valid. □

5. Weyl solution, Weyl function

Let $\Phi(x,\lambda)$ be the solution of equation (1.1) satisfying the conditions $\Phi'(0,\lambda)=1$, $\Phi(\pi,\lambda)=0$. Denote by $C(x,\lambda)$ the solution of equation (1.1) satisfying the initial conditions $C(0,\lambda)=0$, $C'(0,\lambda)=1$. Then, the solution $\psi(x,\lambda)$ can be represented as

$$\psi(x,\lambda) = \psi(0,\lambda)\varphi(x,\lambda) + \Delta(\lambda)C(x,\lambda) \tag{5.1}$$

or

$$\frac{\psi(x,\lambda)}{\Delta(\lambda)} = C(x,\lambda) + \frac{\psi(0,\lambda)}{\Delta(\lambda)}\varphi(x,\lambda). \tag{5.2}$$

Denote

$$M(\lambda) := \frac{\psi(0,\lambda)}{\Delta(\lambda)}. \tag{5.3}$$

It is clear that

$$\Phi(x,\lambda) = C(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{5.4}$$

Functions $\Phi(x,\lambda)$ and $M(\lambda)=\Phi(0,\lambda)$ are respectively called the Weyl solution and the Weyl function of boundary value problem (1.1), (1.2). The Weyl function is a meromorphic function having simple poles at points λ_n being the eigenvalues of boundary value problem (1.1), (1.2). Relations (5.2), (5.4) yield

$$\Phi(x,\lambda)=\frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{5.5}$$

It can be shown that

$$<\varphi(x,\lambda),\Phi(x,\lambda)>=1. \tag{5.6}$$

Theorem 3. *If $M(\lambda)=\widetilde{M}(\lambda)$, then $L=\widetilde{L}$; that is, the boundary value problem (1.1), (1.2) is uniquely determined by the Weyl function.*

Proof. We introduce the matrix $P(x,\lambda)=[P_{ij}(x,\lambda)]_{i,j=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix}\widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda)\end{pmatrix}=\begin{pmatrix}\varphi(x,\lambda) & \Phi(x,\lambda)\\ \varphi'(x,\lambda) & \Phi'(x,\lambda)\end{pmatrix}. \tag{5.7}$$

By (5.7) we have

$$\begin{aligned}\varphi(x,\lambda)&=P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ \Phi(x,\lambda)&=P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda),\end{aligned} \tag{5.8}$$

or

$$\begin{aligned}P_{11}(x,\lambda)&=\varphi(x,\lambda)\widetilde{\Phi}'(x,\lambda)-\Phi(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ P_{12}(x,\lambda)&=-\varphi(x,\lambda)\widetilde{\Phi}(x,\lambda)+\Phi(x,\lambda)\widetilde{\varphi}(x,\lambda).\end{aligned} \tag{5.9}$$

Taking into consideration equations (5.5) and (5.9), we substitute (5.4) into (5.9) to obtain

$$\begin{aligned}P_{11}(x,\lambda)&=1+\tfrac{1}{\Delta(\lambda)}\left[\varphi(x,\lambda)(\widetilde{\psi}'(x,\lambda)-\psi'(x,\lambda))-\psi(x,\lambda)(\widetilde{\varphi}'(x,\lambda)-\varphi'(x,\lambda))\right],\\ P_{12}(x,\lambda)&=\tfrac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\widetilde{\varphi}(x,\lambda)-\varphi(x,\lambda)\widetilde{\psi}(x,\lambda)\right].\end{aligned} \tag{5.10}$$

By (3.1), (3.2), (3.15) and equation (5.10) we get

$$\lim_{|\lambda|\to\infty}\max_{0\leqslant x\leqslant\pi}|P_{11}(x,\lambda)-1|=\lim_{|\lambda|\to\infty}\max_{0\leqslant x\leqslant\pi}|P_{12}(x,\lambda)|=0. \tag{5.11}$$

Hence, if we take into consideration equations (5.4) and (5.9), we get

$$\begin{aligned}P_{11}(x,\lambda)&=\varphi(x,\lambda)\widetilde{C}'(x,\lambda)-C(x,\lambda)\widetilde{\varphi}'(x,\lambda)+(\widetilde{M}(\lambda)-M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda)\\ P_{12}(x,\lambda)&=C(x,\lambda)\widetilde{\varphi}(x,\lambda)-\widetilde{C}(x,\lambda)\varphi(x,\lambda)+(M(\lambda)-\widetilde{M}(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda).\end{aligned}$$

Therefore if $M(\lambda)=\widetilde{M}(\lambda)$, then $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions for each fixed x. It can be easily seen from (5.11) that $P_{11}(x,\lambda)=1$ and $P_{12}(x,\lambda)=0$. Substituting it into (5.8), we get $\varphi(x,\lambda)\equiv\widetilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda)\equiv\widetilde{\Phi}(x,\lambda)$ for each x and λ. Hence, we arrive at $q(x)\equiv\widetilde{q}(x)$. □

Theorem 4. *The expression*

$$M(\lambda)=\frac{1}{2\lambda_0\alpha_0(\lambda_0-\lambda)}+\sum_{n=1}^{\infty}\frac{1}{\alpha_n(\lambda_n^2-\lambda^2)} \tag{5.12}$$

holds true.

Proof. Using (1.9), we calculate

$$\dot{\Delta}(\lambda_n) = 2\lambda_n\alpha_n\beta_n,$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$. Taking into account the last identities, in accordance with (5.3) we calculate:

$$\operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{\psi(0,\lambda_n)}{\dot{\Delta}(\lambda_n)} = \frac{\beta_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{2\lambda_n\alpha_n}. \tag{5.13}$$

Using (3.2), (3.15) and (5.3), we have

$$|M(\lambda)| \leqslant \frac{C_\delta}{|\lambda|}, \lambda \in G_\delta.$$

Thus, we get

$$\lim_{|\lambda|\to\infty} |M(\lambda)| = 0. \tag{5.14}$$

Now, let us consider the contour integral

$$J_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{M(\mu)}{\mu-\lambda}d\mu, \quad \lambda \in Int\Gamma_N,$$

where the contour $\Gamma_N = \left\{\mu : |\mu| = |\lambda_N^0| + \frac{\gamma}{2}\right\}$ is passed counter-clockwise.

Owing to (5.14), we have $\lim_{N\to\infty} J_N(\lambda) = 0$. On the other hand, by the residue theorem, the identity $\lambda_{-n} = -\lambda_n$ and (5.13), we have

$$J_N(\lambda) = M(\lambda) + \sum_{n=-N}^{N} \frac{1}{2\lambda_n\alpha_n(\lambda_n-\lambda)} =$$

$$= M(\lambda) + \frac{1}{2\lambda_0\alpha_0(\lambda_0-\lambda)} + \sum_{n=1}^{N} \frac{1}{\alpha_n(\lambda_n^2-\lambda^2)}$$

and as $N \to \infty$, we arrive at (5.12). □

Theorem 5. *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n \in Z$, then $L = \tilde{L}$. That is, problem (1.1), (1.2) is uniquelly determined by its spectral data.*

Proof. Since $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n \in Z$, considering formula (5.12), we have $M(\lambda) = \tilde{M}(\lambda)$. Using Theorem 9, we arrive at $L = \tilde{L}$. □

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