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PRELIMINARY GROUP CLASSIFICATION OF $(2+1)$-DIMENSIONAL LINEAR ULTRAPARABOLIC KOLMOGOROV–FOKKER–PLANCK EQUATIONS

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Abstract. We make a preliminary group classification of a class of $(2+1)$-dimensional ultraparabolic equations invariant under low-dimensional solvable Lie algebras. It is shown that there exist one, six and four non-equivalent ultraparabolic equations admitting two-, three-, and four-dimensional solvable Lie algebras, respectively.

Keywords: ultraparabolic equation, equivalence transformation, group classification, maximal invariance algebra, solvable Lie algebra.

Mathematics Subject Classification: 35K70, 76M60

1. Introduction

Let us consider the class of $(2+1)$-dimensional linear ultraparabolic Kolmogorov–Fokker–Planck equations:

$$u_t = A(t,x,y)\,u_{xx} + B(t,x,y)\,u_x + C(t,x,y)\,u_y + D(t,x,y)\,u, \tag{1}$$

where A, B, C and D are arbitrary smooth functions in some domain in $\mathbb{R}^3$, $AC \neq 0$.

The equations from class (1) are widely used for describing various processes in physics and in mathematical computations in economics [1], [2], [3], [4], [5]. In particular, among famous equations in class (1) one should mention the following ones:

Kramers equation [1]

$$u_t = -(xu)_y + (V'(y)u)_x + \gamma(xu + u_x)_x, \tag{2}$$

which described a particle motion in a fluctuating media, here function $u = u(t,x,y)$ is the probability density, function $V(y)$ is an external potential, γ is a constant;

for calculating Asian option [5] the equation

$$u_t = -\frac{1}{2}\sigma^2 x^2\, u_{xx} - rx\, u_x + \log x\, u_y + r\, u, \tag{3}$$

or

$$u_t = -\frac{1}{2}\sigma^2 x^2\, u_{xx} - rx\, u_x + \frac{x}{t_0 - T}\, u_y + r\, u, \tag{4}$$

is used, where $u(t,x,y)$ is the value of Asian option depending on a stock rate (variable x) at time t, y is the mean stock rate, T is the period of a contract, t_0 is the beginning of the contract (one usually let $t_0 = 0$), r is the interest rate, σ is the volatility.

As of today, symmetry properties of some sublcasses of class (1) were studied in works [6]–[9]. In particular, in work [7], a group classification of equations in class (2) was made. It was shown in paper [10] that (3) and (4) are reduced respectively to the equations

$$u_t = u_{xx} - xu_y \tag{5}$$

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and

$$u_t = u_{xx} + e^x u_y. \tag{6}$$

We note that for equation (5) obtained by Kolmogorov in 1934, the fundamental solution is known [2]. Moreover, maximal invariance algebras (MIA) were found for equations (5) and (6) [8], [9].

Since each equation in class (1) is linear, for fixed values of functions A, B, C, and D its MIA are infinite-dimensional with an operator $p\,\partial_u$, where function $p\,(t,x,y)$ is an arbitrary smooth solution to equation (1). Thus, in preliminary group classification we exclude from the consideration operators $p\,\partial_u$, i.e., we shall seek only finite-dimensional parts of MIA for equation in class (1). In particular, the **aim of the work** is to find equations (1), whose MIA are solvable and have the dimension at most 4.

In studying symmetry properties of class (1), it is impossible to employ the classical Lie-Ovsyannikov method. The reason is that arbitrary elements A, B, C, and D of the studied class depend on variables t, x, and y. This is why for the preliminary group classification we shall make use of the method by Zhdanov-Lahno proposed in work [11] (for more details see [12]). As of today, the mentioned method is employed in considering wide classes of differential equations [13]–[17].

2. Equivalence transformation and invariance operator for class of equations (1)

An important role in studying of symmetry properties of a class of differential equations by the Zhdanov-Lahno method is played by equivalence transformations (point transforms reducing an arbitrary equation in a chosen class to another equation in the same class). In particular, Zhdanov-Lahno method is effective in studying classes of differential equations admitting a wide group of equivalence transformations.

Theorem 1. *Equivalence transformations for class of equations (1) read as*

$$\bar{t} = T(t,y),\ \bar{x} = X(t,x,y),\ \bar{y} = Y(t,y),\ v = \varphi(t,x,y)u, \tag{7}$$

where T, X, Y and φ are arbitrary smooth functions, $\varphi X_x(T_tY_y - T_yY_t) \neq 0$. They transform an arbitrary equation in class (1) into another equation reading as

$$v_{\bar{t}} = \widetilde{A}(\bar{t},\bar{x},\bar{y})v_{\bar{x}\bar{x}} + \widetilde{B}(\bar{t},\bar{x},\bar{y})v_{\bar{x}} + \widetilde{C}(\bar{t},\bar{x},\bar{y})v_{\bar{y}} + \widetilde{D}(\bar{t},\bar{x},\bar{y})v, \tag{8}$$

where functions $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ are determined by the conditions:

$$(T_t - CT_y)\,\widetilde{A} = X_x^2 A, \tag{9}$$

$$(T_t - CT_y)\,\widetilde{B} = X_yC - X_t + \left(X_{xx} - 2\frac{\varphi_x}{\varphi}\,X_x\right)A + X_xB, \tag{10}$$

$$(T_t - CT_y)\,\widetilde{C} = Y_yC - Y_t, \tag{11}$$

$$(T_t - CT_y)\,\widetilde{D} = D - \frac{\varphi_y}{\varphi}C + \frac{\varphi_t}{\varphi} + \left(2\frac{\varphi_x^2}{\varphi^2} - \frac{\varphi_{xx}}{\varphi}\right)A - \frac{\varphi_x}{\varphi}B. \tag{12}$$

The proof of the theorem is based on a direct method of constructing equivalence transformations, see, for instance, [11].

Remark 1. *Equivalence transformations (7) allow us to simplify class (1), for instance, to let $B = D = 0$. However, in preliminary group classification we shall consider exactly equations (1). It is due to the features of the method employed in the work.*

Remark 2. *In the case $C_x = 0$ there exists a transform reducing equation (1) to the equation with $C = 0$ that is implied by relation (11)). Thus, we shall consider only the case $C_x \neq 0$.*

Before proceeding to applying Zhdanov-Lahno method, let us find the general form of Lie symmetry operator which is admitted by class of equations (1).

Theorem 2. *Invariance operator of class of equations (1) reads as*

$$X = \tau(t,y)\,\partial_t + \xi^1(t,x,y)\,\partial_x + \xi^2(t,y)\,\partial_y + (r(t,x,y)\,u + p(t,x,y))\,\partial_u, \tag{13}$$

where τ, ξ^1, ξ^2, r and p are unknown smooth functions determined by the system of determining equations (SDE)

$$A_x\,\xi^1 + A_y\,\xi^2 + A_t\,\tau + A\left(\tau_t - 2\,\xi^1_x - C\tau_y\right) = 0, \tag{14}$$

$$B_x\,\xi^1 + B_y\,\xi^2 + B_t\,\tau + B\left(\tau_t - \xi^1_x - C\tau_y\right) + A\left(2r_x - \xi^1_{xx}\right) - C\xi^1_y + \xi^1_t = 0, \tag{15}$$

$$C_x\,\xi^1 + C_y\,\xi^2 + C_t\,\tau + C\left(\tau_t - \xi^2_y - C\tau_y\right) + \xi^2_t = 0, \tag{16}$$

$$D_x\,\xi^1 + D_y\,\xi^2 + D_t\,\tau + D\left(\tau_t - C\tau_y\right) + Ar_{xx} + Br_x + Cr_y - r_t = 0, \tag{17}$$

$$p_t = Ap_{xx} + Bp_x + Cp_y + Dp. \tag{18}$$

The proof of the theorem is based on applying the invariance criterion for differential equations (see, for instance, monogrpahs [12], [18], [19], [20], [21]).

In view of the fact that we exclude operators $p\,\partial_u$ responsible for an infinite-dimensional part of MIA for equation (1), the sought Lie symmetry operator (13) becomes

$$X = \tau(t,y)\,\partial_t + \xi^1(t,x,y)\,\partial_x + \xi^2(t,y)\,\partial_y + r(t,x,y)u\,\partial_u. \tag{19}$$

3. Low-dimensional solvable Lie algebras for class of equations (1)

It follows from SDE (14)–(17) that for arbitrary values of functions A, B, C and D, the invariance algebra of equation (1) is one-dimensional with the basis operator $u\partial_u$. Since this operator commutes with operators (19), then among two-dimensional algebras only Abelian algebra [22]

$$2g_1 :\ [e_1, e_2] = 0$$

can be Lie algebra for equation (1).

In order to construct all possible non-equivalent realization of algebra $2g_1$, we apply transformations (7) to operator (19):

$$\begin{aligned}\bar{X} &= \left(\tau T_t + \xi^2 T_y\right)\partial_{\bar{t}} + \left(\tau X_t + \xi^1 X_x + \xi^2 T_y\right)\partial_{\bar{x}}\\ &\quad + \left(\tau Y_t + \xi^2 Y_y\right)\partial_{\bar{y}} + \left(\tau\varphi_t + \xi^1\varphi_x + \xi^2\varphi_y + \varphi\, r\right)u\partial_{\bar{v}}.\end{aligned} \tag{20}$$

It follows from (20) that in the case $(\tau)^2 + (\xi^2)^2 \neq 0$ there exist transformations which reduces an arbitrary operator (19) to the operator $\partial_{\bar{t}}$. In particular, these transformations can be found by solving the equations

$$\tau T_t + \xi^2 T_y = 1,\ \tau X_t + \xi^1 X_x + \xi^2 T_y = 0, \quad \tau Y_t + \xi^2 Y_y = 0,\ \tau\varphi_t + \xi^1\varphi_x + \xi^2\varphi_y + \varphi\, r = 0.$$

In the case $\tau = \xi^2 = 0$, $\xi^1 \neq 0$ we obtain operator $\partial_{\bar{x}}$. If $\tau = \xi^2 = \xi^1 = 0$, then operator (20) reads as $rv\partial_v$ and is reduced to one of the operators:

$$\bar{x}v\partial_v\ (r_x \neq 0),\ \bar{t}\,v\partial_v\ \left((r_t)^2 + (r_y)^2 \neq 0\right),\ \alpha v\partial_v\ (r = \alpha = const).$$

Therefore, the following theorem holds true.

Theorem 3. *An arbitrary operator (19) can be reduced to one of the operators*

$$\partial_t,\ \partial_x,\ u\partial_u,\ tu\partial_u,\ xu\partial_u \tag{21}$$

up transformations (7) and a non-zero multiplicative constant.

Among operators (21), only the operators ∂_t, ∂_x and $u\partial_u$ satisfy SDE (14)–(17). However, operator ∂_x leads us to the condition $C_x = 0$ and is excluded from the consideration. Thus, we obtain one two-dimensional Lie algebra admitting equation (1).

Theorem 4. *Up to equivalence transformations (7), there exists the unique class of equations*

$$u_t = A(x,y)u_{xx} + B(x,y)u_x + C(x,y)u_y + D(x,y)u \tag{22}$$

admitting two-dimensional Lie algebra of symmetry operators (19), namely,

$$2g_1{}^1 = < \partial_t,\ u\partial_u > .$$

For arbitrary values of functions A, B, C and D this algebra is MIA for class of equations (22).

In constructing three-dimensional solvable invariance algebras for class of equations (1), it is sufficient to add one operator (19) to the operators in algebra $2g_1^1$ and to find all possible non-equivalent realizations satisfying the corresponding commutation conditions and SDE (14)–(17). At that, we shall make use the transformation

$$\bar t = t + T(y), \quad \bar x = X(x,y), \quad \bar y = Y(y), \quad v = \varphi(x,y)u \tag{23}$$

preserving the form of operator ∂_t.

In accordance with Mubarakzyanov's classification [22], there exists 7 non-isomorphic three-dimensional solvable Lie algebras. However, we need to consider only those which can contain operator $u\partial_u$, namely,

$$3g_1 :\ [e_i, e_j] = 0\ (i,j = 1,2,3); \quad g_2 \oplus g_1 :\ [e_1, e_2] = e_2; \quad g_{3.1} :\ [e_2, e_3] = e_1.$$

Since constructing of realizations for each of the mentioned algebras is almost the same, we consider only algebra $3g_1$ with basis operators:

$$e_1 = \partial_t, \quad e_2 = u\partial_u, \quad e_3 = X,$$

where X is an arbitrary operator (19).

By the condition $[\partial_t, X] = 0$ we obtain:

$$X = \tau(y)\,\partial_t + \xi^1(x,y)\,\partial_x + \xi^2(y)\,\partial_y + r(x,y)u\,\partial_u.$$

Applying transformations (23) to operator X, we have

$$\bar X = \left(\tau + \xi^2 T_y\right)\partial_{\bar t} + \left(\xi^1 X_x + \xi^2 T_y\right)\partial_{\bar x} + \xi^2 Y_y\,\partial_{\bar y} + \left(\xi^1\varphi_x + \xi^2\varphi_y + \varphi\, r\right)u\partial_{\bar v}. \tag{24}$$

Making calculations similar to ones in Theorem 3, we obtain the operators:

$$\partial_x,\ \partial_y,\ y\partial_t,\ y\partial_t + \partial_x,\ xu\partial_u,\ yu\partial_u,\ y\partial_t + xu\partial_u,\ y\partial_t + r(y)u\partial_u\ (r' \neq 0). \tag{25}$$

Substituting each of operators (25) into SDE (14)–(17) with functions A, B, C and D depending only on variables x and y, we establish that only two realizations of three-dimensional algebra $3g_1$ satisfy the assumption of the problem, namely,

$$3g_1{}^1 = < \partial_t,\ u\partial_u,\ \partial_y >, \qquad u_t = A(x)u_{xx} + B(x)u_x + C(x)u_y + D(x)u;$$

$$3g_1{}^2 = < \partial_t,\ u\partial_u,\ y\partial_t + \partial_x >, \qquad u_t = x^{-1}\left(A(y)u_{xx} + B(y)u_x - u_y + D(y)u\right).$$

Considering algebra $g_2 \oplus g_1$, we obtain the following four non-equivalent (up to transformations (23)) realizations

$$g_2 \oplus g_1{}^1 = < \partial_t,\ e^t\partial_y,\ u\partial_u, >, \qquad g_2 \oplus g_1{}^2 = < -t\partial_t + \partial_x,\ \partial_t,\ u\partial_u >,$$

$$g_2 \oplus g_1{}^3 = < -t\partial_t + \partial_y,\ \partial_t,\ u\partial_u >, \qquad g_2 \oplus g_1{}^4 = < \partial_t,\ e^t(y\partial_t + \partial_x),\ u\partial_u, >$$

satisfying the assumption of the problem.

However, among the mentioned realizations there are equivalent up to transformations (7). Indeed, the changes

$$\bar t = y, \quad \bar y = te^y \quad \text{and} \quad \bar t = -y^{-1}e^{-t}, \quad \bar x = t - xy$$

reduce respectively $g_2 \oplus g_1{}^3$ to $g_2 \oplus g_1{}^1$ and $g_2 \oplus g_1{}^4$ to $g_2 \oplus g_1{}^2$.

Thus, we obtain two non-equivalent (up to transformations (7)) realizations of algebra $g_2 \oplus g_1$ and corresponding classes of equations

$$g_2 \oplus g_1{}^1 = < \partial_t,\ e^t\partial_y,\ u\partial_u, >, \qquad u_t = A(x)u_{xx} + B(x)u_x + \left(C(x) - y\right)u_y + D(x)u;$$

$$g_2 \oplus {g_1}^2 =< -t\partial_t + \partial_x,\ \partial_t,\ u\partial_u >, \qquad u_t = e^x \left(A(y)u_{xx} + B(y)u_x + C(y)u_y + D(y)u\right).$$

While considering algebra $g_{3.1}$ we also obtain two non-equivalent (up to transformations (7)) realizations and the corresponding classes of equations

$$ {g_{3.1}}^1 =< u\partial_u,\ \partial_t,\ \partial_y + tu\partial_u >, \qquad u_t = A(x)u_{xx} + B(x)u_x + C(x)u_y + (D(x) + y)\, u;$$

$$ {g_{3.1}}^2 =< u\partial_u,\ \partial_t,\ y\partial_t + \partial_x + tu\partial_u >, \quad u_t = x^{-1}\left(A(y)u_{xx} + B(y)u_x - u_y + \left(D(y) + \frac{x^2}{2}\right)u\right).$$

Combining the obtained results, we arrive at the theorem.

Theorem 5. *Up to equivalence transformations (7) there exist six classes of equations (1) admitting three-dimensional solvable Lie algebras of symmetry operators (19), namely,*

$$3{g_1}^1 =< \partial_t,\ u\partial_u,\ \partial_y >,$$
$$u_t = A(x)u_{xx} + B(x)u_x + C(x)u_y + D(x)u; \tag{26}$$
$$3{g_1}^2 =< \partial_t,\ u\partial_u,\ y\partial_t + \partial_x >,$$
$$u_t = x^{-1}\left(A(y)u_{xx} + B(y)u_x - u_y + D(y)u\right); \tag{27}$$
$$g_2 \oplus {g_1}^1 =< \partial_t,\ e^t\partial_y,\ u\partial_u >,$$
$$u_t = A(x)u_{xx} + B(x)u_x + \left(C(x) - y\right)u_y + D(x)u; \tag{28}$$
$$g_2 \oplus {g_1}^2 =< -t\partial_t + \partial_x,\ \partial_t,\ u\partial_u >,$$
$$u_t = e^x \left(A(y)u_{xx} + B(y)u_x + C(y)u_y + D(y)u\right); \tag{29}$$
$${g_{3.1}}^1 =< u\partial_u,\ \partial_t,\ \partial_y + tu\partial_u >,$$
$$u_t = A(x)u_{xx} + B(x)u_x + C(x)u_y + (D(x) + y)\,u; \tag{30}$$
$${g_{3.1}}^2 =< u\partial_u,\ \partial_t,\ y\partial_t + \partial_x + tu\partial_u >,$$
$$u_t = x^{-1}\left(A(y)u_{xx} + B(y)u_x - u_y + \left(D(y) + \frac{x^2}{2}\right)u\right). \tag{31}$$

For arbitrary values of functions A, B, C, and D each of the obtained algebras is MIA for the corresponding class of equations.

4. 4-dimensional solvable Lie algebra of symmetry operators (19) for class of equations (1)

Before proceeding to constructing 4-dimensional solvable Lie algebras for class of equations (1), we observe that for some fixed values of functions A, B and D equations (27) and (31) admit algebras of dimension 5 and higher. We exclude such equations from our study. We also note for class of equations (29) it is more effective to employ the direct method of constructing MIA instead of Zhdanov-Lahno method. In particular, applying the transformations

$$\bar{t} = -\int \frac{\exp\left(-\int \frac{B}{C}dy\right)}{C}dy, \quad \bar{y} = -t, \quad \bar{x} = -x + \int \frac{B}{C}dy, \quad v = \exp\left(\int \frac{D}{C}dy\right)u$$

to (29), we obtain

$$u_t = A(t)u_{xx} + e^x u_y. \tag{32}$$

Theorem 6. *For arbitrary value of function $A(t)$, class of equations (32) admit 3-dimensional MIA of symmetry operators (19):*

$$< \partial_y,\ \partial_x + y\partial_y,\ u\partial_u > . \tag{33}$$

Up to equivalence transformations (7), class of equations (32) admits the extension of Lie algebra (33) only in two following cases:

1) function $A(t)$ solves the equation

$$\left(\frac{A'}{A^2}\right)' = aA: \tag{34}$$

$$< \frac{1}{A}\partial_t + \frac{A'}{A^2}\partial_x - \frac{a}{2}xu\partial_u,\ \partial_y,\ \partial_x + y\partial_y,\ u\partial_u >,$$

where $a \neq 0$ is an arbitrary constant;

2) $A(t) = 1: < \partial_t,\ \partial_y,\ \partial_x + y\partial_y,\ u\partial_u,\ 2y\partial_x + y^2\partial_y + (e^x - y)\,\partial_u >$.

Remark 3. *By the transformation*

$$\bar{t} = \frac{a}{2}\int A dt, \quad \bar{x} = x - \ln A,\ \bar{y} = \frac{a}{2}y,\ \bar{a} = \frac{2}{a},$$
$$v = \exp\left(\frac{A'}{2A^2}x - \frac{a}{2}\int A\ln A dt - \frac{1}{4}\int\frac{(A')^2}{A^3}dt\right)u$$

equation (32) with function $A(t)$ being a solution to equation (34) is reduced to the equation

$$u_t = au_{xx} + e^x u_y + xu$$

with MIA of the form

$$< \partial_t,\ \partial_y,\ \partial_x + y\partial_y + tu\partial_u,\ u\partial_u > .$$

Thus, in studying symmetry properties of class of equations (32) we obtain only one equation with 4-dimensional MIA.

We proceed to constructing 4-dimensional solvable Lie algebras for classes of equations (26), (28) and (30). In particular, in accordance with Mubarakzyanov's classification [22] and assumption of the problem, we need to consider the following algebras:

$$\begin{aligned}
4g_1 &: [e_i, e_j] = 0 \quad i,j = 1,2,3,4;\\
g_2 \oplus 2g_1 &: [e_1, e_2] = e_2;\\
g_{3.1} \oplus g_1 &: [e_2, e_3] = e_1;\\
g_{3.2} \oplus g_1 &: [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;\\
g_{3.3} \oplus g_1 &: [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2;\\
g_{3.4} \oplus g_1 &: [e_1, e_3] = e_1, \quad [e_2, e_3] = he_2, \quad -1 \leqslant h < 1,\ h \neq 0;\\
g_{3.5} \oplus g_1 &: [e_1, e_3] = pe_1 - e_2, \quad [e_2, e_3] = e_1 + pe_2, \quad p \geqslant 0;\\
g_{4.1} &: [e_2, e_4] = e_1, \quad [e_3, e_4] = e_2;\\
g_{4.3} &: [e_1, e_4] = e_1, \quad [e_3, e_4] = e_2.
\end{aligned}$$

By the calculations we obtain the following non-equivalent 4-dimensional realizations of solvable Lie algebras and the corresponding equations:

$$g_2 \oplus 2g_1{}^1 = < \partial_t,\ e^t\partial_y,\ \partial_x + \partial_y,\ u\partial_u >,$$
$$u_t = au_{xx} + bu_x + (x - y)u_y + du; \tag{35}$$
$$g_2 \oplus 2g_1{}^2 = < \partial_x - y\partial_y,\ \partial_y,\ \partial_t,\ u\partial_u >,$$
$$u_t = au_{xx} + bu_x + e^{-x}u_y + du; \tag{36}$$
$$g_{3.1} \oplus g_1{}^1 = < \partial_y,\ \partial_t,\ \partial_x + t\partial_y,\ u\partial_u >,$$
$$u_t = au_{xx} + bu_x - xu_y + du; \tag{37}$$
$$g_{3.2} \oplus g_1{}^1 = < \partial_y,\ \partial_t,\ t\partial_t + \partial_x + (t + y)\partial_y,\ u\partial_u >,$$
$$u_t = e^{-x}\left(au_{xx} + bu_x - xe^x u_y + du\right); \tag{38}$$
$$g_{3.2} \oplus g_1{}^2 = < e^t\partial_y,\ e^t(\partial_x + t\partial_y), -\partial_t,\ u\partial_u >,$$
$$u_t = au_{xx} + (b - x)u_x - (x + y)u_y + du; \tag{39}$$

$$g_{3.4} \oplus {g_1}^1 = < e^t\partial_y,\ e^{ht}(\partial_x + \partial_y), -\partial_t,\ u\partial_u >,\ h \neq 0,\ h \neq 1,$$
$$u_t = au_{xx} + (b - hx)u_x + ((1-h)x - y)u_y + du; \tag{40}$$
$$g_{3.4} \oplus {g_1}^2 = < \partial_t,\ \partial_y,\ t\partial_t + \partial_x + hy\partial_y,\ u\partial_u >,\ h \neq 0,\ h \neq 1,$$
$$u_t = e^{-x}\left(au_{xx} + bu_x + e^{hx}u_y + du\right); \tag{41}$$
$$g_{3.5} \oplus {g_1}^1 = < \partial_t,\ \partial_y,\ (y + pt)\partial_t + \partial_x + (py - t)\partial_y,\ u\partial_u >,\ p \geqslant 0,$$
$$u_t = \frac{e^{-px}}{\cos x}\left(au_{xx} + bu_x + \sin x e^{px}u_y + du\right); \tag{42}$$
$${g_{4.1}}^1 = < u\partial_u,\ \partial_y,\ \partial_t,\ \partial_x + t\partial_y + yu\partial_u >,$$
$$u_t = au_{xx} + bu_x - xu_y + \left(d + \frac{x^2}{2}\right)u; \tag{43}$$
$${g_{4.1}}^2 = < u\partial_u,\ \partial_t,\ -y\partial_t + \partial_x - \frac{1}{2}y^2u\partial_u,\ \partial_y + tu\partial_u >,$$
$$u_t = x^{-1}\left(au_{xx} + bu_x + u_y + (d + xy)u\right); \tag{44}$$
$${g_{4.3}}^1 = < \partial_y,\ u\partial_u,\ \partial_t,\ \partial_x + y\partial_y + tu\partial_u >,$$
$$u_t = au_{xx} + bu_x + e^x u_y + xu; \tag{45}$$
$${g_{4.3}}^2 = < e^t\partial_y,\ u\partial_u,\ \partial_x + \partial_y + tu\partial_u,\ -\partial_t >,$$
$$u_t = au_{xx} + bu_x + (x - y)u_y + xu. \tag{46}$$

In equations (35)–(46), $a \neq 0$, b and d are arbitrary constants.

To each of equations (35)–(46) we apply equivalence transformations (7). The result is presented in Table 1.

Remark 4. *In Table 1, in the first column together with the number of a case we provide the number of an equation to which we apply the equivalence transformation.*

Since equations in cases 1, 3, 5, 6, 9, 10 and 12 are related to classes of equations (27) and (31) excluded from the consideration, it remains to study only the equations is cases 4, 7 and 8. As a result, we arrive at the following theorem.

Theorem 7. *Up to equivalence transformations (7), there exist four classes of equations (1), whose MIA are four-dimensional solvable Lie algebras of symmetry operators (19), namely,*

$$u_t = au_{xx} + e^x u_y + xu,$$
$$< \partial_t,\ \partial_y,\ \partial_x + y\partial_y + tu\partial_u,\ u\partial_u >,$$
$$u_t = u_{xx} + \ln x\, u_y + \frac{d}{x^2}u,$$
$$< \partial_t,\ \partial_y,\ 2t\partial_t + x\partial_x + (2y - t)\partial_y,\ u\partial_u >,$$
$$u_t = u_{xx} + x^k\, u_y + \frac{d}{x^2}u,\ (k-1)^2 + d^2 \neq 0,\ k \neq -2,\ 0,$$
$$< \partial_t,\ \partial_y,\ 2t\partial_t + x\partial_x + (2 + k)y\partial_y,\ u\partial_u >,$$
$$u_t = \frac{e^{-px}}{\cos x}\left(u_{xx} + \sin x\, e^{px}u_y + du\right),\ p \geqslant 0,$$
$$< \partial_t,\ \partial_y,\ (y + pt)\partial_t + \partial_x + (py - t)p_y,\ u\partial_u > .$$

Table 1. Simplification of equations (35)–(46) by equivalence transofmation (7).

	Change of variables	Equations after change
1. (35)	$\bar t = e^{-t},\ \bar x = -x - bt,\ \bar y = -e^{-t}(y + bt + b),$ $v = e^{-dt}u,\ a \to -a$	$u_t = \frac{a}{t}\, u_{xx} - xu_y$
2. (36)	$\bar t = at,\ \bar x = -x,\ \bar y = ay,$ $v = \exp\left(\frac{b}{2a}x + \left(\frac{b^2}{4a} - d\right)t\right)u$	$u_t = u_{xx} + e^x u_y$
3. (37)	$\bar t = at,\ \bar y = ay,$ $v = \exp\left(\frac{b}{2a}\,x + \left(\frac{b^2}{4a} - d\right)t\right)u$	$u_t = u_{xx} - xu_y$
4. (38)	$\bar t = \frac{a}{4}t,\ \bar x = \exp\left(\frac{x}{2}\right),\ \bar y = -\frac{a}{8}y,$ $v = \exp\left(\frac{b}{2a}x + \frac{1}{4}x\right)u$	$u_t = u_{xx} + \ln x\, u_y +$ $+\frac{d}{x^2}u$
5. (39)	$\bar t = -2t,\ \bar x = (x - b)e^{-t},$ $\bar y = -2(y + b)e^{-t},$ $v = e^{-dt}u,\ a \to -2a$	$u_t = ae^t u_{xx} - xu_y$
6. (40)	$\bar t = e^{(h-1)t},\ \bar x = (x - \frac{b}{h})e^{-ht},\ \bar y = (y + \frac{h-1}{h}b)e^{-t},$ $v = e^{-dt}u,\ (h-1)k = 1 - 3h,\ k \neq -1,\ -3$	$u_t = at^k u_{xx} - xu_y$
7. (41)	$\bar t = \frac{a}{4}t,\ \bar x = \exp\left(\frac{x}{2}\right),\ \bar y = \frac{a}{4}y,$ $v = \exp\left(\frac{b}{2a}x + \frac{1}{4}x\right)u,\ 2(h-1) = k, k \neq 0, -2$	$u_t = u_{xx} + x^k\, u_y +$ $+\frac{d}{x^2}u$
8. (42)	$\bar t = at,\ \bar y = ay, v = \exp\left(\frac{b}{2a}\,x\right)u$	$u_t = \frac{e^{-p\,x}}{\cos x}(u_{xx} +$ $+ \sin x\, e^{p\,x}u_y + du)$
9. (43)	$v = \exp\left(\frac{b}{2a}x + \left(\frac{b^2}{4a} - d\right)t\right)u$	$u_t = au_{xx} - xu_y +$ $+\frac{x^2}{2}u$
10. (44)	$\bar t = -ay,\ \bar x = x + \frac{ay^3}{3} - by,\ \bar y = at - \frac{a^2y^4}{12} + \frac{aby^2}{2},$ $v = \exp\left(\frac{xy^2}{2} + dy - \frac{by^3}{6} + \frac{ay^5}{20}\right)u$	$u_t = u_{xx} - xu_y$
11. (45)	$\bar x = x - \frac{b^2}{4a},\ \bar y = y\exp\left(-\frac{b^2}{4a}\right),$ $v = \exp\left(\frac{b}{2a}\,x\right)u$	$u_t = au_{xx} + e^x u_y +$ $+xu$
12. (46)	$\bar t = e^{-t},\ \bar x = x + at^2 + bt,$ $\bar y = ye^{-t} - \int \frac{at^2+bt}{e^t}dt,\ a \to -a$ $v = \exp\left(-tx - \frac{1}{2}bt^2 - \frac{a}{3}t^3\right)u,$	$u_t = \frac{a}{t}\, u_{xx} - xu_y$

5. Summary

Employing Zhdanov-Lahno method and known facts from group analysis of differential equations, we made the preliminary group classification of a wide class of (2 + 1)-dimensional ultraparabolic equations (1) w.r.t. solvable Lie algebras up to dimension 4. In particular, it was established that class of equations (1) (up to equivalence transformation (7)) admits one two-dimensional, six three-dimensional and four four-dimensional solvable MIA of symmetry operators (19). In order to complete the classification problem w.r.t. finite-dimensional Lie algebras, one needs to study equations (27) and (31) by the direct method and to find all simple algebras by Zhdanov-Lahno method admitting by class of equations (1).

BIBLIOGRAPHY

1. C.W. Gardiner. *Handbook of stochastic methods.* Springer, Berlin (1985).
2. A.N. Kolmogoroff. *Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung)* // Ann. Math. **35**:1, 116–117 (1934).
3. P. Wilmott, S. Howison, J. Dewynne. *Option pricing: mathematical models and computation.* Oxford financial press, Oxford (1993).
4. A. Pascucci. *Kolmogorov equations in physics and in finance* // Prog. Nonl. Diff. Equat. Appl. **63**, 353–364 (2005).
5. H. Geman, M. Yor. *Bessel processes, Asian options, and perpetuities* // Math. Fin. **3**:4, 349–375 (1993).
6. W.M. Shtelen, V.I. Stogny. *Symmetry properties of one- and two-dimensional Fokker–Planck equations* // J. Phys. A. Math. Gen. **22**:13, L539-L543 (1989).
7. S. Spichak, V. Stogny. *Symmetry analysis of the Kramers equation* // Rep. Math. Phys. **40**:1, 125–130 (1997).
8. S.V. Spichak, V.I. Stogniy, I.M. Kopas. *Symmetry properties and exact solutions of the linear Kolmogorov equation* // Research Bulletin of NTUU “Kyiv Polytechnic Institute”. **4**, 93–97 (2011). (in Ukrainian)
9. S.S. Kovalenko, I.M. Kopas, V.I. Stogniy. *Preliminary group classification of a class of generalized linear Kolmogorov equations* // Research Bulletin of NTUU “Kyiv Polytechnic Institute”. **4**:67–72, (2013). (in Ukrainian)
10. E. Barucci, S. Polidoro, V. Vespri. *Some results on partial differential equations and Asian options* // Math. Models Methods Appl. Sci. **11**:3, 475–497 (2001).
11. R.Z. Zhdanov, V.I. Lahno. *Group classification of heat conductivity equations with a nonlinear source* // J. Phys. A. Math. Gen. **32**:42, 7405-7418 (1999).
12. V.I. Lahno, S.V. Spichak, V.I. Stogny. *Symmetry analysis of evolution type equation.* Insitute of computer studies, Izhevsk (2004). (in Russian).
13. P. Basarab-Horwath, V. Lahno, R. Zhdanov. *The structure of Lie algebras and the classification problem for partial differential equations* // Acta Applicandae Mathematica. **69**:1, 43–94 (2001).
14. V. Lahno, R. Zhdanov. *Group classification of nonlinear wave equations* // J. Math. Phys. **46**:5, id 053301 (2005).
15. R. Zhdanov, V. Lahno. *Group classification of the general second-order evolution equation: semi-simple invariance groups* // J. Phys. A. Math. Theor. **40**:19, 5083-5104 (2007).
16. Q. Huang, V. Lahno, C. Z. Qu, R. Zhdanov. *Preliminary group classification of a class of fourth-order evolution equations* // J. Math. Phys. **50**:2, id 023503 (2009).
17. D.-j. Huang, H.-q. Zhang. *Preliminary group classification of quasilinear third-order evolution equations* // Appl. Math. Mech. **30**:3, 275–292 (2009).
18. N.Kh. Ibragimov. *Transformation groups applied to mathematical physics* Nauka, Moscow (1983). [Mathematics and Its Applications (Soviet Series), **3**. D. Reidel Publishing Company (1985).]
19. L.V. Ovsyannikov. *Group analysis of differential equations.* Nauka, Moscow (1978). (in Russian).
20. G.W. Bluman, S. Kumei. *Symmetries and Differential Equations.* Springer-Verlag, New York (1989).
21. P.J. Olver. *Applications of Lie groups to differential equations.* Springer, Berlin (1986).
22. G.M. Mubarakzyanov. *On solvable Lie algebras* // Izv. Vyssh. Uchebn. Zaved. Mat. 1, 114–123 (1963).

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