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EXISTENCE OF HYPERCYCLIC SUBSPACES FOR TOEPLITZ OPERATORS

A.A. LISHANSKII

In this work we construct a class of coanalytic Toeplitz operators, which have an infinite-dimensional closed subspace, where any non-zero vector is hypercyclic. Namely, if for a function φ which is analytic in the open unit disc $\mathbb{D}$ and continuous in its closure the conditions $\varphi(\mathbb{T})\cap\mathbb{T}\neq\emptyset$ and $\varphi(\mathbb{D})\cap\mathbb{T}\neq\emptyset$ are satisfied, then the operator $\varphi(S^*)$ (where S^* is the backward shift operator in the Hardy space) has the required property. The proof is based on an application of a theorem by Gonzalez, Leon-Saavedra and Montes-Rodriguez.

Keywords: Toeplitz operators, hypercyclic operators, essential spectrum, Hardy space.

1. Introduction

Let X be a separable Banach space (or a Frechet space), and let T be a bounded linear operator in X. If there exists $x\in X$ such that the set $\{T^n x, n\in\mathbb{N}_0\}$ is dense in X, then T is said to be a *hypercyclic operator* and x is called its *hypercyclic vector*. Here $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

The dynamics of linear operators and, as a special case, the theory of hypercyclic operators were actively developed for the last 20 years. A detailed review of the results up to the end of 1990-s is given in paper [1]. For a recent exposition of the theory, see monographs [2, 3].

However, first examples of hypercyclic operators appeared much earlier. In 1929, Birkhoff has shown that the translation operator $T_a: f(z)\mapsto f(z+a), a\in\mathbb{C}$, $a\neq 0$, is hypercyclic in the Frechet space of all entire functions $Hol(\mathbb{C})$ with topology of uniform convergence on the compact sets. Later, McLane proved hypercyclicity of the differentiation operator $D: f\mapsto f'$ on $Hol(\mathbb{C})$. The first example of a hypercyclic operator in the Banach setting was given in 1969 by Rolewicz [4] who showed that for each $\lambda\in\mathbb{C}$, $|\lambda|>1$, the operator λS^* is hypercyclic on $\ell^p(\mathbb{N}_0), 1\leqslant p<\infty$, where S^* is the backward shift on $\ell^p(\mathbb{N}_0)$ transforming a vector $x=(x_0,x_1,\ldots,x_n,\ldots)\in\ell^p(\mathbb{N}_0)$ to the vector $(x_1,x_2,\ldots,x_{n+1},\ldots)$.

Given a hypercyclic operator T, what can be said about the set of its hypercyclic vectors? Clearly, if x is a hypercyclic vector for operator T, then $Tx, T^2x, T^3x,\ldots$ are hypercyclic vectors for T as well. Hence, the set of hypercyclic vectors is dense when it is non-empty.

The following result was proved by Bourdon [5] (a special class of operators commuting with generalized backward shifts was previously considered by Godefroy and Shapiro in [6]).

Theorem (Bourdon, [5]**).** *Let T be a hypercyclic operator acting on a Hilbert space H. Then there exists a dense linear subspace, where any non-zero vector is hypercyclic for T.*

Definition. *Given a hypercyclic operator T, an infinite-dimensional closed subspace, in which every non-zero vector is hypercyclic for T, is called a hypercyclic subspace.*

Montes-Rodriguez [7, Theorem 3.4] proved that the operator λS^*, $|\lambda|>1$, on $\ell^2(\mathbb{N}_0)$ has no hypercyclic subspaces. However, for some class of functions of the backward shift S^* on $\ell^2(\mathbb{N})$ there exists a hypercyclic subspace, and it is the main result of the present paper. To state

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it, we need to introduce some notations. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Recall that the *disc algebra* $A(\mathbb{D})$ is the space of all functions continuous in the closed disc $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$ (with the norm $\max_{z\in\overline{\mathbb{D}}} |\varphi(z)|$).

Main Theorem. *For each function $\varphi \in A(\mathbb{D})$ such that $\varphi(\mathbb{T}) \cap \mathbb{T} \neq \emptyset$ and $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ the operator $\varphi(S^*)$ on $\ell^2(\mathbb{N}_0)$ has a hypercyclic subspace.*

Note that the $\varphi(z) = \lambda z$, $|\lambda| > 1$, does not satisfy this condition.

The examples of applying the Main Theorem may be interpreted as certain Toeplitz operator on the Hardy space. The Hardy space $H^2 = H^2(\mathbb{D})$ is the space of all functions of the form $f(z) = \sum_{n\geqslant 0} c_n z^n$ with $\{c_n\} \in \ell^2(\mathbb{N}_0)$, and thus is naturally identified with $\ell^2(\mathbb{N}_0)$. Recall that for a function $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with the symbol φ is defined as $T_\varphi f = P_+(\varphi f)$, where P_+ stands for the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. Then the backward shift on S^* may be identified with the Toeplitz operator $T_{\overline{z}}$. It was shown in [6] that each coanalytic Toeplitz operator $T_{\overline{\varphi}}$ (i.e., φ is a bounded analytic function in $\mathbb{D}$) is hypercyclic whenever $\varphi(\mathbb{D})$ intersects $\mathbb{T}$. Our Main Theorem provides a class of coanalytic Toeplitz operators having a hypercyclic subspace.

A general sufficient condition for the existence of a hypercyclic subspace was given by Gonzalez, Leon-Saavedra and Montes-Rodriguez in [8]. To state it, we need the following stronger version of hypercyclicity:

Definition. *An operator T acting on a separable Banach space $\mathcal{B}$ is said to be hereditarily hypercyclic if there exists a sequence of non-negative integers $\{n_k\}$ such that for each subsequence $\{n_{k_i}\}$ there exists a vector x such that the sequence $\{T^{n_{k_i}} x\}$ is dense in $\mathcal{B}$.*

We also need to recall the notion of the essential spectrum.

Definition. *An operator U is called Fredholm if* Ran U *is closed and has a finite codimension and* Ker U *is finite-dimensional. The essential spectrum of an operator T is defined as*

$$\sigma_e(T) = \{\lambda : T - \lambda I \text{ is non-Fredholm}\}.$$

Theorem (Gonzalez, Leon-Saavedra, Montes-Rodriguez, [8, Theorem 3.2]**).** *Let T be a hereditary hypercyclic bounded linear operator on a separable Banach space $\mathcal{B}$. Let the essential spectrum of T intersect the closed unit disc. Then there exists a hypercyclic subspace for operator T.*

We intend to use this result in the proof of the Main Theorem.

Let us mention some other results on this topic. In [9], Shkarin proved that the differentiation operator on the standard Frechet space $Hol(\mathbb{C})$ has a hypercyclic subspace. In [10, Corollary 5.5], Quentin Menet generalized this result: he proved that for each non-constant polynomial P the operator $P(D)$ has a hypercyclic subspace. He also obtained some results concerning weighted shifts on ℓ^p.

2. On essential spectra of linear operators

The following lemma is well known. We give its proof for the convenience of the reader.

Lemma. *Essential spectrum of the operator S^* is the unit circle.*

Proof. Let us consider three cases:

Case 1: $|\lambda| > 1$. The operator $S^* - \lambda I = -\lambda(I - \frac{1}{\lambda}S^*)$ is invertible and, thus, it is Fredholm.

Case 2: $|\lambda| < 1$. We have $S^* - \lambda I = S^*(I - \lambda S)$. Since the operator S^* is Fredholm (its kernel is one-dimensional, its image is the whole space ℓ^2), and $I - \lambda S$ is invertible, their composition is also a Fredholm operator.

Case 3: $|\lambda| = 1$. The operator $S^* - \lambda I$ is not Fredholm because its image has an infinite codimension.

Indeed, the pre-image of the sequence $(\lambda y_1, \lambda^2 y_2, \lambda^3 y_3, \lambda^4 y_4, \ldots) \in \ell^2$ is given by $(a, \lambda(y_1 + a), \lambda^2(y_1+y_2+a), \ldots)$ and the identity $a = -\sum_{i=1}^{+\infty} y_i$ is necessary for the inclusion of this sequence into ℓ^2.

Then the pre-image of the sequence

$$\Big(1, \frac{1}{2}, \underbrace{0, \ldots, 0}_{\geqslant 2^2-1 \text{ times}}, \frac{1}{4}, \underbrace{0, \ldots, 0}_{\geqslant 2^4-1 \text{ times}}, \ldots, \frac{1}{2^n}, \underbrace{0, \ldots, 0}_{\geqslant 2^{2^n}-1 \text{ times}}, \ldots\Big), \tag{1}$$

multiplied componentwise by $(\lambda, \lambda^2, \lambda^3, \ldots)$, is given by

$$\Big(-2, -1, \underbrace{-\frac{1}{2}, \ldots, -\frac{1}{2}}_{\geqslant 2^2 \text{ times}}, \underbrace{-\frac{1}{4}, \ldots, -\frac{1}{4}}_{\geqslant 2^4 \text{ times}}, \ldots, \underbrace{-\frac{1}{2^n}, \ldots, -\frac{1}{2^n}}_{\geqslant 2^{2^n} \text{ times}}, \ldots\Big),$$

multiplied componentwise by $(1, \lambda, \lambda^2, \ldots)$, but such sequences do not belong to ℓ^2. All sequences of the form (1), as is easily seen, form an infinite-dimensional subspace in ℓ^2. □

The following important theorem about the mapping of the essential spectra can be found, e.g., in [11, p. 107].

Essential Spectrum Mapping Theorem. *For each bounded linear operator T in a Hilbert space H and for each polynomial P, one has $\sigma_e(P(T)) = P(\sigma_e(T))$.*

3. Proof of the Main Theorem

In the proof of hereditary hypercyclicity of operator $\varphi(S^*)$ we will use the following well-known criterion due to Godefroy and Shapiro [6] (for the explicit statement see, e.g., [3, Theorem 3.1]):

Theorem (Godefroy–Shapiro criterion). *Let T be a bounded linear operator in a separable Banach space. Suppose that the subspaces*

$$X_0 = \operatorname{span}\{x \in X : Tx = \lambda x \textit{ for some } \lambda \in \mathbb{C},\ |\lambda| < 1\},$$
$$Y_0 = \operatorname{span}\{x \in X : Tx = \lambda x \textit{ for some } \lambda \in \mathbb{C},\ |\lambda| > 1\},$$

are dense in X. Then T is hereditarily hypercyclic.

Proof of the Main Theorem. We should verify two conditions of the theorem of Gonzalez, Leon-Saavedra and Montes-Rodriguez.

Each function φ in the disc-algebra can be approximated uniformly in $\overline{\mathbb{D}}$ by a sequence of polynomials P_n. Thus, $P_n(S^*)$ tends to $\varphi(S^*)$ in the operator norm.

We need to show that $\sigma_e(\varphi(S^*))$ intersects the closed unit disc. Since $\varphi(\mathbb{T}) \cap \mathbb{T} \neq \emptyset$, there exist λ, $\mu \in \mathbb{T}$ such that $\varphi(\lambda) = \mu$. Then $\mu_n = P_n(\lambda)$ tends to μ. By the Essential Spectrum Mapping Theorem, for each polynomial P one has $\sigma_e(P(S^*)) = P(\sigma_e(S^*)) = P(\mathbb{T})$. In particular, $\mu_n = P_n(\lambda) \in \sigma_e(P_n(S^*))$ for each n, and therefore, $P_n(S^*) - \mu_n I$ is not Fredholm.

Since the set of Fredholm operators is open in the operator norm (see, e.g., [12, Theorem 4.3.11]), the set of non-Fredholm operators is closed. Hence, the limit of $P_n(S^*) - \mu_n I$, which is equal to $\varphi(S^*) - \mu I$, is not Fredholm, and μ belongs to the essential spectrum of $\varphi(S^*)$. The first condition of the theorem by Gonzalez, Leon-Saavedra and Montes-Rodriguez is verified.

It is well known that the condition $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ implies that $\varphi(S^*)$ satisfies the Godefroy–Shapiro criterion. Let us briefly recall this argument.

Recall that the point spectrum of S^* equals $\sigma_p(S^*) = \{\lambda : |\lambda| < 1\}$ and an eigenvector is given by $(1, \lambda, \lambda^2, \cdots) \in \ell^2(\mathbb{N}_0)$, or, if we pass to the Hardy space $H^2(\mathbb{D})$ using the natural identification of H^2 with $\ell^2(\mathbb{N}_0)$, by

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z} = \sum_{n\geqslant 0} \lambda^n z^n.$$

These are the Cauchy kernels being reproducing kernels of H^2. Clearly, k_λ, $\lambda \in \mathbb{D}$, are also eigenvectors of $\varphi(S^*)$ associated with the eigenvalues $\varphi(\lambda)$.

By the condition $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, we know that $\varphi(\mathbb{D})$ is an open set intersecting both $\mathbb{D}$ and $\mathbb{C}\setminus\overline{\mathbb{D}}$. Clearly, both of the sets $X_0 = \{k_\lambda, \lambda \in \mathbb{D} : |\varphi(\lambda)| > 1\}$ and $Y_0 = \{k_\lambda, \lambda \in \mathbb{D} : |\varphi(\lambda)| < 1\}$ are dense in H^2. Indeed, $f \in H^2$ is orthogonal to k_λ if and only if $f(\lambda) = 0$ and both $\{\lambda \in \mathbb{D} : |\varphi(\lambda)| > 1\}$ and $\{\lambda \in \mathbb{D} : |\varphi(\lambda)| < 1\}$ are open sets. Thus the conditions of the Godefroy–Shapiro criterion are satisfied and the hereditarily hypercyclicity of operator $\varphi(S^*)$ follows.

Thus, by the theorem of Gonzalez, Leon-Saavedra and Montes-Rodriguez, the operator $\varphi(S^*)$ has a hypercyclic subspace. □

In conclusion, we formulate one open question. It would be interesting to generalize the statement of Montes-Rodriguez that the operator λS^*, $|\lambda| > 1$, on $\ell^2(\mathbb{N}_0)$, has no hypercyclic subspaces. A natural conjecture is:

Conjecture. Let $B = p(S^*)$, where p is a polynomial such that $|p(\lambda)| > 1$ for $|\lambda| = 1$. Then operator B has no hypercyclic subspaces.

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Andrei Alexandrovich Lishanskii,
SPbSU, Chebyshev laboratory,
14th Line 29B, Vasilyevsky Island,
St.Petersburg 199178, RUSSIA
E-mail: `Lishanskiyaa@gmail.com`