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ON LEFSCHETZ FORMULAS FOR FLOWS ON FOLIATED MANIFOLDS

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Abstract. The paper is devoted to the Lefschetz formulas for flows on compact manifolds, preserving a codimension one foliation and having fixed points. We develop an approach to the Lefschetz formulae based on the notion of the regularized trace on some algebra of singular integral operators introduced in a previous paper. The Lefschetz formula is proved in the case when the flow preserves a foliation given by the fibers of a fiber bundle over a circle. For a particular example of a flow on a two-dimensional torus, preserving a Reeb type foliation, we prove an analogue of the McKean-Singer formula for smoothed regularized Lefschetz functions.

Keywords: Lefschetz formula, flow, closed orbits, fixed points, foliated manifold, regularized trace.

Mathematics Subject Classification: 58J22, 37C25, 58J42

1. Introduction

The paper is devoted to the Lefschetz formulas for flows on compact manifolds, preserving a codimension one foliation. Here, by Lefschetz formulas for a flow, we mean such quantitative relations, which relate some global invariants of the flow with its closed orbits (for a general information on Lefschetz formulas for flows, see, for instance, [6, 7, 2] and references therein). A recent interest in Lefschetz type formulas for flows preserving a foliation is connected with the approach to a proof of the Riemann hypothesis suggested by Deninger in a plenary talk at the International Congress of Mathematicians in Berlin in [3]. This approach is based on an analogy of some Lefschetz formulas for flows, preserving a foliation with explicit trace formulas in number theory (for a more detailed information see also [4, 5, 11] and references therein). In particular, Deninger suggested as a conjectures a Lefschetz formula for flows preserving a foliation. In the case when the flow has no fixed points, such a formula was proved by the first author jointly with J. Alvarez Lopez in [1]. Let us describe briefly this formula.

Let X be a compact manifold of dimension n and $\mathcal{F}$ be a smooth codimension one foliation on X. Suppose that $T = \{T_t : X \to X : t \in \mathbb{R}\}$ is a flow on X satisfying the following conditions:

(P1) For every $t \in \mathbb{R}$, diffeomorphism T_t maps each leaf of $\mathcal{F}$ to a (possibly, another) leaf.

(P2) Orbits of T are transverse to the leaves of $\mathcal{F}$: i.e., for each $x \in M$,

$$T_xX = \mathbb{R}\,V(x) \oplus T_x\mathcal{F},$$

where $V \in C^\infty(X, TX)$ is the infinitesimal generator of the flow.

In particular, flow T has no fixed points.

Let us consider the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ given by the space $\Omega(\mathcal{F}) = C^\infty(X, \Lambda T^*\mathcal{F})$ of smooth leafwise differential forms on X and by the leafwise de Rham differential $d_{\mathcal{F}} : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$. Let g be a Riemannian metric on X. Denote by $\delta_{\mathcal{F}} = d^*_{\mathcal{F}}$ the corresponding leafwise de Rham codifferential and by $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplace operator. For each $u = 0, \dots, n-1$, we

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denote by $P_{\mathcal{H}^u(\mathcal{F})}$ the orthogonal projection in $L^2\Omega^u(\mathcal{F})$ on the subspace $\mathcal{H}^u(\mathcal{F}) = \ker \Delta^u_{\mathcal{F}}$ of leafwise harmonic u-forms.

For every $f \in C_0^\infty(\mathbb{R})$, we define the operator T_f in $\Omega(\mathcal{F})$ by

$$T_f = \int_{\mathbb{R}} T_t^* \cdot f(t)\,dt\,,$$

where T_t^* is the operator in $\Omega(\mathcal{F})$ induced by the action of flow T_t.

The Lefschetz function of T is defined as a distribution $L(T)$ on the real line by the identity

$$\langle L(T), f\rangle = \sum_{u=0}^{n-1}(-1)^u \operatorname{Tr}(T_f \circ P_{\mathcal{H}^u(\mathcal{F})}), \quad f \in C_0^\infty(\mathbb{R}). \tag{1}$$

Theorem 1 ([1]). *In a neighborhood of* 0 *in* $\mathbb{R}$*, the identity*

$$L(T) = \chi_\Lambda(\mathcal{F}) \cdot \delta_0$$

holds true.

Here $\delta_0 \in \mathcal{D}'(\mathbb{R})$ is the delta-function at 0 and $\chi_\Lambda(\mathcal{F}) \in \mathbb{R}$ is the L^2-Euler characteristic of $\mathcal{F}$ introduced by Connes. Note that in [1], function $L(T)$ is denoted by $\chi_{\text{dis}}(\mathcal{F})$ and is called the distributional Euler characteristic of $\mathcal{F}$.

Recall that the set $O_x = \{T_t(x) \in X : t \in \mathbb{R}\} \subset X$ is called the trajectory or the orbit of T, passing through $x \in X$. A point x is called a fixed point of flow T, if $T_t(x) = x$ for every $t \in \mathbb{R}$. In this case, $O_x = \{x\}$. An orbit O_x is called closed (or periodic), if there exists $\tau \in \mathbb{R}$ (a period) such that $T_\tau(x) = x$, and $O_x \neq \{x\}$. The least positive period of a closed orbit is called a primitive period.

A closed orbit O_x of the flow T is called simple if

$$\det(id - dT_\tau(x) : T_x\mathcal{F}^* \to T_x\mathcal{F}^*) \neq 0$$

for each period τ. We let

$$\varepsilon(\tau) = \operatorname{sgn}\, \det\,(id - dT_\tau(x) : T_x\mathcal{F}^* \to T_x\mathcal{F}^*)\,.$$

Theorem 2 ([1]). *Suppose that all closed orbits of the flow* T *are simple. Then*

$$L(T) = \sum_c \tau(c) \sum_{k\neq 0} \varepsilon(k\tau(c)) \cdot \delta_{k\tau(c)}$$

on $\mathbb{R}\setminus\{0\}$*, where* c *runs over the set of all closed orbits of the flow* T*,* $\tau(c)$ *denotes the primitive period of* c*, and* x *is an arbitrary point in* c*.*

The Lefschetz formula of [1] can be considered as an index theorem for a certain differential operator, transversally elliptic with respect to the flow (see [9] for relations with transverse index theory on foliated manifolds). The following analytic result plays an essential role in [1]. Let K be a leafwise smoothing operator on X, that is, an operator in $C^\infty(X)$ given by a family of proper integral operators with smooth kernel acting along the leaves of the foliation. Then, for each function $f \in C_0^\infty(\mathbb{R})$, operator $T_f \circ K$ is an integral operator with smooth kernel. This fact allows us to define a distribution $f \in C_0^\infty(\mathbb{R}) \mapsto \operatorname{Tr} T_f \circ K$ on $\mathbb{R}$, where Tr denotes the trace of $T_f \circ K$ as a trace class operator in Hilbert space $L^2(X)$.

The arguments adduced in [3, 4] show that, if we want to treat the explicit trace formula for the Riemann zeta function as a Lefschetz formula for a certain flow preserving a foliation, then the flow necessarily must have fixed points. In this paper we are interested in generalizations of the Lefschetz formula for a flow on a compact manifold with codimension one foliation to the case when the flow has fixed points. More precisely, we consider a flow T on a compact manifold X of dimension n equipped with a codimension one foliation $\mathcal{F}$ satisfying Condition (P1) as well as the following conditions:

(P3) The flow T has finitely many fixed points being simple.

(P4) The orbits of T are transverse to all leaves of the foliations, except the leaves containing the fixed points of the flow.

Recall that a fixed point x of the flow T is called simple if for each $t > 0$

$$\det(Id - dT_t(x) : T_xX \to T_xX) \neq 0.$$

The present work is a part of a joint project with J. Alvarez Lopez and E. Leichtnam, in which there was proposed a new approach to Lefschetz formulas for the flows satisfying Conditions (P1), (P3), (P4). This approach is as follows. First of all, we observe that, under Conditions (P1), (P3), (P4), an operator of the form $T_f \circ K$, where K is a leafwise smoothing operator, is not, in general, an operator with smooth kernel, because its kernel may have singularities near the leaves containing fixed points of the flow. The idea consists in extending the trace functional defined on operators with smooth kernel to a functional r-Tr, called the regularized trace, defined on a wider class of integral operators, which includes operators of the form $T_f \circ K$. Unfortunately, projections $P_{\mathcal{H}^u(\mathcal{F})}$ are not always nice leafwise smoothing operators. However, if this is the case and the regularized trace of an operator of the form $T_f \circ P_{\mathcal{H}^u(\mathcal{F})}$ is well defined, then we can define an analogue of the Lefschetz function of flow T, the regularized Lefschetz function $L(T) \in \mathcal{D}'(\mathbb{R})$, as

$$\langle L(T), f\rangle = \sum_{u=0}^{n-1} (-1)^u \,\text{r-Tr}(T_f \circ P_{\mathcal{H}^u(\mathcal{F})}), \quad f \in C_0^\infty(\mathbb{R}). \tag{2}$$

In the general case, one can try to use well elaborated methods of the index theory, replacing $P_{\mathcal{H}^u(\mathcal{F})}$ with an operator of the form $\psi(t\Delta_{\mathcal{F}})$, where ψ is a sufficiently nice function on the real line and $t > 0$; for instance, with the operator $e^{-t\Delta_{\mathcal{F}}}$.

Regularized trace functional r-Tr and the corresponding algebra of singular integral operators $\mathcal{K}(X, X^0)$ on an arbitrary compact manifold X having singularities at some smooth codimension one submanifold X^0 were constructed in [10] by using ideas of the papers by Melrose [12, 13, 14], in particular, a geometric approach to constructing and studying algebras of singular integral operators suggested in these papers. It should be noted that functional r-Tr does not have the trace property, but there is a formula for the regularized trace of the commutator of operators in terms of some integral operators with smooth kernel on X^0.

The purpose of this paper is a realization of the approach mentioned above for a flow T satisfying Conditions (P1), (P3), (P4) in two cases.

First of all, we consider the case when foliation $\mathcal{F}$ is given by the fibers of a fibration over the circle S^1. It is shown in this case that, for each $f \in C_0^\infty(\mathbb{R})$ and for each leafwise smoothing operator K, operator $T_f \circ K$ belongs to the algebra $\mathcal{K}(X, X^0)$, where X^0 is the union of all leaves containing fixed points of the flow. Moreover, operator $P_{\mathcal{H}^u(\mathcal{F})}$ is a leafwise smoothing operator. These facts allow us to define regularized Lefschetz function $L(T) \in \mathcal{D}'(\mathbb{R})$ by (2). By an explicit computation of the regularized trace of $T_f \circ K$, we obtain the following Lefschetz formula (see Theorem 7 for a more detailed formulation).

Theorem 3. *We have the formula*

$$L(T) = C\chi(F)\delta_0,$$

where $\chi(F)$ is the Euler characteristic of a fiber F and C is some constant, depending only on the flow.

Then we consider the simplest particular example of a foliation $\mathcal{F}$, which is not given by the fibers of a fiber bundle. Namely, we consider the foliation $\mathcal{F}$ on the two-dimensional torus $X = \mathbb{T}^2 = \mathbb{R}^2/(2\mathbb{Z}\times\mathbb{Z})$, whose lift on $\mathbb{R}^2$ under the natural map $\mathbb{R}^2 \to \mathbb{Z}^2$, a foliation $\widetilde{\mathcal{F}}$ on $\mathbb{R}^2$, is given by the level sets of the map $p : \mathbb{R}^2 \to \mathbb{R}$, $p(y, z) = e^z \cos\left(\frac{\pi}{2}y\right)$. This foliation has one compact leaf $X^0 = \{(y, z) \in \mathbb{T}^2 : y = 1\}$, all other leaves are non-compact and wrap it. Note that $\mathcal{F}$ is not transversely orientable. One can explicitly construct a class of flows on X satisfying Conditions (P1), (P3), (P4). Fixed points of each such flow belong to X^0.

In this case, one can also prove that, for each $f \in C_0^\infty(\mathbb{R})$ and for each leafwise smoothing operator K, operator $T_f \circ K$ belongs to the algebra $\mathcal{K}(X, X^0)$, where X^0 is the union of all leaves, containing fixed points of the flow, and, therefore, its regularized trace is well defined. Nevertheless, an explicit computation of the regularized trace of $T_f \circ K$ seems to be a quite complicated problem.

Another problem is that, in this case, projection $P_{\mathcal{H}^u(\mathcal{F})}$ is not, in general, a leafwise smoothing operator. This is why, instead of it, we consider operator $\psi(t\Delta_{\mathcal{F}})^2$, where ψ is a sufficiently nice function on the real line and $t>0$, and the corresponding smoothed regularized Lefschetz function $L_{t,\psi}\in\mathcal{D}'(\mathbb{R})$ given by (in the case under consideration $n=2$)

$$\langle L_{t,\psi}, f\rangle = \sum_{u=0}^{n-1}(-1)^u \,\text{r-Tr}(T_f\circ\psi(t\Delta^u_{\mathcal{F}})^2). \tag{3}$$

An important fact in the proof of the Lefschetz formula in [1] is that the smoothed Lefschetz function $L_{t,\psi}$ (defined by means of the usual trace) is independent of t (an analogue of the well known McKean-Singer formula). This fact makes an essential use of the trace property of Tr. Since r-Tr does not have trace property, the smoothed regularized Lefschetz function $C_{t,\psi,f}$, in general, depends on t. Using the formula for the regularized trace of the commutator obtained in [10], in Proposition 6 we prove a formula for the derivative of $C_{t,\psi,f}$ with respect to t in terms of traces of some integral operators on the circle. A more explicit formula for $\frac{d}{dt}C_{t,\psi,f}$ is obtained in the case when $\psi(x)=e^{-\frac{x}{2}}$, that is, when $\psi(t\Delta^u_{\mathcal{F}})^2=e^{-t\Delta^u_{\mathcal{F}}}$ (see Theorem 8 below). The next natural step is the study of the asymptotic behavior of $C_{t,\psi,f}$ as $t\to 0$ and as $t\to+\infty$, which is a rather difficult problem and it will not be discussed in this paper.

The outline of the paper is as follows. In Section 2, we recall some necessary facts from [10]. In Section 3, we treat the case when the foliation $\mathcal{F}$ is given by the fibers of a fiber bundle over the circle S^1. Section 4 is devoted to the study of a flow on the Reeb foliation. At the end of Section 4, we discuss an analogue of the McKean-Singer formula for the example considered in Section 3.

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2. Preliminaries

In this section, we remind necessary basic notions from [10].

2.1. Algebra $\mathcal{K}(X,X^0;E)$. Let X be a smooth compact manifold of dimension n, X^0 its smooth codimension one submanifold, E a Hermitian vector bundle on X. Suppose that there is given a Riemannian metric g_X on X and $N(X^0)$ is oriented.

We shall consider operator, acting on half-densities. Denote by $\Omega^{\frac12}_X$ the half-density bundle on X. Consider an operator $A: C^\infty_0(X\setminus X^0, E\otimes\Omega^{\frac12}_X)\to C^\infty(X\setminus X^0, E\otimes\Omega^{\frac12}_X)$ with the kernel

$$k_A\in C^\infty\left((X\times X)\setminus(\{X^0\times X\}\cup\{X\times X^0\}), \mathcal{L}(E)\otimes\Omega^{\frac12}_{X\times X}\right),$$

where $\mathcal{L}(E)$ is the vector bundle on $X\times X$, whose fiber at $(p_1,p_2)\in X\times X$ consists of linear maps from E_{p_2} to E_{p_1}. The action of A on a half-density $\mu\in C^\infty_0(X\setminus X^0, E\otimes\Omega^{\frac12}_X)$ is given by

$$A\mu=\int_X k_A\mu. \tag{4}$$

From now on, $|dx^0|$ is a fixed smooth positive density on X^0.

We shall need some special coordinates on X defined near X^0. Let $\exp: N(X^0):=TX/TX^0\cong (TX^0)^\perp\to X$ be the exponential map of the Riemannian metric g_X for the submanifold X^0. We shall identify X^0 with the zero section of the bundle $N(X^0)$, that allows us to consider it as a submanifold of $N(X^0)$. It is well known that there exists a neighborhood $U\supset X^0$ in $N(X^0)$ such that the restriction $\exp|_U$ to U is a diffeomorphism of U on some neighborhood $\exp(U)=:V$ of X^0 in X called a tubular neighborhood of X^0. One can associate to every $p\in V$ a pair $(x,x^0)\in N(X^0)$, where $x^0\in X^0$ and $x\in N_{x^0}(X^0)$, $\exp(x)=p$. Since the normal bundle $N(X^0)$ is oriented, the Riemannian metric gives rise to an isomorphism $N_{x^0}(X^0)\cong\mathbb{R}$ and one can assume that $x\in\mathbb{R}$. Thus, the point p is uniquely determined by the pair (x,x^0), where $x\in\mathbb{R}$, $x^0\in X^0$. Without loss of generality, one can assume that $p\in V$ if and only if $x\in(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$. We shall call the map $V\to(-\varepsilon,\varepsilon)\times X^0$, $p\mapsto(x,x^0)$ a normal coordinate system near X^0.

If E is a Hermitian vector bundle on X, then we can assume that there is an isomorphism $E_{(x,x^0)} \cong E_{(0,x^0)} = E^0_{x^0}$ for each $x \in (-\varepsilon, \varepsilon)$, where E^0 is the restriction of E to X^0. In this case, we shall say that $\exp(U)$ is endowed with an adapted trivialization of the bundle E.

Let us take a normal coordinate system with coordinates $(x, x^0) \in (-\varepsilon, \varepsilon) \times X^0$ and an adapted trivialization of E in a tubular neighborhood V of X^0. Let (x_1, x_2, x^0_1, x^0_2) be the corresponding coordinates on $V \times V$. Kernel k_A is written as

$$k_A = K_A(x_1, x_2, x^0_1, x^0_2) \left| \frac{dx_1}{x_1} \frac{dx_2}{x_2} dx^0_1 dx^0_2 \right|^{\frac{1}{2}},$$

where $K_A(x_1, x_2, x^0_1, x^0_2)$ is a linear map from $E_{(x_2,x^0_2)} \cong E^0_{x^0_2}$ to $E_{(x_1,x^0_1)} \cong E^0_{x^0_1}$ for each (x_1, x_2, x^0_1, x^0_2). Introduce a coordinate system $(x, s, x^0_1, x^0_2) \in ((-\varepsilon, \varepsilon)\setminus\{0\}) \times (\mathbb{R}\setminus\{0\}) \times X^0 \times X^0$ on $(V \setminus X^0) \times (V \setminus X^0)$ by

$$x = x_1, \quad s = \frac{x_1}{x_2}. \tag{5}$$

In the local coordinate system (x, s, x^0_1, x^0_2), kernel k_A is written as

$$k_A = \widetilde{K}_A(x, s, x^0_1, x^0_2) \left| \frac{dx}{x} \frac{ds}{s} dx^0_1 dx^0_2 \right|^{\frac{1}{2}},$$

where, for each $(x, s, x^0_1, x^0_2) \in (-\varepsilon, \varepsilon) \times (\mathbb{R} \setminus \{0\}) \times X^0 \times X^0$,

$$\widetilde{K}_A(x, s, x^0_1, x^0_2) = K_A(x, \frac{x}{s}, x^0_1, x^0_2) : E^0_{x^0_2} \to E^0_{x^0_1}. \tag{6}$$

Let $\mu \in C^\infty_0(X, E \otimes \Omega^{\frac{1}{2}}_X)$, $\operatorname{supp} \mu \subset V$. Write $\mu = u(x, x^0) \left| \frac{dx}{x} dx^0 \right|^{\frac{1}{2}}$, where $u \in C^\infty_0(V, E) \cong C^\infty_0((-\varepsilon, \varepsilon) \times X^0, E^0)$. Then

$$A\mu\Big|_V = \left(\int_{X^0} \int_{-\infty}^{+\infty} \widetilde{K}_A(x, s, x^0_1, x^0_2) u\left(\frac{x}{s}, x^0_2\right) \frac{ds}{s} dx^0_2 \right) \left| \frac{dx}{x} dx^0_1 \right|^{\frac{1}{2}}.$$

Definition 1. *We say that $A \in \mathcal{K}(X, X^0, E)$, if the following conditions hold:*

1. *For each $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\varrho(x, X^0) > \varepsilon$, $\varrho(y, X^0) < \delta$ or $\varrho(y, X^0) > \varepsilon$, $\varrho(x, X^0) < \delta$, then $k_A(x, y) = 0$ (here ϱ is the geodesic distance defined by g_X).*
2. *Function $\widetilde{K}_A(x, s, x^0_1, x^0_2)$ defined by (6) is smooth on $(-\varepsilon, \varepsilon) \times (\mathbb{R} \setminus \{0\}) \times X^0 \times X^0$.*
3. *There exist m, M, $0 < m < M < \infty$ such that function $\widetilde{K}_A$ is supported in the set of all $(x, s, x^0_1, x^0_2) \in ((-\varepsilon, \varepsilon) \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \times X^0 \times X^0$ such that $m < |s| < M$.*

Set $\mathcal{K}(X, X^0, E)$ is an algebra.

2.2. Regularized trace. Operators from $\mathcal{K}(X, X^0, E)$, in general, are not trace class. In this section we introduce a functional on $\mathcal{K}(X, X^0, E)$, called the regularized trace functional, which coincides with the trace functional on trace class operators.

Let X be a compact manifold, X^0 its smooth codimension one submanifold, E a Hermitian vector bundle on X, g_X a Riemannian metric on X. A density μ on X is called a smooth relative density, if, in a normal coordinate system near X^0, it is written as

$$\mu = u(x, x^0) \left| \frac{dx}{x} dx^0 \right|, \quad (x, x^0) \in (-\varepsilon, \varepsilon) \times X^0, \tag{7}$$

where $|dx^0|$ is a fixed smooth density on X^0 and u is a smooth function on $(-\varepsilon, \varepsilon) \times X^0$. For a smooth relative density μ, define a density $\mu\,|_{X^0}$ on X^0 as follows. If μ is written in the form (7), then

$$\mu\,|_{X^0} = u(0, x^0)|dx^0|.$$

Define a continuous function r on X by the formula $r(p) = \varrho(p, X^0)$, where ϱ is the geodesic distance from p to X^0.

Definition 2. *The regularized integral of a smooth relative density μ over X is defined by*

$$\int\limits_X^r \mu = \lim_{\varepsilon \to 0} \Big(\int\limits_{\substack{X \\ r(p)>\varepsilon}} \mu + 2 \ln \varepsilon \int\limits_{X^0} \mu \,|_{X^0} \Big). \tag{8}$$

For kernel k_A of an operator $A \in \mathcal{K}(X, X^0; E)$, one can naturally define its restriction to the diagonal $\Delta = \{(p, p) \in X \times X : p \in X\} \cong X$ as a matrix-valued density $k_A \,|_\Delta \in C^\infty \big(X \setminus X^0, \mathcal{L}(E) \otimes \Omega_X \big)$ on $X \setminus X^0$. One can prove that its pointwise trace $\operatorname{tr} k_A \,|_\Delta \in C^\infty \big(X \setminus X^0, \Omega_X \big)$ is a smooth relative density on X that allows us to give the following definition.

Definition 3. *The regularized trace of an operator $A \in \mathcal{K}(X, X^0; E)$ with kernel k_A is defined by*

$$\text{r-Tr}(A) = \int\limits_X^r \operatorname{tr} k_A \,|_\Delta \,.$$

As mentioned above, there is a formula for the regularized trace of the commutator of operators $A, B \in \mathcal{K}(X, X^0; E)$ in terms of certain families of integral operators with smooth kernel on X^0 associated with A and B.

Let $A \in \mathcal{K}(X, X^0; E)$ and let $\widetilde{K}_A$ be defined by (6). Then there exists the limit

$$\lim_{x \to 0} \widetilde{K}_A(x, s, x_1^0, x_2^0) =: \widetilde{K}_A(0, s, x_1^0, x_2^0) : E^0_{x_2^0} \to E^0_{x_1^0}. \tag{9}$$

Definition 4. *The indicial operator associated with an operator $A \in \mathcal{K}(X, X^0, E)$ is the operator $I(A)$ acting on a half-density $\mu \in C_0^\infty((\mathbb{R} \setminus \{0\}) \times X^0, \pi_2^* E^0 \otimes \Omega^{\frac12}_{(R \setminus \{0\}) \times X^0})$ of the form $\mu = u(x, x^0) \left| \frac{dx}{x} dx^0 \right|^{\frac12}$ by the formula*

$$I(A)\mu = I(A) u(x, x^0) \left| \frac{dx}{x} dx^0 \right|^{\frac12} \in C_0^\infty((\mathbb{R} \setminus \{0\}) \times X^0, \pi_2^* E^0 \otimes \Omega^{\frac12}_{\mathbb{R} \setminus \{0\} \times X^0}),$$

where

$$I(A) u(x, x^0) = \int\limits_{X^0} \int\limits_{-\infty}^{+\infty} \widetilde{K}_A(0, s, x^0, x_1^0) u\left(\frac{x}{s}, x_1^0 \right) \frac{ds}{s} dx_1^0, \quad x \in \mathbb{R} \setminus \{0\}, x^0 \in X^0.$$

Here $\pi_2^* E^0$ denotes the bundle on $(\mathbb{R} \setminus \{0\}) \times X^0$, which is the pull-back of the bundle E^0 under the projection $\pi_2 : (\mathbb{R} \setminus \{0\}) \times X^0 \to X^0$: $(\pi_2^* E^0)_{(x, x^0)} = E^0_{x^0}$.

Definition 5. *The indicial families of an operator $A \in \mathcal{K}(X, X^0, E)$ are the families $\{I^\pm(A, \lambda) : \lambda \in \mathbb{C}\}$ of integral operators in the space $C^\infty(X^0, E^0)$ with smooth kernels given by*

$$K_{I^+(A,\lambda)}(x_1^0, x_2^0) = \int\limits_0^{+\infty} s^{-i\lambda} \widetilde{K}_A(0, s, x_1^0, x_2^0) \frac{ds}{s} : E^0_{x_2^0} \to E^0_{x_1^0},$$

$$K_{I^-(A,\lambda)}(x_1^0, x_2^0) = \int\limits_{-\infty}^{0} |s|^{-i\lambda} \widetilde{K}_A(0, s, x_1^0, x_2^0) \frac{ds}{|s|} : E^0_{x_2^0} \to E^0_{x_1^0}.$$

Using the Paley-Wiener theorem, it is easy to show that the functions $K_{I^\pm(A,\lambda)}$ are well defined for each $\lambda \in \mathbb{C}$ and are entire functions of λ.

The following properties of the indicial operators hold:

1. $I(A \circ B) = I(A) \circ I(B)$.
2. $I^\pm(A \circ B, \lambda) = I^+(A, \lambda) \circ I^\pm(B, \lambda) + I^-(A, \lambda) \circ I^\mp(B, \lambda)$.

Theorem 4. *If $A \in \mathcal{K}(X, X^0, E)$ and $B \in \mathcal{K}(X, X^0, E)$, then*

$$\text{r-Tr}([A,B]) = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} \text{Tr}(\partial_\lambda I^+(A,\lambda)\circ I^+(B,\lambda) + \partial_\lambda I^-(A,\lambda)\circ I^-(B,\lambda))d\lambda,$$

where Tr *denotes the trace of integral operator in $C^\infty(X^0, E^0)$.*

Integrating by parts in λ, one can rewrite the formula as

$$\text{r-Tr}([A,B]) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \text{Tr}(I^+(A,\lambda)\circ \partial_\lambda I^+(B,\lambda) + I^-(A,\lambda)\circ \partial_\lambda I^-(B,\lambda))d\lambda.$$

3. Flows on fiber bundles

3.1. A flow on a fiber bundle and operators associated with it. Suppose that a compact manifold X of dimension n is a total space of a fiber bundle $\pi : X \to S^1$ over the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with a compact manifold F as fiber and a foliation $\mathcal{F}$ is given by the fibers of π.

Suppose that a flow T on X satisfies Conditions (P1), (P3), (P4). Then there exists a flow $\overline{T}$ on S^1 such that $\pi(T_t(x)) = \overline{T}_t(\pi(x))$ for each $x \in X$. We write the infinitesimal generator $\overline{V} = \frac{d}{dt}\overline{T}_t|_{t=0}$ of the flow $\overline{T}$ as

$$\overline{V}(y) = a(y)\frac{\partial}{\partial y}, \qquad y \in S^1,$$

where $a \in C^\infty(S^1)$. By Condition (P3), flow $\overline{T}$ has finitely many fixed points $\alpha_1, \ldots, \alpha_k \in S^1$, which are non-degenerate, that is, $a(\alpha_j) = 0$ and $a'(\alpha_j) \neq 0$ for each $j = 1, \ldots, k$. The points α_j corresponds to the leaves of $\mathcal{F}$ containing fixed points of flow T.

In a trivialization $\pi^{-1}(U) \cong F \times U$ of the fiber bundle π, the infinitesimal generator $V = \frac{d}{dt}T_t|_{t=0}$ of T reads as

$$V(y) = v_0(y) + a(y)\frac{\partial}{\partial y}, \quad y \in U, \tag{10}$$

where v_0 is a vector field on $\pi^{-1}(U)$ tangent to the fibers of π (that is, a smooth family of vector fields on F parametrized by $y \in U$). In other words, V is a transversally projectable vector field on X.

Let E be a complex vector bundle on X. Suppose that there is given a flow T^E on E, which covers the flow T on X, and, in addition, the map $T^E_t(x) : E_x \to E_{T_t(x)}$ in the fibers of E induced by the flow is linear. Denote by $T^*_t : C^\infty(X, E) \to C^\infty(X, E)$ the operator induced by flow T:

$$T^*_t u(x) = r_t(x)[u(T_t(x))],$$

where $r_t(x) = T^E_{-t}(T_t(x)) : E_{T_t(x)} \to E_x$.

Denote by $\mathcal{V} \subset TX$ the subbundle of the tangent bundle TX consisting of vectors tangent to the fibers of π. Let $|\mathcal{V}|^s$ be the s-density bundle associated with $\mathcal{V}$ (the bundle of fiberwise s-densities). Fix a smooth positive section α of the bundle of fiberwise densities $|\mathcal{V}|$. It is given by a smooth family $\{\alpha_y : y \in S^1\}$, where α_y is a smooth positive density on F_y. The product of α with $|dy|$ gives rise to a well defined smooth positive density $|d\nu| = |d\alpha| \otimes |dy|$ on X. We shall identify the space of smooth half-densities $C^\infty(X, \Omega^{\frac{1}{2}}_X)$ with $C^\infty(X)$ by using the fixed half-density $|d\nu|^{1/2} = |d\alpha|^{1/2} \otimes |dy|^{1/2}$.

For a function $f \in C^\infty_0(\mathbb{R})$, define a linear operator T_f in $C^\infty(X, E \otimes \Omega^{\frac{1}{2}}_X)$ by the formula: for $\mu = u|d\alpha|^{1/2} \otimes |dy|^{1/2} \in C^\infty(X, E \otimes \Omega^{\frac{1}{2}}_X)$

$$T_f\mu = \left(\int_{-\infty}^{+\infty} f(t)T^*_t u\,dt\right)|d\alpha|^{1/2} \otimes |dy|^{1/2}. \tag{11}$$

3.2. Leafwise smoothing operators. By definition, a leafwise smoothing operator $K : C^\infty(X, E\otimes\Omega_X^{\frac{1}{2}}) \to C^\infty(X, E\otimes\Omega_X^{\frac{1}{2}})$ is given by an element k of the space $C^\infty(X\times_\pi X, \mathcal{L}(E)\otimes|\mathcal{V}|^{\frac{1}{2}}\otimes|\mathcal{V}|^{\frac{1}{2}})$ by the formula:

$$K\mu = \int\limits_F k\mu, \tag{12}$$

where $X\times_\pi X = \{(p_1,p_2)\in X\times X : \pi(p_1)=\pi(p_2)\}$ and $\mathcal{L}(E)$ is the vector bundle on $X\times_\pi X$, whose fiber at $(p_1,p_2)\in X\times_\pi X$ consists of linear maps from E_{p_2} to E_{p_1}, $\mu\in C^\infty(X, E\otimes\Omega_X^{\frac{1}{2}})$, and the integral should be understood as follows.

Let $\pi^{-1}(U)\subset X\cong F\times U$ be a trivialization of π over some open set $U\subset S^1$. The choice of a trivialization determines a diffeomorphism $\pi^{-1}(U)\times_\pi\pi^{-1}(U)\cong F\times F\times U$. Namely, to each pair $(p_1,p_2)\in\pi^{-1}(U)\times_\pi\pi^{-1}(U)$ we associate a triplet (x_1^0,x_2^0,y), where $(x_1^0,y_1)\in F\times U$ corresponds to p_1, and $(x_2^0,y_2)\in F\times U$ to p_2, moreover, $y_1=y_2=y$. Fix a smooth positive density $|dx^0|$ on F. The half-densities k and μ can be written as

$$k = k(x_1^0,x_2^0,y)|dx_1^0|^{\frac{1}{2}}|dx_2^0|^{\frac{1}{2}},\quad \mu = u(x_2^0,y)|dx^0|^{\frac{1}{2}}|dy|^{\frac{1}{2}},\quad x_1^0,x_2^0\in F,\quad y\in U.$$

Let

$$\int\limits_F k\mu = \left(\int\limits_F k(x_1^0,x_2^0,y)u(x_2^0,y)|dx_2^0|\right)|dx_1^0|^{\frac{1}{2}}|dy|^{\frac{1}{2}},\quad x_1^0\in F,\quad y\in U,$$

where the integral in the right-hand side of the equality should be understood as the integral of a smooth density on F with values in $E_{(x_1^0,y)}$. It is easy to see that this definition is correct, that is, independent of the choice of the trivialization and density $|dx^0|$.

3.3. Operators $T_f\circ K$. Fix the standard Riemannian metric $g_{S^1}=dy^2$ on S^1. Choose a Riemannian metric g_X on X such that the map $\pi:(X,g_X)\to(S^1,g_{S^1})$ is a Riemannian submersion, that is, for each tangent vector $V\in T_pX$ orthogonal to the fiber of π and passing through p, we have the identity $\|V\|_{g_X}=\|d\pi_p(V)\|_{g_{S^1}}$. Consider the smooth codimension one submanifold $X^0=\bigcup\limits_{j=1}^k F_{\alpha_j}$. The following statement holds.

Theorem 5. *Suppose that flow T satisfies Conditions (P1), (P3), (P4). Then, for each leafwise smoothing operator K and for each function $f\in C_0^\infty(\mathbb{R})$, operator $T_f\circ K$ belongs to $\mathcal{K}(X,X^0,E)$.*

Proof. For each $\ell=1,\dots,k$, the set $U_\ell=S^1\setminus\alpha_\ell$ is contractible. Therefore, the restriction of π to U_ℓ is trivial, that is, there exists a diffeomorphism $\pi^{-1}(U_\ell)\cong F\times U_\ell$ such that $F_\alpha=\pi^{-1}(\alpha)\cong F\times\{\alpha\}$.

To be definite, put $\ell=1$, $U_\ell=U$ and write the action of T_f on a half-density μ in the trivialization. For each $t\in\mathbb{R}$ and for each $y\in(\alpha_j,\alpha_{j+1})$, $j=1,\dots,k$ $(\alpha_{k+1}=\alpha_1)$, we have $\overline{T}_t(y)\in(\alpha_j,\alpha_{j+1})$. Therefore, diffeomorphism T_t is written as

$$T_t(x^0,y)=(S_t(x^0,y),\overline{T}_t(y)),\quad (x^0,y)\in F\times U,$$

and the leafwise density α as $d\alpha=w(x^0,y)|dx^0|$. Then, for each half-density $\mu=u(x^0,y)|dx^0|^{\frac{1}{2}}|dy|^{\frac{1}{2}}$, we have

$$(T_f\circ K)\mu=[(T_f\circ K)u(x^0,y)]|dx^0|^{\frac{1}{2}}|dy|^{\frac{1}{2}},$$

where

$$(T_f\circ K)u(x^0,y)=\int\limits_{-\infty}^{+\infty} f(t)\int\limits_F r_t(x^0,y)k(S_t(x^0,y),x_1^0,\overline{T}_t(y))$$
$$\cdot\, u(x_1^0,\overline{T}_t(y))|w(x^0,y)|^{1/2}|w(x_1^0,\overline{T}_t(y))|^{\frac{1}{2}}dt|dx_1^0|, \tag{13}$$

where $k=k(x^0,x_1^0,y)|w(x^0,y)|^{\frac{1}{2}}|w(x_1^0,y)|^{\frac{1}{2}}|dx^0|^{\frac{1}{2}}|dx_1^0|^{\frac{1}{2}}$, $(x^0,y)\in F\times U$, is the kernel of the leafwise smoothing operator K. For a fixed $y\in S^1\setminus\{\alpha_1,\dots,\alpha_k\}$, say, $y\in(\alpha_1,\alpha_2)$, we make the change of variables $t\in(-\infty,+\infty)\mapsto y_1=\overline{T}_t(y)\in(\alpha_1,\alpha_2)$ in the integral in the right hand side of (13). Since $\frac{dy_1}{dt}=a(\overline{T}_t(y))=a(y_1)\neq 0$, t can be expressed in terms of y_1:

$$t=\Psi(y,y_1):=\int\limits_y^{y_1}\frac{dz}{a(z)},\quad y,y_1\in(\alpha_1,\alpha_2). \tag{14}$$

It is easy to see that $\Psi\in C^\infty((\alpha_1,\alpha_2)\times(\alpha_1,\alpha_2))$. Therefore, we get:

$$(T_f\circ K)u(x^0,y)=\int\limits_{[\alpha_1,\alpha_2]} f(\Psi(y,y_1))\int\limits_F r_{\Psi(y,y_1)}(x^0,y)k(S_{\Psi(y,y_1)}(x^0,y),x_1^0,y_1)$$
$$\cdot\, u(x_1^0,y_1)|w(x^0,y)|^{1/2}|w(x_1^0,y_1)|^{\frac12}\frac{1}{|a(y_1)|}|dx_1^0||dy_1|.$$

Thus, the kernel of $T_f\circ K$ reads as

$$k_f(x^0,x_1^0,y,y_1)=f(\Psi(y,y_1))r_{\Psi(y,y_1)}(x^0,y)k(S_{\Psi(y,y_1)}(x^0,y),x_1^0,y_1)$$
$$\cdot\,|w(x^0,y)|^{1/2}|w(x_1^0,y_1)|^{\frac12}\frac{1}{|a(y_1)|}|dx_1^0|^{\frac12}|dy_1|^{\frac12}|dx^0|^{\frac12}|dy|^{\frac12}, \tag{15}$$

if $x^0,x_1^0\in F$, $y,y_1\in(\alpha_1,\alpha_2)$, and

$$k_f(x^0,x_1^0,y,y_1)=0,$$

if $x^0,x_1^0\in F$, $y\in(\alpha_1,\alpha_2)$, $y_1\notin(\alpha_1,\alpha_2)$. In particular, the kernel of $T_f\circ K$ is a smooth half-density on $X\times X\setminus(X^0\times X)\cup(X\times X^0)$.

In this case, Condition (1) of Definition 1 follows immediately from the fact that function f is compactly supported and, for each $y\in(\alpha_j,\alpha_{j+1})$,

$$\lim_{y_1\to\alpha_j}\Psi(y,y_1):=\int\limits_y^{\alpha_j}\frac{dz}{a(z)}=\infty,\quad \lim_{y_1\to\alpha_{j+1}}\Psi(y,y_1):=\int\limits_y^{\alpha_{j+1}}\frac{dz}{a(z)}=\infty.$$

Fix $j=1,\dots,k$. Since the map π is a Riemannian submersion, the normal coordinates $(x,x^0)\in(-\varepsilon,\varepsilon)\times F_{\alpha_j}$ determine a trivialization of π over the ball $B(\alpha_j,\varepsilon)\subset S^1$ of radius ε centered in $\alpha_j\in S^1$. Namely, the trivialization $V=\pi^{-1}(B(\alpha_j,\varepsilon))\cong F_{\alpha_j}\times B(\alpha_j,\varepsilon)$ maps a point $p\in\pi^{-1}(B(\alpha_j,\varepsilon))$ with normal coordinates $(x,x^0)\in(-\varepsilon,\varepsilon)\times F_{\alpha_j}$ into $(x^0,x+\alpha_j)\in F_{\alpha_j}\times B(\alpha_j,\varepsilon)$. It is easy to see that the formula (15) holds in this trivialization for each $x^0,x_1^0\in F_{\alpha_j}$ and $y,y_1\in B(\alpha_j,\varepsilon)$. We shall assume that $B(\alpha_j,\varepsilon)\subset(\alpha_{j-1},\alpha_{j+1})$

Let (x_1,x_2,x_1^0,x_2^0) be the corresponding coordinates on $V\times V$. Introduce a coordinate system $(x,s,x_1^0,x_2^0)\in((-\varepsilon,\varepsilon)\setminus\{0\})\times(\mathbb{R}\setminus\{0\})\times F_{\alpha_j}\times F_{\alpha_j}$ on $(V\setminus F_{\alpha_j})\times(V\setminus F_{\alpha_j})$ by

$$x=x_1=y-\alpha_j,\quad s=\frac{y-\alpha_j}{y_1-\alpha_j}. \tag{16}$$

We consider the function $\widetilde{K}_f$ on $(-\varepsilon,\varepsilon)\times(\mathbb{R}\setminus\{0\})\times X^0\times X^0$, defined by kernel k_f via formula (6) . Since k_f is supported in $\bigsqcup_j\pi^{-1}(\alpha_j,\alpha_{j+1})\times\pi^{-1}(\alpha_j,\alpha_{j+1})$, $\widetilde{K}_f(x,s,x^0,x_1^0)=0$ for $s<0$.

For $s>0$, function $\widetilde{K}_f$ reads as

$$\widetilde{K}_f(x,s,x^0,x_1^0)=f(\Psi(\alpha_j+x,\alpha_j+\frac{x}{s}))r_{\Psi(\alpha_j+x,\alpha_j+\frac{x}{s})}(x^0,\alpha_j+x)$$
$$\cdot\, k(S_{\Psi(\alpha_j+x,\alpha_j+\frac{x}{s})}(x^0,\alpha_j+x),x_1^0,\alpha_j+\frac{x}{s})|w(x^0,\alpha_j+x)|^{1/2}$$
$$\cdot\,|w(x_1^0,\alpha_j+\frac{x}{s})|^{\frac12}\frac{\frac{|x|}{s^{1/2}}}{|a(\alpha_j+\frac{x}{s})|}. \tag{17}$$

Using formula (17), it is easy to check the validity of Conditions (2) and (3) of Definition 1. □

3.4. Regularized trace of $T_f\circ K$. Let K be a leafwise smoothing operator with kernel $k\in C^\infty(X\times_\pi X,\mathcal{L}(E)\otimes|\mathcal{V}|^{\frac12}\otimes|\mathcal{V}|^{\frac12})$. Let $\pi^{-1}(U)\subset X\cong F\times U$ be a trivialization of π over some open set $U\subset S^1$. Fix a smooth positive half-density $|dx^0|$ on F. Then k can be written as

$$k=k(x_1^0,x_2^0,y)|dx_1^0|^{\frac12}|dx_2^0|^{\frac12},\quad x_1^0,x_2^0\in F,\quad y\in U.$$

We define a smooth leafwise density k_X on $\pi^{-1}(U)\cong F\times U$ by

$$k_X=\operatorname{tr}k(x^0,x^0,y)|dx^0|,\quad x^0\in F,\quad y\in U.$$

It is easy to see that this definition determines a well defined smooth leafwise density $k_X\in C^\infty(X,|\mathcal{V}|)$ on X, that is, it is independent of the choice of density $|dx^0|$ and the trivialization of π.

Let $V^*\in\Omega^1(X\setminus X^0)$ be a differential 1-form on X such that $V^*(V)=1$ and $V^*(W)=0$ for each $W\in\mathcal{V}$. It gives rise to a transverse density $|V^*|\in C^\infty(X\setminus X^0,|TX/\mathcal{V}|)$. The product of leafwise density $k_X\in C^\infty(X,|\mathcal{V}|)$ and transverse density $|V^*|\in C^\infty(X\setminus X^0,|TX/\mathcal{V}|)$ is well defined as a density $k_X|V^*|\in C^\infty(X\setminus X^0,|TX|)$ on $X\setminus X^0$.

Theorem 6. *Suppose that flow T satisfies Conditions (P1), (P3), (P4). Then $k_X|V^*|$ is a smooth relative density on manifold X with distinguished submanifold X^0, and the following formula*

$$\operatorname{r-Tr}(T_f\circ K)=f(0)\,{\int\limits_X}^{r}k_X|V^*| \tag{18}$$

holds true.

Proof. Let $k\in C^\infty(X\times_\pi X,\mathcal{L}(E)\otimes|\mathcal{V}|^{\frac12}\otimes|\mathcal{V}|^{\frac12})$ be the kernel of leafwise smoothing operator K. Using an appropriate cover of S^1 and a partition of unity subordinate to it, one can reduce the proof to the case when k is supported in $\pi^{-1}(U)$, where $U\subset S^1$ is an interval, containing a single point α_j for some j. Choose a trivialization $\pi^{-1}(U)\subset X\cong F\times U$ of π over U. We write the vector field V in the form (10). Then $V^*=\frac{1}{a(y)}dy$ and

$$k_X|V^*|=\frac{\operatorname{tr}k(x^0,x^0,y)}{|a(y)|}|dx^0||dy|,\quad x^0\in F,\quad y\in U.$$

By definition,

$$\operatorname{r-Tr}(T_f\circ K)={\int\limits_X}^{r}\operatorname{tr}k_f\,|_\Delta\ ,$$

where k_f is given by (15). From (15), using that $\Psi(y,y)=0$, we get:

$$k_f\,|_\Delta\,=f(0)k(x^0,x^0,y)|w(x^0,y)|\frac{1}{|a(y)|}|dx^0||dy|=f(0)k_X|V^*|.$$

It is easy to see that $k_X|V^*|$ is a smooth relative density on X. Therefore, the regularized integral ${\int\limits_X}^{r}\operatorname{tr}k_f\,|_\Delta$ exists and formula (18) is valid. □

3.5. The Lefschetz formula for a flow on a fiber bundle. In this section, we prove the Lefschetz formula described in Theorem 3.

First of all, we give a description of various objects associated with the foliation $\mathcal{F}$ given by the fibers of the fiber bundle $\pi:X\to S^1$. We recall that fiber $T_p\mathcal{F}$ of bundle $T\mathcal{F}$ at $p\in X$ is the tangent bundle to the leaf passing through p. In the case under consideration, the bundle $T\mathcal{F}$ coincides with the vertical subbundle $\mathcal{V}$ of the tangent bundle TX.

An arbitary element of the space $\Omega^u(\mathcal{F})$ of leafwise differential u-forms on X can be represented as a family $\omega=\{\omega(y)\in\Omega^u(F_y):y\in S^1\}$, where $\omega(y)$ is a differential form on fiber F_y smoothly depending on y. In other words, it can be said that $\omega\in C^\infty(S^1,\Omega^u_F)$ is a smooth section of the infinite-dimensional bundle Ω^u_F, whose fiber at y is the space $\Omega^u(F_y)$ of smooth differential u-forms on F_y.

The leafwise de Rham differential $d_{\mathcal{F}} : \Omega^u(\mathcal{F}) \to \Omega^{u+1}(\mathcal{F})$ is described as follows: for $\omega = \{\omega(y) : y \in S^1\} \in \Omega^u(\mathcal{F})$, we have

$$d_{\mathcal{F}}\omega = \{d[\omega(y)] : y \in S^1\} \in \Omega^{u+1}(\mathcal{F}),$$

where $d : \Omega^u(F_y) \to \Omega^{u+1}(F_y)$ is the de Rham differential on fiber F_y.

Let $g_{\mathcal{F}}$ be a leafwise Riemannian metric on X. It is given by a family of Riemannian metrics $g(y)$ on the fibers F_y, smoothly depending on $y \in S^1$. Metric $g_{\mathcal{F}}$ gives rise to an inner product on space $\Omega^u(\mathcal{F}) \cong C^\infty(S^1, \Omega^u_F)$:

$$(\omega', \omega'') = \int_{S^1} (\omega'(y), \omega''(y))_{g(y)} dy.$$

It is easy to show that the adjoint operator $\delta_{\mathcal{F}} : \Omega^{u+1}(\mathcal{F}) \to \Omega^u(\mathcal{F})$ of $d_{\mathcal{F}}$ is given as follows:

$$\delta_{\mathcal{F}}\omega(y) = d^*[\omega(y)], \quad \omega \in \Omega^{u+1}(\mathcal{F}),$$

where $d^* : \Omega^{u+1}(F_y) \to \Omega^u(F_y)$ is the adjoint operator of d on F_y with respect to the metric $g(y)$. Leafwise Laplace operator $\Delta^u_{\mathcal{F}}$ reads as

$$\Delta^u_{\mathcal{F}}\omega(y) = \Delta^u_{g(y)}[\omega(y)], \quad \omega \in \Omega^u(\mathcal{F}),$$

where $\Delta^u_{g(y)}$ is the Laplace operator on F_y given by metric $g(y)$.

The space $\mathcal{H}^u(\mathcal{F}) = \ker \Delta^u_{\mathcal{F}}$ of leafwise harmonic differential u-forms is described as follows:

$$\mathcal{H}^u(\mathcal{F}) = \{\omega \in C^\infty(S^1, \Omega^u_F) : (\forall y \in S^1)\omega(y) \in \mathcal{H}^u_{g(y)}\},$$

where $\mathcal{H}^u_{g(y)} = \ker \Delta^u_{g(y)}$ is the space of harmonic differential u-forms on F_y. Finally, projection $P_{\mathcal{H}^u(\mathcal{F})}$ on the space of leafwise harmonic u-forms is the operator $P_{\mathcal{H}^u(\mathcal{F})} : L^2(S^1, L^2\Omega^u_F) \to L^2(S^1, L^2\Omega^u_F)$ given, for each $\omega \in L^2(S^1, L^2\Omega^u_F)$, by

$$P_{\mathcal{H}^u(\mathcal{F})}\omega(y) = P_{\mathcal{H}^u_{g(y)}}[\omega(y)], \quad y \in S^1,$$

where, for each $y \in S^1$, operator $P_{\mathcal{H}^u_{g(y)}}$ is the orthogonal projection in the space $L^2\Omega^u(F_y)$ of L^2 differential u-forms on F_y on subspace $\mathcal{H}^u_{g(y)}$. Operator $P_{\mathcal{H}^u_{g(y)}}$ is an operator with smooth kernel $k^u_y \in C^\infty(\Lambda^u T^*\mathcal{F} \boxtimes (\Lambda^u T^*\mathcal{F})^*)$. Hence, projection $P_{\mathcal{H}^u(\mathcal{F})}$ is given by

$$P_{\mathcal{H}^u(\mathcal{F})}\omega(x^0, y) = \int_{F_y} k^u_y(x^0, x^0_1)\omega(x^0_1, y)dx^0_1.$$

It is well known that kernel $k^u_y(x^0, x^0_1)$ depends smoothly on y, and, therefore, operator $P_{\mathcal{H}^u(\mathcal{F})}$ is a leafwise smoothing operator.

Suppose that flow T satisfies Conditions (P1), (P3), (P4). Consider $X^0 = \bigcup_{j=1}^{k} F_{\alpha_j}$ as a smooth codimension one submanifold. We define the regularized Lefschetz function $L(T) \in \mathcal{D}'(\mathbb{R})$:

$$\langle L(T), f\rangle = \sum_{u=0}^{n-1} (-1)^u \operatorname{r-Tr}(T_f \circ P_{\mathcal{H}^u(\mathcal{F})}), \quad f \in C^\infty_0(\mathbb{R}).$$

By Theorem 5, the regularized Lefschetz function $L(T) \in \mathcal{D}'(\mathbb{R})$ is well defined.

Recall that the Euler characteristic of a compact manifold M of dimension d is the number

$$\chi(M) = \sum_{u=0}^{d} (-1)^u \dim H^u(M),$$

where $H^u(M)$ is the de Rham cohomology of M.

Theorem 7. *Suppose that flow T satisfies Conditions (P1), (P3), (P4). Then the formula*

$$L(T) = \chi(F) \int_{S^1}^{r} \frac{dy}{|a(y)|}\delta_0$$

holds true.

Proof. By Theorem 6, we have

$$\text{r-Tr}(T_f \circ P_{\mathcal{H}^u(\mathcal{F})}) = f(0) \int\limits_X^r (k_{P_{\mathcal{H}^u(\mathcal{F})}})_X |V^*|.$$

For each leafwise smoothing operator K with kernel k, the following formula of integration along the fibers of π holds:

$$\int\limits_X^r k_X |V^*| = \int\limits_{S^1}^r (\operatorname{tr}_{\mathcal{F}} K) \frac{1}{|a(y)|} dy,$$

where $K(y) : C^\infty(F_y) \to C^\infty(F_y)$ is the restriction of K to fiber F_y, $\operatorname{tr}_{\mathcal{F}} K$ is the function on S^1, which maps each $y \in S^1$ to trace $\operatorname{tr} K(y)$ of $K(y)$ (the fiberwise trace of K). By this formula, we get

$$\langle L(T), f \rangle = f(0) \sum_{u=0}^{n-1} (-1)^u \int\limits_{S^1}^r \frac{1}{|a(y)|} \operatorname{tr} P_{\mathcal{H}^u_{g(y)}} dy = f(0) \sum_{u=0}^{n-1} (-1)^u \int\limits_{S^1}^r \frac{1}{|a(y)|} \dim \mathcal{H}^u_{g(y)} dy.$$

For each $y \in S^1$, there is the Hodge isomorphism $\mathcal{H}^u_{g(y)} \cong H^u(F_y)$. Therefore, we obtain

$$\langle L(T), f \rangle = f(0) \sum_{u=0}^{n-1} (-1)^u \int\limits_{S^1}^r \frac{1}{|a(y)|} \dim H^u(F_y) = f(0) \int\limits_{S^1}^r \frac{1}{|a(y)|} \chi(F_y) dy.$$

Since all fibers F_y are diffeomorphic to F, $\chi(F_y) = \chi(F)$ for each $y \in S^1$, it completes immediately the proof of the theorem. □

4. Reeb foliation

4.1. A foliation on the torus and flows on it. We consider the submersion $p : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$p(y, z) = e^z \cos\left(\frac{\pi}{2} y\right), \quad (y, z) \in \mathbb{R}^2.$$

The level sets of p determine a foliation $\widetilde{\mathcal{F}}$ on $\mathbb{R}^2$ with leaves of the form $L_v = \{p(y, z) = v\}$ with $v \in \mathbb{R}$. For each $(k, \ell) \in \mathbb{Z}^2$, the map $R_{(k,\ell)} : \mathbb{R}^2 \to \mathbb{R}^2, (y, z) \mapsto (y + 2k, z + \ell)$ maps L_v into $L_{(-1)^k v e^\ell}$. Therefore, foliation $\widetilde{\mathcal{F}}$ defines a foliation $\mathcal{F}$ on the two-dimensional torus $X = \mathbb{R}^2/(2\mathbb{Z} \times \mathbb{Z})$. Note that all leaves of $\mathcal{F}$ are non-compact except for the leaf $L_0 = \{(y, z) \in X : y = 1\}$ corresponding to $v = 0$. Fix the standard Riemannian metric $g = dy^2 + dz^2$ on X.

As a rule, we shall work with the manifold $\widehat{X} = (\mathbb{R}/2\mathbb{Z}) \times \mathbb{R}$, which is a covering over X, and with the corresponding lift $\widehat{\mathcal{F}}$ of foliation $\mathcal{F}$ to $\widehat{X}$. Let us define a foliated coordinate system on $(\widehat{X}, \widehat{\mathcal{F}})$ with coordinates (u, v):

$$u = z; \quad v = e^z \cos\left(\frac{\pi}{2} y\right), \quad (y, z) \in (0, 2) \times \mathbb{R} \subset \widehat{X}, \tag{19}$$

and also a foliated coordinate system on $(\widehat{X}, \widehat{\mathcal{F}})$ with coordinates (u_0, v_0):

$$u_0 = y_0; \quad v_0 = e^{z_0} \cos\left(\frac{\pi}{2} y_0\right), \quad (y_0, z_0) \in (-1, 1) \times \mathbb{R} \subset \widehat{X}. \tag{20}$$

The coordinate systems introduced in such a way give rise to an atlas of the foliation $(\widehat{X}, \widehat{\mathcal{F}})$.

We shall identify functions on X with functions on $\widehat{X}$ satisfying the condition $f(y, z + 1) = f(y, z)$ for each $y \in \mathbb{R}/2\mathbb{Z}, z \in \mathbb{R}$. The formulas (19) define a diffeomorphism ψ from $\widetilde{X} = (0, 2) \times \mathbb{R} \subset \widehat{X}$ onto $S = \{(u, v) \in \mathbb{R}^2 : -e^u < v < e^u\}$. In coordinates (u, v), the periodicity condition for f is written as $f(u + 1, ev) = f(u, v)$ for each $(u, v) \in S$.

By direct computation we get:

$$\frac{\partial}{\partial u} = \frac{2}{\pi} \cot\left(\frac{\pi}{2} y\right) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial v} = -\frac{2}{\pi} e^{-z} \csc\left(\frac{\pi}{2} y\right) \frac{\partial}{\partial y}. \tag{21}$$

Let T be a flow on $(X,\mathcal{F})$ satisfying Conditions (P1), (P3) and (P4). We shall denote also by T the lifted flow on $(\widehat{X},\widehat{\mathcal{F}})$. Then, by Condition (P1), infinitesimal generator V of flow T in coordinates (19) reads as

$$V = A(u,v)\frac{\partial}{\partial u} + B(v)\frac{\partial}{\partial v}, \quad (u,v)\in S, \tag{22}$$

where A and B are smooth functions on S satisfying the conditions:

$$A(u+1,ev) = A(u,v), \quad B(ev) = eB(v).$$

Since B is a smooth function at 0, it is easy to see that there exists $\alpha\in\mathbb{R}$ such that $B(v)=\alpha v$ for each $v\in\mathbb{R}$. It follows from here that the infinitesimal generator of a flow on X satisfying Condition (P1) is tangent to the compact leaf L_0. Moreover, if flow T satisfies Conditions (P1), (P3) and (P4), then $\alpha\neq 0$ and the flow has at least one fixed point, which necessarily belongs to L_0.

Writing the vector field V in the coordinates (y,z), one can show that it is smooth on X if and only if the function

$$f(y,z) = \begin{cases} A\left(z, -e^z\cos\frac{\pi}{2}y\right) & \text{if } (y,z)\in(-1,0)\times\mathbb{R};\\ A\left(z, e^z\cos\frac{\pi}{2}y\right) & \text{if } (y,z)\in(0,1)\times\mathbb{R}\end{cases}$$

is smooth for $y=0$. As an example of the function A, one can take

$$A(u,v) = f_0(v^2e^{-2u}) + \sum_{k=1}^{\infty}(f_k(ve^{-u})\cos ku + g_k(ve^{-u})\sin ku),$$

where $f_k, g_k\in C^\infty([0,1])$ are even functions, $f_0(1)=\alpha$, $f_k(1)=g_k(1)=0$.

Flow T of vector field V can be written as

$$T_t(y,z) = (Y(t,y,z), Z(t,y,z)).$$

In (u,v)-coordinates, flow T is given by the system of differential equations $\dot{u} = A(u,v), \dot{v} = \alpha v$. Therefore, in (u,v)-coordinates it reads as

$$T_t(u,v) = (U(t,u,v), e^{\alpha t}v).$$

The restriction of T to L_0 is given by the equation $\dot{z} = A(z,0)$. It is easy to see that the circle $y=0$ is a periodic orbit with a period $\tau = \frac{1}{\alpha}$.

In the sequel, we shall consider a complex vector bundle E on X. We shall assume that E is trivial as a vector bundle, that is, $E\cong X\times\mathbb{C}^N$ for some N. We shall also assume that E is equipped with a flow T^E, which covers the flow T on X, moreover, the map $T^E_t(y,z): E_{(y,z)}\to E_{T_t(y,z)}$ in the fibers of E induced by this flow is linear. We denote by T^*_t the operator in $C^\infty(X,E)$ induced by flow T:

$$T^*_t u(y,z) = r_t(y,z)[u(T_t(y,z))],$$

where $r_t(y,z) = T^E_{-t}(T_t(y,z)) : E_{T_t(y,z)}\to E_{(y,z)}$.

We shall be interested, mainly, in two particular cases. The first case is related with the space of smooth functions. In this case, $E = X\times\mathbb{C}$ and $r_t(y,z)=1$. Another important case is related with the space $\Omega^1(\mathcal{F})$ of leafwise differential 1-forms on X. In this case, $E = T^*\mathcal{F}\otimes\mathbb{C}$.

The operator $T_f : C^\infty(X, E\otimes\Omega_X^{\frac12})\to C^\infty(X,E\otimes\Omega_X^{\frac12})$, where $f\in C^\infty_0(\mathbb{R})$, is given by the formula: for $\mu = a(y,z)|dydz|^{1/2}$,

$$T_f\mu = \left(\int_{-\infty}^{+\infty} f(\tau)r_\tau(y,z)[a(T_\tau(y,z))]d\tau\right)|dydz|^{1/2}. \tag{23}$$

4.2. Analogue of the McKean-Singer formula. In this subsection, we introduce the smoothed regularized Lefschetz function and formulate the main results of Section 4.

We shall begin with a general setting. Let $(X,\mathcal{F})$ be an arbitrary n-dimensional compact manifold with codimension one foliation and T be a flow on X satisfying Conditions (P1), (P3) and (P4). We denote by X^0 the submanifold of X, which consists of the leaves containing fixed points of the flow. Fix an arbitrary Riemannian metric on X.

Denote by $\mathcal{A}$ the set of functions $\phi : \mathbb{R} \to \mathbb{C}$, which can be extended to entire functions on the complex plane, and moreover, for each compact subset $K \subset \mathbb{R}$, the set of functions $x \mapsto \phi(x+iy)$, $y \in K$, is bounded in Schwartz space $S(\mathbb{R})$.

Fix an even function $\phi \in \mathcal{A}$ of the form $\phi(x) = \psi(x^2), x \in \mathbb{R}$, where $\psi \in C^\infty(\mathbb{R})$. For each $t > 0$ and $f \in C_0^\infty(\mathbb{R})$, define operators $C^u_{t,\psi,f} : \Omega^u(\mathcal{F}) \to \Omega^u(\mathcal{F}), u = 0, \ldots, n$ and $C_{t,\psi,f} : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$ by

$$C^u_{t,\psi,f} = T_f \circ \psi(t\Delta^u_{\mathcal{F}})^2, \quad C_{t,\psi,f} = T_f \circ \psi(t\Delta_{\mathcal{F}})^2.$$

Suppose that operator $\psi(t\Delta^u_{\mathcal{F}})^2 : \Omega^u(\mathcal{F}) \to \Omega^u(\mathcal{F})$ is a leafwise smoothing operator, operator $C^u_{t,\psi,f} : \Omega^u(\mathcal{F}) \to \Omega^u(\mathcal{F})$ belongs to $\mathcal{K}(X, X^0, \Lambda^u(T\mathcal{F}))$, and, therefore, there exists the regularized trace r-Tr$(C^u_{t,\psi,f})$. Formula (3) defines the smoothed regularized Lefschetz function $L_{t,\psi} \in \mathcal{D}'(\mathbb{R})$. Let us rewrite this formula as

$$\langle L_{t,\psi}, f\rangle = \text{r-Tr}^s(C_{t,\psi,f}),$$

where r-Tr$^s(C_{t,\psi,f})$ is the regularized supertrace of operator $C_{t,\psi,f}$:

$$\text{r-Tr}^s(C_{t,\psi,f}) = \sum_{u=0}^{n-1} (-1)^u \,\text{r-Tr}\left(C^u_{t,\psi,f}\right).$$

An explicit computation of $L_{t,\psi}$ seems to be a quite complicated problem even in the particular example described in Section 4.1. In the present paper, we make only the first step in this computation, namely, we prove existence of function $L_{t,\psi}$ and compute its derivative with respect to t in this case. The results of the computation for an arbitrary function ψ are given in Proposition 6. A more concrete formula is obtained in the case when $\psi(x) = e^{-\frac{x}{2}}$, that is, when operator $C_{t,\psi,f}$ reads as

$$C_{t,\psi,f} = B_{t,f} := T_f \circ e^{-t\Delta_{\mathcal{F}}},$$

and the corresponding smoothed regularized Lefschetz function L_t is given by

$$\langle L_t, f\rangle = \text{r-Tr}^s(T_f \circ e^{-t\Delta_{\mathcal{F}}}).$$

Theorem 8. *Let $(X, \mathcal{F}, g)$ be a compact foliated Riemannian manifold described in Section 4.1 and T be a flow on X satisfying Conditions (P1), (P3) and (P4). Then, for each $t > 0$, the formula*

$$\frac{1}{2}\langle \frac{d}{dt} L_t, f\rangle = \sum_{n\in\mathbb{Z}} c_n(t) f\left(\frac{n}{\alpha}\right), \quad f \in C_0^\infty(\mathbb{R})$$

holds true, where

$$c_n(t) = \int_0^1 \left[h_+\left(t, z - Z\left(\frac{n}{\alpha}, 1, z\right) + n\right) + h_-\left(t, Z\left(-\frac{n}{\alpha}, 1, z\right) - z + n\right)\right] dz,$$

and functions h_+ and h_- are given by

$$h_+(t,k) = -\frac{1}{\alpha}\frac{1}{4\sqrt{\pi}\sqrt{t}}\left(\frac{k^3}{4t^2} + \frac{k}{2t} - \frac{k}{4}\right) e^{-\frac{(k+t)^2}{4t}},$$

$$h_-(t,k) = \frac{1}{\alpha}\frac{1}{4\sqrt{\pi}\sqrt{t}}\left(\frac{k^3}{4t^2} - \frac{k}{2t} - \frac{k}{4} + 1\right) e^{-\frac{(k+t)^2}{4t}}.$$

In particular, we have

$$c_0(t) = \int_0^1 \left[h_+(t,0) + h_-(t,0)\right] dz = \frac{1}{\alpha}\frac{1}{4\sqrt{\pi}\sqrt{t}} e^{-\frac{t}{4}}.$$

The rest of this section is devoted to the proof of this theorem. We shall follow the lines of [1] with some modifications related with the fact that the regularized trace functional does not have the trace property.

We begin with a general formula for the derivative of the regularized supertrace of $C_{t,\psi,f}$ with respect to t, which holds for an arbitrary n-dimensional compact Riemannian manifold $(X, \mathcal{F}, g)$ with codimension 1 foliation and a flow T on X satisfying Conditions (P1), (P3) and (P4) (cf. [1, Lemma 3.3]).

Proposition 1. *The formula*

$$\begin{aligned}\frac{d}{dt}\,\text{r-Tr}^s\, C_{t,\psi,f} =& 2\,\text{r-Tr}\left[T_f\circ d_{\mathcal F}^-\circ\psi'(t\Delta_{\mathcal F}^-),\psi(t\Delta_{\mathcal F}^-)\circ\delta_{\mathcal F}^+\right]\\ &+2\,\text{r-Tr}\left[\psi(t\Delta_{\mathcal F}^-)\circ\delta_{\mathcal F}^+d_{\mathcal F}^-,T_f\circ\psi'(t\Delta_{\mathcal F}^-)\right]\\ &-2\,\text{r-Tr}\left[T_f\circ d_{\mathcal F}^+\circ\psi'(t\Delta_{\mathcal F}^+),\psi(t\Delta_{\mathcal F}^+)\circ\delta_{\mathcal F}^-\right]\\ &-2\,\text{r-Tr}\left[\psi(t\Delta_{\mathcal F}^+)\circ\delta_{\mathcal F}^-d_{\mathcal F}^+,T_f\circ\psi'(t\Delta_{\mathcal F}^+)\right]\end{aligned}\qquad(24)$$

holds true.

Proof. Space $\Omega(\mathcal F)$ can be represented as a direct sum $\Omega(\mathcal F)=\Omega^+(\mathcal F)\oplus\Omega^-(\mathcal F)$, where

$$\Omega^+(\mathcal F)=\bigoplus_{u\text{ even}}\Omega^u(\mathcal F),\quad \Omega^-(\mathcal F)=\bigoplus_{u\text{ odd}}\Omega^u(\mathcal F).$$

With respect to this decomposition, operators $d_{\mathcal F}$, $\delta_{\mathcal F}$, $\Delta_{\mathcal F}$ can be written as 2×2 block matrices:

$$d_{\mathcal F}=\begin{pmatrix}0 & d_{\mathcal F}^-\\ d_{\mathcal F}^+ & 0\end{pmatrix},\quad \delta_{\mathcal F}=\begin{pmatrix}0 & \delta_{\mathcal F}^-\\ \delta_{\mathcal F}^+ & 0\end{pmatrix},\quad \Delta_{\mathcal F}=\begin{pmatrix}\Delta_{\mathcal F}^+ & 0\\ 0 & \Delta_{\mathcal F}^-\end{pmatrix}.$$

We write $\psi(t\Delta_{\mathcal F})$ in the block form:

$$\psi(t\Delta_{\mathcal F})=\begin{pmatrix}\psi(t\Delta_{\mathcal F}^+) & 0\\ 0 & \psi(t\Delta_{\mathcal F}^-)\end{pmatrix}.$$

Multiplying operators $d_{\mathcal F}$ and $\delta_{\mathcal F}$ in the block form, it is easy to show that

$$\Delta_{\mathcal F}^+=d_{\mathcal F}^-\delta_{\mathcal F}^++\delta_{\mathcal F}^-d_{\mathcal F}^+;\quad \Delta_{\mathcal F}^-=d_{\mathcal F}^+\delta_{\mathcal F}^-+\delta_{\mathcal F}^+d_{\mathcal F}^-.$$

We get

$$\begin{aligned}\frac{d}{dt}\,\text{r-Tr}^s\, C_{t,\psi,f} =& 2\,\text{r-Tr}^s\left(T_f\circ\Delta_{\mathcal F}\circ\psi'(t\Delta_{\mathcal F})\circ\psi(t\Delta_{\mathcal F})\right)\\ =&2\,\text{r-Tr}\left(T_f\circ(d_{\mathcal F}^-\delta_{\mathcal F}^++\delta_{\mathcal F}^-d_{\mathcal F}^+)\circ\psi'(t\Delta_{\mathcal F}^+)\circ\psi(t\Delta_{\mathcal F}^+)\right)\\ &-2\,\text{r-Tr}\left(T_f\circ(d_{\mathcal F}^+\delta_{\mathcal F}^-+\delta_{\mathcal F}^+d_{\mathcal F}^-)\circ\psi'(t\Delta_{\mathcal F}^-)\circ\psi(t\Delta_{\mathcal F}^-)\right).\end{aligned}$$

Since operators $d_{\mathcal F}^-$ and T_s^* commute, we have the identity:

$$I_1:=\text{r-Tr}\left(T_f\circ d_{\mathcal F}^-\delta_{\mathcal F}^+\circ\psi'(t\Delta_{\mathcal F}^+)\circ\psi(t\Delta_{\mathcal F}^+)\right)=\text{r-Tr}\left(d_{\mathcal F}^-\circ T_f\circ\delta_{\mathcal F}^+\circ\psi'(t\Delta_{\mathcal F}^+)\circ\psi(t\Delta_{\mathcal F}^+)\right).$$

Since $\delta_{\mathcal F}^+\circ\Delta_{\mathcal F}^+=\Delta_{\mathcal F}^-\circ\delta_{\mathcal F}^+$ and $d_{\mathcal F}^-\circ\Delta_{\mathcal F}^-=\Delta_{\mathcal F}^+\circ d_{\mathcal F}^-$, we get

$$\begin{aligned}I_1=&\,\text{r-Tr}\left(d_{\mathcal F}^-\circ T_f\circ\psi'(t\Delta_{\mathcal F}^-)\circ\psi(t\Delta_{\mathcal F}^-)\circ\delta_{\mathcal F}^+\right)\\ =&\,\text{r-Tr}\left[T_f\circ d_{\mathcal F}^-\circ\psi'(t\Delta_{\mathcal F}^-),\psi(t\Delta_{\mathcal F}^-)\circ\delta_{\mathcal F}^+\right]\\ &+\text{r-Tr}\left[\psi(t\Delta_{\mathcal F}^-)\circ\delta_{\mathcal F}^+d_{\mathcal F}^-,T_f\circ\psi'(t\Delta_{\mathcal F}^-)\right]+\text{r-Tr}\left(T_f\circ\delta_{\mathcal F}^+d_{\mathcal F}^-\circ\psi'(t\Delta_{\mathcal F}^-)\circ\psi(t\Delta_{\mathcal F}^-)\right).\end{aligned}$$

Similarly, one can show that

$$I_2:=\text{r-Tr}\left(T_f\circ d_{\mathcal F}^+\delta_{\mathcal F}^-\circ\psi'(t\Delta_{\mathcal F}^-)\circ\psi(t\Delta_{\mathcal F}^-)\right)=\text{r-Tr}\left(d_{\mathcal F}^+\circ T_f\circ\delta_{\mathcal F}^-\circ\psi'(t\Delta_{\mathcal F}^-)\circ\psi(t\Delta_{\mathcal F}^-)\right).$$

Since $\delta_{\mathcal F}^-\circ\Delta_{\mathcal F}^-=\Delta_{\mathcal F}^+\circ\delta_{\mathcal F}^-$ and $d_{\mathcal F}^+\circ\Delta_{\mathcal F}^+=\Delta_{\mathcal F}^-\circ d_{\mathcal F}^+$, we get

$$\begin{aligned}I_2=&\,\text{r-Tr}\left(d_{\mathcal F}^+\circ T_f\circ\psi'(t\Delta_{\mathcal F}^+)\circ\psi(t\Delta_{\mathcal F}^+)\circ\delta_{\mathcal F}^-\right)\\ =&\,\text{r-Tr}\left[T_f\circ d_{\mathcal F}^+\circ\psi'(t\Delta_{\mathcal F}^+),\psi(t\Delta_{\mathcal F}^+)\circ\delta_{\mathcal F}^-\right]\\ &+\text{r-Tr}\left[\psi(t\Delta_{\mathcal F}^+)\circ\delta_{\mathcal F}^-d_{\mathcal F}^+,T_f\circ\psi'(t\Delta_{\mathcal F}^+)\right]+\text{r-Tr}\left(T_f\circ\delta_{\mathcal F}^-d_{\mathcal F}^+\circ\psi'(t\Delta_{\mathcal F}^+)\circ\psi(t\Delta_{\mathcal F}^+)\right).\end{aligned}$$

□

In the situation described in Theorem 8, the leafwise de Rham complex reads as

$$0\longrightarrow\Omega^0(\mathcal F)\xrightarrow{d_{\mathcal F}}\Omega^1(\mathcal F)\longrightarrow 0.$$

Therefore, we have

$$d_{\mathcal F}^+=d_{\mathcal F}:\Omega^0(\mathcal F)\longrightarrow\Omega^1(\mathcal F),\quad d_{\mathcal F}^-=0,\quad \delta_{\mathcal F}^-=d_{\mathcal F}^*:\Omega^1(\mathcal F)\longrightarrow\Omega^0(\mathcal F),\quad \delta_{\mathcal F}^+=0.$$

$$\Delta_{\mathcal F}^+=d_{\mathcal F}^*d_{\mathcal F}:\Omega^0(\mathcal F)\longrightarrow\Omega^0(\mathcal F),\quad \Delta_{\mathcal F}^-=d_{\mathcal F}d_{\mathcal F}^*:\Omega^1(\mathcal F)\longrightarrow\Omega^1(\mathcal F).$$

Formula (24) becomes

$$\begin{aligned}\frac{d}{dt}\,\text{r-Tr}^s\, C_{t,\psi,f} =& 2\,\text{r-Tr}\left[T_f\circ\psi'(t\Delta^+_{\mathcal F}), \psi(t\Delta^+_{\mathcal F})\circ\Delta^+_{\mathcal F}\right]\\ &-2\,\text{r-Tr}\left[T_f\circ d^+_{\mathcal F}\circ\psi'(t\Delta^+_{\mathcal F}),\psi(t\Delta^+_{\mathcal F})\circ\delta^-_{\mathcal F}\right].\end{aligned}\tag{25}$$

From now on, we shall consider the compact foliated Riemannian manifold $(X,\mathcal F,g)$ and flow T on X satisfying Conditions (P1), (P3) and (P4) described in Section 4.1.

4.3. Leafwise smoothing operators. To describe leafwise smoothing operators on foliated manifold $(X,\mathcal F)$, we shall use the representation $X=\widehat X/\mathbb Z$ and the corresponding representation of leafwise smoothing operators on $(X,\mathcal F)$ as $\mathbb Z$-invariant leafwise smoothing operators on $(\widehat X,\widehat{\mathcal F})$.

Consider the set

$$\widehat{\mathcal R}=\{(y_1,z_1,y_2,z_2)\in\widehat X\times\widehat X : (y_1,z_1)\sim_{\mathcal F}(y_2,z_2)\}.$$

A topology on $\widehat{\mathcal R}$ is defined as follows. Set $\widehat{\mathcal R}$ is represented as a disjoint union $\widehat{\mathcal R}=\widehat{\mathcal R}_1\bigsqcup\widehat{\mathcal R}_2$, where $\widehat{\mathcal R}_1=\{(y_1,z_1,y_2,z_2)\in\widehat{\mathcal R} : y_1\neq 1, y_2\neq 1\}$ is homeomorphic to $(-1,1)\times(-1,1)\times(0,+\infty)$, $\widehat{\mathcal R}_2=\{(1,z_1,1,z_2) : z_1,z_2\in\mathbb R\}\cong\mathbb R^2$, and a sequence $(y_1^{(n)},z_1^{(n)},y_2^{(n)},z_2^{(n)})\in\widehat{\mathcal R}_1$ converges to $(1,z_1,1,z_2)\in\widehat{\mathcal R}_2$ if and only if it converges in $\widehat X\times\widehat X$ and for sufficiently large n the points $(y_1^{(n)},z_1^{(n)})$ and $(y_2^{(n)},z_2^{(n)})$ are in the same (depending on n) component of the set $\widehat X\setminus(\{0\}\times\mathbb R\cup\{1\}\times\mathbb R)$ (the last condition is also equivalent to that the distance between these points along the leaves of $\widehat{\mathcal F}$ is uniformly bounded in n).

Proposition 2. *Set $\widehat{\mathcal R}$ can be endowed with a structure of a smooth manifold.*

Proof. An atlas on $\widehat{\mathcal R}$ consists of two charts. The first chart associates to a point $(y_1,z_1,y_2,z_2)\in U_1=\widehat{\mathcal R}\cap(-1,1)\times\mathbb R\times(-1,1)\times\mathbb R$ its coordinates $X_1(y_1,z_1,y_2,z_2)=(y_1,y_2,v)\in(-1,1)\times(-1,1)\times(0,+\infty)$, where $v=p(y_1,z_1)=p(y_2,z_2)$. The second chart associates to a point $(y_1,z_1,y_2,z_2)\in U_2=(\widehat{\mathcal R}\cap(0,1]\times\mathbb R\times(0,1]\times\mathbb R)\cup(\widehat{\mathcal R}\cap(1,2)\times\mathbb R\times(1,2)\times\mathbb R)$ its coordinates

$$X_2(y_1,z_1,y_2,z_2)=(z_1,z_2,v)\in\{(z_1,z_2,v)\in\mathbb R^3 : |v|<e^{z_1}, |v|<e^{z_2}\},$$

where $v=p(y_1,z_1)=p(y_2,z_2)$. ☐

Denote by $\widehat L_{(y_1,z_1)}$ the leaf passing through (y_1,z_1). On each leaf of $\widehat{\mathcal F}$, we fix the positive density $|dz|$, which is the lift of the density $|dz|$ on the real line $\mathbb R$ by the map $\widehat L\to\mathbb R$, $(y,z)\mapsto z$. This density is smooth everywhere on $\widehat L$ except $y=0$.

A leafwise smoothing operator $\widehat K : C^\infty(\widehat X)\to C^\infty(\widehat X)$ is given by the formula

$$\widehat K u(y_1,z_1)=\int\limits_{\widehat L_{(y_1,z_1)}} k(y_1,z_1,y_2,z_2)u(y_2,z_2)|dz_2|.$$

The kernel of leafwise smoothing operator $\widehat K$ is given by a function k on $\widehat{\mathcal R}$ satisfying the conditions:

- $k(y_1,z_1+1,y_2,z_2+1)=k(y_1,z_1,y_2,z_2)$.
- There exists a constant C such that $k(y_1,z_1,y_2,z_2)=0$ for each $(y_1,z_1,y_2,z_2)\in\widehat{\mathcal R}$ with $d((y_1,z_1),(y_2,z_2))>C$, where d is the leafwise distance between (y_1,z_1) and (y_2,z_2).
- Density $k(y_1,z_1,y_2,z_2)|dz_2|$ is a smooth density on $\widehat L_{(y_1,z_1)}$ smoothly depending on $(y_2,z_2)\in\widehat X$: for each $u\in C^\infty(\widehat X)$, the function

$$(y_1,z_1)\in\widehat X\mapsto\int\limits_{\widehat L_{(y_1,z_1)}} k(y_1,z_1,y_2,z_2)u(y_2,z_2)|dz_2|$$

is a smooth function on $\widehat X$.

For each $(y_1, z_1, z_2) \in (0,2) \times \mathbb{R} \times \mathbb{R}$ such that $z_2 \geqslant \ln|p(y_1, z_1)|$, we let

$$Y_2(y_1, z_1, z_2) = \frac{2}{\pi} \arccos\left(p(y_1, z_1)e^{-z_2}\right). \tag{26}$$

We observe that $Y_2(y_1, z_1, z_2)$ is the unique solution to the equation $p(y_1, z_1) = p(y_2, z_2)$, belonging to the interval $(0,2)$, and $2 - Y_2(y_1, z_1, z_2)$ is the unique solution to the equation $p(y_1, z_1) = -p(y_2, z_2)$ belonging to the interval $(0,2)$. The intersection of the leaf $\widehat{L}_{(y_1,z_1)}$ with the coordinate neighborhood $(0,2) \times \mathbb{R}$ reads as

$$\widehat{L}_{(y_1,z_1)} \cap (0,2) \times \mathbb{R} = \{(Y_2(y_1, z_1, z_2), z_2) : z_2 \in (\ln|p(y_1, z_1)|, +\infty)\}$$
$$\bigcup \{(Y_2(y_1, z_1, z_2), z_2) : z_2 \in (\ln|p(y_1, z_1)|, +\infty)\}.$$

Therefore, in coordinates on $(0,2) \times \mathbb{R}$, operator $\widehat{K}$ can be written as follows:

$$\begin{aligned} Kb(y_1, z_1) = & \int\limits_{\ln|p(y_1,z_1)|}^{+\infty} K_+(z_1, z_2, p(y_1, z_1)) b(Y_2(y_1, z_1, z_2), z_2) |dz_2| \\ & + \int\limits_{\ln|p(y_1,z_1)|}^{+\infty} K_-(z_1, z_2, p(y_1, z_1)) b(2 - Y_2(y_1, z_1, z_2), z_2) |dz_2|, \end{aligned} \tag{27}$$

where functions $K_\pm(z_1, z_2, v)$ are defined for $|v| < e^{z_1}, |v| < e^{z_2}$: for $v \neq 0$

$$K_\pm(z_1, z_2, v) = k(y_1, z_1, y_2, z_2),$$

and $y_1, y_2 \in (0,2)$ are such that $p(y_1, z_1) = v$, $p(y_2, z_2) = \pm v$,

$$K_+(z_1, z_2, 0) = k(1, z_1, 1, z_2), \quad K_-(z_1, z_2, 0) = 0.$$

The leafwise distance $d((y_1, z_1), (y_2, z_2))$ between the points (y_1, z_1) and (y_2, z_2) in the same leaf $\widehat{L}_v = \{(y,z) \in (0,2) \times \mathbb{R} : p(y,z) = v\}$ of $\widehat{\mathcal{F}}$ is given by

$$d((y_1, z_1), (y_2, z_2)) = d(z_1, z_2) = \int_{\ln|v|}^{z_2} \sqrt{1 + \frac{4}{\pi^2} \frac{v^2 e^{-2z}}{1 - v^2 e^{-2z}}}\, dz \mp \int_{\ln|v|}^{z_1} \sqrt{1 + \frac{4}{\pi^2} \frac{v^2 e^{-2z}}{1 - v^2 e^{-2z}}}\, dz,$$

where we take the sign '+' if the points are in the same component of the intersection of the leaf with the coordinate neighborhood ($p(y_1, z_1)p(y_2, z_2) > 0$), and the sign '-' if the points are in different components of the intersection of the leaf with the coordinate neighborhood ($p(y_1, z_1)p(y_2, z_2) < 0$).

The conditions on k are rewritten as follows:

- $K_\pm(z_1 + 1, z_2 + 1, v) = K_\pm(z_1, z_2, v)$.
- There exists a constant C such that $K_\pm(z_1, z_2, v) = 0$ for all (z_1, z_2, v) with $d(z_1, z_2) > C$.
- Functions $K_\pm(z_1, z_2, v)$ are smooth for $|v| < e^{z_1}, |v| < e^{z_2}$.

Note that Condition 2) implies

$$\lim_{v \to 0} K_-(z_1, z_2, v) = 0.$$

The corresponding operator on half-densities is written as follows: for $\mu = b(y,z)|dy|^{1/2}|dz|^{1/2}$,

$$K\mu = Kb(y,z)|dy|^{1/2}|dz|^{1/2}, \tag{28}$$

where Kb is given by (27).

4.4. Operator $T_f \circ K$. Consider the smooth codimension one submanifold $X^0 = \{(y,z) \in X : y = 1\}$.

Theorem 9. *For each leafwise smoothing operator K and for each function $f \in C_0^\infty(\mathbb{R})$, operator $T_f \circ K$ belongs to $\mathcal{K}(X, X^0, E)$.*

To prove this theorem, we compute the kernel of $T_f \circ K$ in the coordinates on $(0,2)\times\mathbb{R}$. For $\mu = a(y,z)|dydz|^{1/2}$, denoting $(T_f\circ K)\mu = (T_f\circ K)a(y,z)|dydz|^{1/2}$, we obtain:

$$(T_f\circ K)a(y,z) = \int\limits_{-\infty}^{+\infty} f(\tau) \int\limits_{\ln|p(y,z)|}^{+\infty} r_\tau(y,z)[K_+(Z(\tau,y,z),z_2,p(Y(\tau,y,z),Z(\tau,y,z)))]$$
$$\cdot\, a(Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2),z_2)|dz_2|d\tau$$
$$+\int\limits_{-\infty}^{+\infty} f(\tau) \int\limits_{\ln|p(y,z)|}^{+\infty} r_\tau(y,z)[K_-(Z(\tau,y,z),z_2,p(Y(\tau,y,z),Z(\tau,y,z))))]$$
$$\cdot\, a(2-Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2),z_2)|dz_2|d\tau.$$

Let us make a change of variables $(\tau,z_2)\to(y_2,z_2)$, which, for the integral in the first term, reads as

$$y_2 = Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2) = Y_2(y,z+\alpha\tau,z_2)$$
$$\Leftrightarrow \tau(y,z,y_2,z_2) = \int\limits_{p(y,z)}^{p(y_2,z_2)} \frac{dw}{B(w)} = \frac{1}{\alpha}\left(z_2 - z + \ln\left|\cos\left(\frac{\pi}{2}y_2\right)\right| - \ln\left|\cos\left(\frac{\pi}{2}y\right)\right|\right),$$

and, for the integral in the second term, reads as

$$y_2 = 2 - Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2)$$
$$\Leftrightarrow \tau(y,z,y_2,z_2) = \int\limits_{p(y,z)}^{-p(y_2,z_2)} \frac{dw}{B(w)} = \frac{1}{\alpha}\left(z_2 - z + \ln\left|\cos\left(\frac{\pi}{2}y_2\right)\right| - \ln\left|\cos\left(\frac{\pi}{2}y\right)\right|\right).$$

Taking into account that

$$\frac{\partial Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2)}{\partial\tau} = -\frac{2}{\pi}\alpha\cot\left(\frac{\pi}{2}Y_2(Y(\tau,y,z),Z(\tau,y,z),z_2)\right),$$

we obtain:

$$(T_f\circ K)a(y,z) = \frac{\pi}{2|\alpha|}\int\limits_{(0,1)\times\mathbb{R}} f(\tau)r_\tau(y,z)[K_+(Z(\tau,y,z),z_2,p(y_2,z_2))]a(y_2,z_2)\left|\tan\left(\frac{\pi}{2}y_2\right)\right||dy_2||dz_2|$$
$$+\frac{\pi}{2|\alpha|}\int\limits_{(1,2)\times\mathbb{R}} f(\tau)r_\tau(y,z)[K_-(Z(\tau,y,z),z_2,-p(y_2,z_2))]a(y_2,z_2)\left|\tan\left(\frac{\pi}{2}y_2\right)\right||dy_2||dz_2|,$$

where

$$\tau(y,z,y_2,z_2) = \frac{1}{\alpha}\left(z_2 - z + \ln\left|\cos\left(\frac{\pi}{2}y_2\right)\right| - \ln\left|\cos\left(\frac{\pi}{2}y\right)\right|\right).$$

Kernel k_f of $T_f\circ K$, as an operator on X, is given by

$$\begin{aligned} k_f(y,z,y_2,z_2) = \sum_{n\in\mathbb{Z}} f(\tau+\frac{n}{\alpha})r_{\tau+\frac{n}{\alpha}}(y,z)[K_pm(Z(\tau+\frac{n}{\alpha},y,z),z_2+n,p(y_2,z_2+n))] \\ \cdot\frac{\pi}{2|\alpha|}\left|\tan\left(\frac{\pi}{2}y_2\right)\right||dy|^{1/2}|dz|^{1/2}|dy_2|^{1/2}|dz_2|^{1/2}, \end{aligned} \tag{29}$$

where $\tau=\tau(y,z,y_2,z_2)$, the sign '+' is taken if $p(y,z)p(y_2,z_2)>0$ and the sign '-' corresponds to the case $p(y,z)p(y_2,z_2)<0$. Using this formula, one can easily complete the proof of Theorem 9.

4.5. Indicial family associated with $T_f \circ K$. To compute the indicial family associated with operator $T_f \circ K$, we make the change of variables $y = 1+x$, $y_2 = 1+x_2$ in (29). Since $p(y,z)p(y_2,z_2) = e^{z+z_2} \sin\frac{\pi}{2}x \cdot \sin\frac{\pi}{2}x_2$, the sign of $p(y,z)p(y_2,z_2)$ coincides with the sign of xx_2. We obtain that

$$\begin{aligned} k_f = & \sum_{n\in\mathbb{Z}} f(\tau(1+x,z,1+x_2,z_2+n))r_\tau(1+x,z) \\ & \cdot [K_\pm(Z(\tau,1+x,z),z_2+n,p(Y(\tau,1+x,z),Z(\tau,1+x,z)))] \\ & \cdot \frac{\pi}{2|\alpha|}|x|^{1/2}|x_2|^{1/2}\left|\cot\left(\frac{\pi}{2}x_2\right)\right|\left|\frac{dx}{x}\right|^{1/2}|dz|^{1/2}\left|\frac{dx_2}{x_2}\right|^{1/2}|dz_2|^{1/2}, \end{aligned}$$

where, if $xx_2 > 0$, we take the sign '+'and, if $xx_2 < 0$, the sign is '-'.

We make the change of variables $x = x, x_2 = \frac{x}{s}$. Since $xx_2 = \frac{x^2}{s}$, the sign of xx_2 coincides with the sign of s. Function $\widetilde{K}_f$ defined by function k_f via formula (6) is given by

$$\begin{aligned} \widetilde{K}_f(x,s,z,z_2) = & \sum_{n\in\mathbb{Z}} f(\tau(1+x,z,1+\frac{x}{s},z_2+n))r_\tau(1+x,z) \\ & \cdot [K_\pm(Z(\tau,1+x,z),z_2+n,p(Y(\tau,1+x,z),Z(\tau,1+x,z)))]\frac{\pi}{2|\alpha|}|x||s|^{-1/2}\left|\cot\left(\frac{\pi x}{2s}\right)\right|, \end{aligned}$$

where, for $s > 0$, we take the sign $+$ and, for $s < 0$, the sign $-$.

In the limit $x \to 0$, we get

$$K_+(Z(\tau,1+x,z),z_2+n,p(Y(\tau,1+x,z),Z(\tau,1+x,z))) \to K(z_1,z_2+n,0),$$

and

$$K_-(Z(\tau,1+x,z),z_2+n,p(Y(\tau,1+x,z),Z(\tau,1+x,z))) \to 0.$$

Moreover, for $s > 0$, we get

$$\lim_{x\to 0}\tau(1+x,z,1+\frac{x}{s},z_2+n) = \frac{1}{\alpha}(z_2+n-z-\ln s).$$

Therefore, for $s > 0$

$$\widetilde{K}_f(0,s,z,z_2) = \lim_{x\to 0}\widetilde{K}_f(x,s,z,z_2) = \sum_{n\in\mathbb{Z}} f(\tau_0)r_{\tau_0}(1,z)[K(Z(\tau_0,1,z),z_2+n,0)]\frac{s^{1/2}}{|\alpha|},$$

where $\tau_0 = \frac{1}{\alpha}(z_2+n-z-\ln s)$, and for $s < 0$

$$\widetilde{K}_f(0,s,z,z_2) = 0. \tag{30}$$

Denote by E^0 the restriction of E to $X^0 \cong \mathbb{R}$. The bundle E^0 is trivial, $E^0 \cong X^0 \times \mathbb{C}^N$, moreover, the flow T^E on E induces a flow on E^0 with the corresponding map $r_t(1,z) : E^0_{Z(t,1,z)} \to E^0_z$.

By (30), the indicial operator $I_-(T_f \circ K, \lambda)$ equals zero.

The kernel of indicial operator $I_+(T_f \circ K, \lambda)$ acting on $C^\infty(S^1, E^0 \otimes \Omega^{\frac{1}{2}}_{S^1})$ reads as

$$K_{I_+(T_f\circ K,\lambda)}(z,z_2) = \int_0^\infty s^{-i\lambda}\widetilde{K}_f(0,s,z,z_2)\frac{ds}{s} = \sum_{n\in\mathbb{Z}}\int_0^\infty s^{-i\lambda}f(\tau_0)r_{\tau_0}(1,z)[K(Z(\tau_0,1,z),z_2+n,0)]\frac{s^{1/2}}{|\alpha|}\frac{ds}{s}.$$

Making the change of variables $t = \tau_0(z,z_2,s) = \frac{1}{\alpha}(z_2+n-z-\ln s)$, we get:

$$K_{I_+(T_f\circ K,\lambda)}(z,z_2) = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e^{(i\lambda-\frac{1}{2})(z-z_2+\alpha t)}f(t)r_t(1,z)[K(Z(t,1,z),z_2+n,0)]dt|dz|^{1/2}|dz_2|^{1/2}. \tag{31}$$

For each $\lambda \in \mathbb{C}$, we define a one-parameter group $\{T^{S^1,\lambda}_t : t \in \mathbb{R}\}$ of bounded operators in $L^2(S^1, E^0 \otimes \Omega^{\frac{1}{2}}_{S^1})$ by

$$T^{S^1,\lambda}_t[u(z)|dz|^{1/2}] = e^{(i\lambda-\frac{1}{2})(z-Z(t,1,z)+\alpha t)}r_t(1,z)[u(Z(t,1,z))]|dz|^{1/2}.$$

In the scalar case, we have $E^0 = \mathbb{R}\times\mathbb{C}$, $r_t(1,z)=1$. We denote by $T_t^{(0),S^1,\lambda}$ the corresponding group of operators:

$$T_t^{(0),S^1,\lambda}u(z) = e^{(i\lambda-\frac{1}{2})(z-Z(t,1,z)+\alpha t)}u(Z(t,1,z)). \tag{32}$$

In the case of the space of leafwise differential one forms $E^0 = \mathbb{R}\times\mathbb{C}$, but the action of the flow is given by the formula

$$r_t(1,z) = \frac{\partial Z}{\partial z}(t,1,z) = e^{\int\limits_0^t A_U(Z(\tau_1,1,z),0)d\tau_1}.$$

We denote by $T_t^{(0),S^1,\lambda}$ the corresponding group of operators acting on space $\Omega^1(S^1)$ of smooth differential 1-forms on S^1:

$$T_t^{(1),S^1,\lambda}(u(z)dz) = e^{(i\lambda-\frac{1}{2})(z-Z(t,1,z)+\alpha t)}\frac{\partial Z}{\partial z}(t,1,z)u(Z(t,1,z))dz. \tag{33}$$

For each $f\in C_0^\infty(\mathbb{R})$, we let

$$T_f^{S^1,\lambda} = \int_{-\infty}^{+\infty} f(t)T_t^{S^1,\lambda}dt. \tag{34}$$

The restriction of operator K to $L_0 = \{(y,z) : y=1\}\cong S^1$ is integral operator K_{S^1} in space $L^2(S^1, E\otimes\Omega^{\frac{1}{2}}_{S^1})$ with kernel

$$k_{S^1}(z,z_2) = \sum_{n\in\mathbb{Z}} K(z,z_2+n,0)|dz|^{1/2}|dz_2|^{1/2}.$$

For each $\lambda\in\mathbb{C}$, we introduce integral operator $K_{S^1}(\lambda)$ in $L^2(S^1,E^0\otimes\Omega^{\frac{1}{2}}_{S^1})$ with kernel

$$k_{S^1,\lambda}(z,z_2) = \sum_{n\in\mathbb{Z}} e^{(i\lambda-\frac{1}{2})(z-z_2-n)}K(z,z_2+n,0)|dz|^{1/2}|dz_2|^{1/2}.$$

Proposition 3. *For each leafwise smoothing operator K, indicial family $I_+(T_f\circ K,\lambda)$: $C^\infty(S^1,E^0\otimes\Omega^{\frac{1}{2}}_{S^1})\to C^\infty(S^1,E^0\otimes\Omega^{\frac{1}{2}}_{S^1})$ is given by*

$$I_+(T_f\circ K,\lambda) = T_f^{S^1,\lambda}\circ K_{S^1}(\lambda). \tag{35}$$

4.6. Indicial family associated with K. In this section, we introduce a notion of the indicial family associated with a leafwise smoothing operator K in such a way that an analogue of Theorem 4 holds true (see Proposition 4 below).

Definition 6. *The indicial family $I_+(K,\lambda) : C^\infty(S^1,E^0\otimes\Omega^{\frac{1}{2}}_{S^1})\to C^\infty(S^1,E^0\otimes\Omega^{\frac{1}{2}}_{S^1})$ associated with a leafwise smoothing operator K is defined by the formula*

$$I_+(K,\lambda) = K_{S^1}(\lambda). \tag{36}$$

The kernel of operator $I_+(K,\lambda)$ is given by the formula

$$K_{I_+(K,\lambda)}(z,z_2) = \sum_{n\in\mathbb{Z}} e^{(i\lambda-\frac{1}{2})(z-z_2-n)}K(z,z_2+n,0)|dz|^{1/2}|dz_2|^{1/2}. \tag{37}$$

Proposition 4. *Let K_1, K_2 be leafwise smoothing operators. Then the formula*

$$\text{r-Tr}[T_f\circ K_1, K_2] = -\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\text{tr}(\partial_\lambda I_+(T_f\circ K_1,\lambda)\circ I_+(K_2,\lambda))d\lambda \tag{38}$$

holds true.

The proof of Proposition 4 is given in Appendix A.

We denote by $T_f^{(0)}$ (resp. $T_f^{(1)}$) the operator acting on $\Omega^0(\mathcal{F})$ (resp. $\Omega^1(\mathcal{F})$), by formula (23). By Proposition 4, formula (25) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\text{r-Tr}^s\, C_{t,\psi,f} =& \frac{1}{\pi i}\int_{-\infty}^{+\infty} \mathrm{Tr}(I_+(T_f^{(0)}\circ\psi'(t\Delta^+_{\mathcal{F}}),\lambda)\circ\partial_\lambda I_+(\psi(t\Delta^+_{\mathcal{F}})\circ\Delta^+_{\mathcal{F}},\lambda))d\lambda \\ &- \frac{1}{\pi i}\int_{-\infty}^{+\infty} \mathrm{Tr}(I_+(T_f^{(1)}\circ d^+_{\mathcal{F}}\circ\psi'(t\Delta^+_{\mathcal{F}}),\lambda)\circ\partial_\lambda I_+(\psi(t\Delta^+_{\mathcal{F}})\circ\delta^-_{\mathcal{F}},\lambda))d\lambda.\end{aligned} \tag{39}$$

The next problem is the computation of the indicial families in formula (39).

4.7. Operator $\psi(t\Delta^+_{\mathcal{F}})$ and its indicial family. In this section, we give a description of operator $\psi(t\Delta^+_{\mathcal{F}})$ as a leafwise smoothing operator and compute the indicial family associated with this operator.

First of all, we shall give a description of various objects associated with $\mathcal{F}$. Consider the coordinate neighborhood $\widetilde{X}=(0,2)\times\mathbb{R}$ with coordinates (y,z) or with foliated coordinates (u,v) given by (19). Then the tangent space $T\mathcal{F}$ of $\mathcal{F}$ is generated by vector $\frac{\partial}{\partial u}$ defined by (21). The leafwise de Rham differential $d_{\mathcal{F}}:\Omega^0(\mathcal{F})\to\Omega^1(\mathcal{F})$ reads as

$$d_{\mathcal{F}}f=\frac{\partial f}{\partial u}(y,z)du=\left(\frac{2}{\pi}\cot\left(\frac{\pi}{2}y\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial z}\right)f(y,z)du,$$

where $du\in\Omega^1(\mathcal{F})$ is determined by the condition $\langle du,\frac{\partial}{\partial u}\rangle=1$.

The leafwise Riemannian metric $g_{\mathcal{F}}$ induced by the standard Riemannian metric $g=dy^2+dz^2$ is given by

$$g_{\mathcal{F}}(y,z)=G(y,z)du^2,\quad G(y,z)=\frac{4\cot^2(\frac{\pi}{2}y)+\pi^2}{\pi^2}.$$

It is easy to see that the leafwise de Rham codifferential $d^*_{\mathcal{F}}:\Omega^1(\mathcal{F})\to\Omega^0(\mathcal{F})$ is given by

$$d^*_{\mathcal{F}}(gdu)=-G(y,z)^{-1}\frac{\partial g}{\partial u}+h(y)g, \tag{40}$$

where

$$h(y)=-\frac{d}{dy}\left(\frac{2\pi\cot\left(\frac{\pi}{2}y\right)}{4\cot^2(\frac{\pi}{2}y)+\pi^2}\right)=-\frac{\pi^2}{\sin^2\frac{\pi}{2}y}\frac{4\cot^2\frac{\pi}{2}y-\pi^2}{(4\cot^2\frac{\pi}{2}y+\pi^2)^2}.$$

It follows from here that the leafwise Laplacian $\Delta^+_{\mathcal{F}}:\Omega^0(\mathcal{F})\longrightarrow\Omega^0(\mathcal{F})$ reads as

$$\Delta^+_{\mathcal{F}}=-G(y,z)^{-1}\frac{\partial^2}{\partial u^2}+h(y)\frac{\partial}{\partial u}.$$

The leaves of $\widetilde{\mathcal{F}}$ can be parameterized by $v\in[0,\infty)$. The leaf $\widehat{L}_v$, corresponding to $v\in[0,\infty)$, in the coordinate neighborhood $\tilde{X}$ reads as

$$\widehat{L}_v=\{(y,z)\in(0,2)\times\mathbb{R}:|p(y,z)|=v\}.$$

The projection $(y,z)\mapsto z$ identifies the intersection of $\widehat{L}_v$ with the coordinate neighborhood $(0,2)\times\mathbb{R}$ with the disjoint union of two copies of the semi-axis $(\ln|v|,+\infty)$. Therefore, we have the decomposition

$$L^2(\widehat{L}_v)\cong L^2(\mathbb{R},\sqrt{G(u,v)}du)\oplus L^2(\mathbb{R},\sqrt{G(u,v)}du).$$

As above, let ψ be a smooth function on $\mathbb{R}$ such that the function $\phi(x)=\psi(x^2)$ belongs to $\mathcal{A}$. Then one can define the operator $\psi(t\Delta_v)$ as a bounded operator in $L^2(\widehat{L}_v)$ for each $v\in[0,\infty)$ and the operator $\psi(t\Delta^+_{\mathcal{F}})$ as a bounded operator in $L^2(\widehat{X})$ (see [8, 15]).

Operator $\psi(t\Delta_v)$ is written as follows: for $f=f_+\oplus f_-\in L^2(\mathbb{R},\sqrt{G(u,v)}du)\oplus L^2(\mathbb{R},\sqrt{G(u,v)}du)$, we have

$$\psi(t\Delta_v)f_\pm(u_1)=\int_{\ln|v|}^{+\infty}(K^+_\psi(t,u_1,u_2,v)f_\pm(u_2)+K^-_\psi(t,u_1,u_2,v)f_\mp(u_2))\sqrt{G(u_2,v)}du_2.$$

The following proposition is a straightforward consequence of general results proved in [8, 15].

Proposition 5. *(1) The action of $\psi(t\Delta^+_{\mathcal{F}})$ on $b \in C^\infty((0,2)\times\mathbb{R})$ is given by*

$$\psi(t\Delta^+_{\mathcal{F}})b(y_1,z_1) = \int\limits_{\ln|p(y_1,z_1)|}^{+\infty} \Big(K^+_\psi(t,z_1,z_2,p(y_1,z_1))b(Y_2(y_1,z_1,z_2),z_2) \\ + K^-_\psi(t,z_1,z_2,p(y_1,z_1))b(2-Y_2(y_1,z_1,z_2),z_2)\Big)\sqrt{G(z_2,p(y_1,z_1))}|dz_2|.$$

(2) Kernel K_ψ of $\psi(t\Delta^+_{\mathcal{F}})$ is a smooth function on $\widehat{\mathcal{R}}$.

As it was shown in [15], if the Fourier transform of the function $\phi(x)=\psi(x^2)$ is compactly supported (note that, by Paley-Wiener theorem, such function ϕ belongs to $\mathcal{A}$), then, for each $t>0$, there exists a constant C such that $K^\pm_\psi(t,z_1,z_2,v)=0$ for all (z_1,z_2,v) such that $d(z_1,z_2)>C$. Therefore, in this case, operator $\psi(t\Delta^+_{\mathcal{F}})$ is a leafwise smoothing operator with the kernel

$$K_\pm(t,z_1,z_2,v) = K^\pm_\psi(t,z_1,z_2,v)\sqrt{G(Y_2(y_1,z_1,z_2),z_2)}.$$

By definition, operator $\psi(t\Delta^+_{\mathcal{F}})_{S^1}(\lambda)$ is the integral operator on S^1 with the kernel

$$k_{S^1,\lambda}(z_1,z_2) = \sum_{n\in\mathbb{Z}} e^{(i\lambda-\frac{1}{2})(z_1-z_2-n)}K_\psi(t,z_1,z_2+n,0)|dz_1|^{1/2}|dz_2|^{1/2}. \tag{41}$$

It is easy to check that $h(1)=1$. Hence, for $v=0$ (or $y=1$)

$$d^*_{\mathcal{F}}(gdu) = -\frac{\partial g}{\partial z}+g, \quad \Delta^+_0 = -\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial z}.$$

Therefore, $\psi(t\Delta^+_0)$ is a pseudodifferential operator with complete symbol $\psi(t(\xi^2+i\xi))$:

$$\psi(t\Delta^+_0)f(z) = \frac{1}{2\pi}\int e^{i(z-z_1)\xi}\psi(t(\xi^2+i\xi))f(z_1)dz_1d\xi.$$

Kernel $K_\psi(t,z_1,z_2,0)$ of $\psi(t\Delta^+_0)$ is given by

$$K_\psi(t,z_1,z_2,0) = \frac{1}{2\pi}\int e^{i(z_1-z_2)\xi}\psi(t(\xi^2+i\xi))d\xi.$$

Since $\psi(t(\xi^2+i\xi)) = \psi(t[(\xi+\frac{i}{2})^2+\frac{1}{4}])$, the function $\xi\mapsto\psi(t(\xi^2+i\xi))$ belongs to Schwartz space S, and, therefore, the integral in the right hand side of the last identity converges absolutely. By (41), we get

$$I_+(\psi(t\Delta^+_{\mathcal{F}}),\lambda) = (\psi(t\Delta^+_{\mathcal{F}}))_{S^1}(\lambda) = \psi\left(t\left((D_z-\lambda)^2+\frac{1}{4}\right)\right). \tag{42}$$

By a straightforward computation, one can show that, for each leafwise smoothing operator $K: C^\infty(X)\to C^\infty(X)$

$$I_+(T^{(1)}_f\circ d_{\mathcal{F}}\circ K,\lambda) = \left(iD_z - i\lambda+\frac{1}{2}\right)\circ I_+(T_f\circ K,\lambda),$$

and, for each leafwise smoothing operator $K:\Omega^1(\mathcal{F})\to\Omega^1(\mathcal{F})$

$$I_+(T_f\circ K\circ d^*_{\mathcal{F}},\lambda) = I_+(T_f\circ K,\lambda)\circ\left(-iD_z+i\lambda+\frac{1}{2}\right).$$

Using the above computations of indicial families, by (39) we arrive at the following statement.

Proposition 6. *If the Fourier transform of the function* $\phi(x)=\psi(x^2)$ *is compactly supported, then, for each* $t>0$ *and* $f\in C_0^\infty(\mathbb{R})$, *function* $\operatorname{r-Tr}^s C_{t,\psi,f}$ *is well-defined and formula*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\operatorname{r-Tr}^s C_{t,\psi,f} = &-\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\Bigg(T_f^{(0),S^1,\lambda}\circ\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)\\ &\circ 2t\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)(D_z-\lambda)\left((D_z-\lambda)^2+\frac14\right)\Bigg)d\lambda\\ &-\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\Bigg(T_f^{(0),S^1,\lambda}\circ\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)\\ &\circ 2\psi\left(t\left((D_z-\lambda)^2+\frac14\right)\right)(D_z-\lambda)\Bigg)d\lambda\\ &+\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\Bigg(T_f^{(1),S^1,\lambda}\circ\left(iD_z-i\lambda+\frac12\right)\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)\\ &\circ 2t\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)(D_z-\lambda)\left(-iD_z+i\lambda+\frac12\right)\Bigg)d\lambda\\ &-\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\Bigg(T_f^{(1),S^1,\lambda}\circ\left(iD_z-i\lambda+\frac12\right)\psi'\left(t\left((D_z-\lambda)^2+\frac14\right)\right)\\ &\circ i\psi\left(t\left((D_z-\lambda)^2+\frac14\right)\right)\Bigg)d\lambda\end{aligned}\tag{43}$$

holds true.

In the particular case when $\psi(x)=e^{-\frac{x}{2}}$, formula (43) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\operatorname{r-Tr}^s B_{t,f} = &-\frac{t}{2\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\left(T_f^{(0),S^1,\lambda}\circ e^{-t\left((D_z-\lambda)^2+\frac14\right)}(D_z-\lambda)\left((D_z-\lambda)^2+\frac14\right)\right)d\lambda\\ &+\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\left(T_f^{(0),S^1,\lambda}\circ e^{-t\left((D_z-\lambda)^2+\frac14\right)}\right)(D_z-\lambda)d\lambda\\ &+\frac{t}{2\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\left(T_f^{(1),S^1,\lambda}\circ e^{-t\left((D_z-\lambda)^2+\frac14\right)}(D_z-\lambda)\left((D_z-\lambda)^2+\frac14\right)\right)d\lambda\\ &+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}\left(T_f^{(1),S^1,\lambda}\circ\left(-(D_z-\lambda)+\frac{i}{2}\right)e^{-t\left((D_z-\lambda)^2+\frac14\right)}\right)d\lambda.\end{aligned}\tag{44}$$

Observe that in this case, the Fourier transform of the function $\phi(x)=e^{-\frac{x^2}{2}}$ is not compactly supported, and, therefore, Proposition 6 can not be directly applied. The expression in the right hand side of (43) depends continuously on ψ in the topology of the Schwartz space and can be considered as a definition of function $\operatorname{r-Tr}^s C_{t,\psi,f}$ in the case when ψ does not satisfy the assumptions of Proposition 6, in particular, when $\psi(x)=e^{-\frac{x}{2}}$. In other words, as in [1, 15], we can take an arbitrary sequence $\psi_n\in C^\infty(\mathbb{R})$ such that, for each n, the Fourier transform of $\phi_n(x)=\psi_n(x^2)$ is compactly supported and ψ_n converges to $e^{-\frac{x^2}{2}}$ in the topology of the Schwartz space and put, by definition, $\operatorname{r-Tr}^s B_{t,f}=\lim_{n\to\infty}\operatorname{r-Tr}^s C_{t,\psi,f}$.

Thus, the proof of Theorem 8 is reduced to the computation of integrals of the form $\int_{-\infty}^{+\infty} \mathrm{tr}(T_f^{S^1,\lambda} \circ \phi(D_z-\lambda))d\lambda$ given in Appendix B. The use of the results of Appendix B immediately allows us to compute the expression in the right hand side of (44) and complete the proof of Theorem 8.

4.8. Flows on fiber bundles. In this section, as an illustration, we describe analogues of the notions introduced above for the example of the flows on fiber bundles considered in Section 3. We shall use notation introduced in Section 3.

Let K be a leafwise smoothing operator in $C^\infty(X, E\otimes\Omega_X^{\frac{1}{2}})$ given by kernel $k\in C^\infty(X\times_\pi X, \mathcal{L}(E)\otimes|\mathcal{V}|^{\frac{1}{2}}\otimes|\mathcal{V}|^{\frac{1}{2}})$. First of all, by (17), we have: for $s>0$

$$\widetilde{K}_f(0,s,x^0,x^0_1)=\lim_{x\to 0}\widetilde{K}_f(x,s,x^0,x^0_1)=f(-\frac{1}{a'(\alpha_j)}\ln s)r_{-\frac{1}{a'(\alpha_j)}\ln s}(x^0,\alpha_j)$$
$$\cdot\, k(S_{-\frac{1}{a'(\alpha_j)}\ln s}(x^0,\alpha_j),x^0_1,\alpha_j)|w(x^0,\alpha_j)|^{1/2}|w(x^0_1,\alpha_j)|^{\frac{1}{2}}\frac{s^{1/2}}{|a'(\alpha_j)|},$$

and, for $s<0$,

$$\widetilde{K}_f(0,s,x^0,x^0_1)=0,\quad x^0,x^0_1\in F_{\alpha_j}.$$

Therefore, the kernel of indicial operator $I_+(T_f\circ K,\lambda)$ is given by

$$K_{I_+(T_f\circ K,\lambda)}(x^0,x^0_1)=\int_0^\infty s^{-i\lambda}\widetilde{K}_f(0,s,x^0,x^0_1)\frac{ds}{s}$$
$$=\int_0^{+\infty} s^{-i\lambda}f\Big(-\frac{1}{a'(\alpha_j)}\ln s\Big)r_{-\frac{1}{a'(\alpha_j)}\ln s}(x^0,\alpha_j)k\left(S_{-\frac{1}{a'(\alpha_j)}\ln s}(x^0,\alpha_j),x^0_1,\alpha_j\right)$$
$$\cdot\,|w(x^0,\alpha_j)|^{1/2}|w(x^0_1,\alpha_j)|^{\frac{1}{2}}\frac{s^{1/2}}{|a'(\alpha_j)|}\frac{ds}{s}|dx^0|^{\frac{1}{2}}|dx^0_1|^{\frac{1}{2}},\quad x^0,x^0_1\in F_{\alpha_j}.$$

Making the change of variables $t=-\frac{1}{a'(\alpha_j)}\ln s$ in the last integral, we get

$$K_{I_+(T_f\circ K,\lambda)}(x^0,x^0_1)=\int_{-\infty}^{+\infty} e^{ia'(\alpha_j)\lambda t}e^{-\frac{1}{2}a'(\alpha_j)t}f(t)r_t(x^0,\alpha_j)$$
$$\cdot\, k[S_t(x^0,\alpha_j),x^0_1,\alpha_j]|w(x^0,\alpha_j)|^{1/2}|w(x^0_1,\alpha_j)|^{\frac{1}{2}}dt|dx^0|^{\frac{1}{2}}|dx^0_1|^{\frac{1}{2}}. \tag{45}$$

Since $\widetilde{K}_f(0,s,x^0,x^0_1)=0$ for $s<0$, we obtain $I_-(T_f\circ K,\lambda)=0$.

Let us describe operator $I_+(T_f\circ K,\lambda)$. Since each α_j is a fixed point of flow $\bar{T}$, flow T maps the fiber F_{α_j} into itself. Denote by $E^{(\alpha_j)}$ the restriction of E to F_{α_j}, by $T^{(\alpha_j)}$ the restriction of T to F_{α_j} and by $r_t^{(\alpha_j)}: E^{(\alpha_j)}_{T_t^{(\alpha_j)}(x)}\to E^{(\alpha_j)}_x$ the map given by r_t. Let $(T_t^{(\alpha_j)})^*$ be the operator on $C^\infty(F_{\alpha_j},E^{(\alpha_j)})$, induced by T:

$$(T_t^{(\alpha_j)})^*u(x)=r_t^{(\alpha_j)}(x)[u(T_t^{(\alpha_j)}(x))].$$

For each $g\in C_0^\infty(\mathbb{R})$, we define operator $T_g^{(\alpha_j)}$ in $C^\infty(F_{\alpha_j},E^{(\alpha_j)}\otimes\Omega^{\frac{1}{2}}_{F_{\alpha_j}})$ by the following formula: for $\mu=u|d\alpha_{\alpha_j}|^{1/2}$

$$T_g^{(\alpha_j)}=\left(\int_{-\infty}^{+\infty} g(t)(T_t^{(\alpha_j)})^*u\,dt\right)|d\alpha_{\alpha_j}|^{1/2}.$$

For each $\alpha\in S^1$, one can naturally define the restriction of K to F_α as an integral operator $K(\alpha)$ in $C^\infty(F_\alpha,E^{(\alpha)}\otimes\Omega^{\frac{1}{2}}_{F_\alpha})$.

We have:

$$C^\infty(X^0, E_{X^0}\otimes \Omega^{\frac12}_{X^0}) = \bigoplus_{j=1}^k C^\infty(F_{\alpha_j}, E^{(\alpha_j)}\otimes \Omega^{\frac12}_{F_{\alpha_j}}).$$

Using formulas (13) and (45), it is easy to show that the operator $I_+(T_f\circ K,\lambda)$ maps each subspace $C^\infty(F_{\alpha_j}, E^{(\alpha_j)}\otimes \Omega^{\frac12}_{F_{\alpha_j}})$ into itself, and its restriction to $C^\infty(F_{\alpha_j}, E^{(\alpha_j)}\otimes \Omega^{\frac12}_{F_{\alpha_j}})$ reads as

$$I_+(T_f\circ K,\lambda)\Big|_{C^\infty(F_{\alpha_j}, E^{(\alpha_j)}\otimes \Omega^{\frac12}_{F_{\alpha_j}})} = T_{f_\lambda^{(\alpha_j)}}^{(\alpha_j)}\circ K(\alpha_j), \tag{46}$$

where $f_\lambda^{(\alpha_j)}(t) = e^{ia'(\alpha_j)\lambda t}e^{-\frac12 a'(\alpha_j)t}f(t)$.

In this case, one can prove an analogue of Proposition 4 by the same method as was used in the proof of this proposition above in this section. In particular, indicial family $I_+(K,\lambda)$ associated with a leafwise smoothing operator $K: C^\infty(X, E\otimes\Omega^{\frac12}_X)\to C^\infty(X,E\otimes \Omega^{\frac12}_X)$ is defined as

$$I_+(K,\lambda) = K_{X^0}.$$

Here K_{X^0} is the restriction of K to X^0. This operator maps each subspace $C^\infty(F_{\alpha_j}, E^{(\alpha_j)}\otimes \Omega^{\frac12}_{F_{\alpha_j}})$ into itself, and its restriction to this subspace coincides with $K(\alpha_j)$.

Thus, in the case under consideration, indicial family $I(K,\lambda)$ is independent of λ. Therefore, thanks to an analogue of Proposition 4 for this case, we get for each leafwise smoothing operators K_1 and K_2

$$\text{r-Tr}[T_f\circ K_1, K_2] = 0.$$

By this fact, it follows from (24) that $\text{r-Tr}^s\, C_{t,\psi,f}$ is independent of t:

$$\frac{d}{dt}\,\text{r-Tr}^s\, C_{t,\psi,f} = 0.$$

A. Proof of Proposition 4

Suppose that a function $g\in C_0^\infty(\mathbb{R})$ satisfies the following conditions: g is an even function, $g(s)\geqslant 0$ for each s, $\operatorname{supp} g\subset(-1,1)$ and $\int_{-\infty}^{+\infty} g(s)ds = 1$. For each natural m, we let $g_m(t) = mg(mt)$. The operator $T_{g_m}\circ K_2$ belongs to $\mathcal{K}(X,X^0)$, therefore, the following formula

$$\text{r-Tr}[T_f\circ K_1, T_{g_m}\circ K_2] = -\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}(\partial_\lambda I_+(T_f\circ K_1,\lambda)\circ I_+(T_{g_m}\circ K_2,\lambda))d\lambda \tag{47}$$

holds true.

Let us show that, as $m\to\infty$, the left and right hand sides of (47) converge to the left and right hand sides of (38), respectively. The difference R_m of the right hand sides of (47) and (38) is written as:

$$R_m = -\frac{1}{\pi i}\int_{-\infty}^{+\infty}\operatorname{tr}(\partial_\lambda I_+(T_f\circ K_1,\lambda)\circ(I_+(T_{g_m}\circ K_2,\lambda) - I_+(K_2,\lambda))d\lambda.$$

Using properties of trace class operators, we get

$$|R_m| \leqslant \frac{1}{\pi}\int_{-\infty}^{+\infty}\|\partial_\lambda I_+(T_f\circ K_1,\lambda)\|_2\cdot\|I_+(T_{g_m}\circ K_2,\lambda) - I_+(K_2,\lambda)\|_2 d\lambda, \tag{48}$$

where $\|A\|_2 = \operatorname{tr}(A^*A)$ is the Hilbert-Schmidt norm of the operator A.

Using (31) and (37), it is easy to prove the following estimate:

$$\sup_{z,z_2}|K_{I_+(T_{g_m}\circ K_2,\lambda)}(z,z_2) - K_{I_+(K_2,\lambda)}(z,z_2)| \leqslant C_1\frac{|\lambda|}{m}.$$

It implies that

$$\|I_+(T_{g_m}\circ K_2,\lambda)-I_+(K_2,\lambda)\|_2 = \left(\int_0^1\int_0^1 |K_{I_+(T_{g_m}\circ K_2,\lambda)}(z,z_2)-K_{I_+(K_2,\lambda)}(z,z_2)|^2dzdz_2\right)^{1/2}\leqslant C_1\frac{|\lambda|}{m}.$$

Therefore, by (48), we get

$$|R_m|\leqslant \frac{C_2}{m}\int\limits_{-\infty}^{+\infty}\|\partial_\lambda I_+(T_f\circ K_1,\lambda)\|_2|\lambda|d\lambda. \tag{49}$$

The kernel of $\partial_\lambda I_+(T_f\circ K_1,\lambda)$ as an operator in $\mathbb{R}$ reads as

$$K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)=\int\limits_{-\infty}^{\infty} e^{(i\lambda-\frac12)(z-z_2+\alpha t)}i(z-z_2+\alpha t)f(t)r_t(1,z)[K(Z(t,1,z),z_2,0)]dt.$$

Consider the function $h\in C_0^\infty(\mathbb{R}^3)$ given by

$$h(z,z_2,t)=e^{-\frac12\alpha t}i(z-z_2+\alpha t)f(t)r_t(1,z)[K(Z(t,1,z),z_2,0)].$$

Note that

$$K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)=e^{(i\lambda-\frac12)(z-z_2)}\hat h(z,z_2,\alpha\lambda),$$

where $\hat h\in\mathcal{S}(\mathbb{R})$ is the Fourier transform of h with respect to t. It is easy to see that there exists a constant $d>0$ such that $h(z,z_2,t)=0$ for $|z-z_2|>d$ or $|t|>d$, and $h(z+1,z_2+1,t)=h(z,z_2,t)$ for each z,z_2,t. Therefore, $K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)=0$ for $|z-z_2|>d$, $K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z+1,z_2+1)=K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)$ for each z,z_2,λ and, as $\lambda\to\infty$, kernel $K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)$ converges to zero faster than any power of λ uniformly on z,z_2: for each N

$$\sup_{z,z_2,\lambda}(1+\lambda^2)^N\left|K_{\partial_\lambda I_+(T_f\circ K_1,\lambda)}(z,z_2)\right|<\infty$$

Hence, as $\lambda\to\infty$, the Hilbert-Schmidt norm of operator $\partial_\lambda I_+(T_f\circ K_1,\lambda)$ in $L^2(S^1)$ converges to zero faster than any power of λ: for each N

$$\sup_{x,y,\lambda}(1+\lambda^2)^N\|\partial_\lambda I_+(T_f\circ K_1,\lambda)\|_2<\infty.$$

Therefore, the integral in the right hand side of inequality (49) converges, and, by (49), we obtain that $|R_m|\to 0$ as $m\to\infty$.

Consider the left hand side of (47). Let us show that $\text{r-Tr}(T_f\circ K_1\circ T_{g_m}\circ K_2)$ tends to $\text{r-Tr}(T_f\circ K_1\circ K_2)$ and $\text{r-Tr}(T_{g_m}\circ K_2\circ T_f\circ K_1)$ tends to $\text{r-Tr}(K_2\circ T_f\circ K_1)$ as $m\to\infty$. Formula (29) implies that the restriction of the kernel of $T_f\circ K_1\circ T_{g_m}\circ K_2$ to the diagonal is represented as (for $p(y,z)>0\Leftrightarrow y\in(0,1)$):

$$\begin{aligned}
&k_{T_f\circ K_1\circ T_{g_m}\circ K_2}\,|_\Delta\,(y,z,y,z)\\
&=\frac{\pi^2}{4\alpha^2}\sum_{n,n_2\in\mathbb{Z}}\Bigg(\int\limits_0^1\int\limits_0^1 f(\tau+\frac{n}{\alpha})r_{\tau+\frac{n}{\alpha}}(y,z)[K_{1+}(Z(\tau+\frac{n}{\alpha},y,z),z_2+n,p(y_2,z_2+n))]\\
&\cdot\left|\tan\left(\frac{\pi}{2}y_2\right)\right|g_m(\tau_1+\frac{n_2}{\alpha})r_{\tau_1+\frac{n_2}{\alpha}}(y_2,z_2)[K_{2+}(Z(\tau_1+\frac{n_2}{\alpha},y_2,z_2),z+n_2,p(y,z+n_2))]\\
&\cdot\left|\tan\left(\frac{\pi}{2}y\right)\right|dy_2dz_2\Bigg)|dy||dz|\\
&+\frac{\pi^2}{4\alpha^2}\sum_{n,n_2\in\mathbb{Z}}\Bigg(\int\limits_0^1\int\limits_{-1}^0 f(\tau+\frac{n}{\alpha})r_{\tau+\frac{n}{\alpha}}(y,z)[K_{1-}(Z(\tau+\frac{n}{\alpha},y,z),z_2+n,p(y_2,z_2+n))]\\
&\cdot\left|\tan\left(\frac{\pi}{2}y_2\right)\right|g_m(\tau_1+\frac{n_2}{\alpha})r_{\tau_1+\frac{n_2}{\alpha}}(y_2,z_2)[K_{2-}(Z(\tau_1+\frac{n_2}{\alpha},y_2,z_2),z+n_2,-p(y,z+n_2))]
\end{aligned}$$

$$\cdot \left|\tan\left(\frac{\pi}{2}y\right)\right| dy_2dz_2\Bigg)|dy||dz|,$$

where $\tau = \tau(y,z,y_2,z_2)$ and $\tau_1 = \tau(y_2,z_2,y,z) = -\tau(y,z,y_2,z_2)$. Note that, in the first term, $p(y,z)$ and $p(y_2,z_2)$ are of the same sign, and, in the second one, are of different signs. By means of some simple transformations, this formula can be rewritten as follows:

$$k_{T_f\circ K_1\circ T_{g_m}\circ K_2}\mid_\Delta (y,z,y,z) = \left|\tan\left(\frac{\pi}{2}y\right)\right| \left(\int\limits_{-\infty}^{+\infty} g_m(t)v(t,y,z)dt\right)|dy||dz|,$$

where

$$\begin{aligned} v(t,y,z) =& \frac{\pi}{2|\alpha|}\sum_{N\in\mathbb{Z}}\int\limits_{\alpha t'+N+\ln|p(y,z)|}^{+\infty} dz_2' f\left(t'+\frac{N}{\alpha}\right) \\ &\cdot\Bigg[r_{t'+\frac{N}{\alpha}}(y,z)\left[K_{1+}\left(Z\left(t'+\frac{N}{\alpha},y,z\right),z_2',p(Y_3',z_2')\right)\right)\Bigg] \\ &\cdot r_{-t'}(Y_3',z_2')\left[K_{2+}(Z(-t',Y_3',z_2'),z+N,p(y,z+N))\right] \\ &+ r_{t'+\frac{N}{\alpha}}(y,z)\left[K_{1-}(Z(t'+\frac{N}{\alpha},y,z),z_2',p(Y_3',z_2'))\right] \\ &\cdot r_{-t'}(Y_3',z_2')[K_{2-}(Z(-t',Y_3',z_2'),z+N,-p(y,z+N))]\Bigg]. \end{aligned}$$

By (29), the kernel of $T_f\circ K_1\circ K_2$ is given by:

$$\begin{aligned} &k_{T_f\circ K_1\circ K_2}(y,z,y_2,z_2) \\ &\quad= \frac{\pi}{2|\alpha|}\sum_{n\in\mathbb{Z}} f\left(\tau+\frac{n}{\alpha}\right) r_{\tau+\frac{n}{\alpha}}(y,z)\left[K_+\left(Z\left(\tau+\frac{n}{\alpha},y,z\right),z_2+n,p(y_2,z_2+n)\right)\right] \\ &\quad\cdot\left|\tan\left(\frac{\pi}{2}y_2\right)\right| |dy|^{1/2}|dz|^{1/2}|dy_2|^{1/2}|dz_2|^{1/2} \\ &\quad+ \frac{\pi}{2|\alpha|}\sum_{n\in\mathbb{Z}} f\left(\tau+\frac{n}{\alpha}\right) r_{\tau+\frac{n}{\alpha}}(y,z)\left[K_-\left(Z\left(\tau+\frac{n}{\alpha},y,z\right),z_2+n,p(y_2,z_2+n)\right)\right] \\ &\quad\cdot\left|\tan\left(\frac{\pi}{2}y_2\right)\right| |dy|^{1/2}|dz|^{1/2}|dy_2|^{1/2}|dz_2|^{1/2}, \end{aligned}$$

where K_+ and K_- define the kernel of $K_1\circ K_2$:

$$K_\pm(z_1,z_2,v) = \int\limits_{\ln|v|}^{+\infty} K_{1+}(z_1,z_3,v)K_{2\pm}(z_3,z_2,v)|dz_3| + \int\limits_{\ln|v|}^{+\infty} K_{1-}(z_1,z_3,v)K_{2\mp}(z_3,z_2,-v)|dz_3|.$$

It is easy to see that, for $p(y,z)>0$, one can write:

$$k_{T_f\circ K_1\circ K_2}\mid_\Delta (y,z,y,z) = \left|\tan\left(\frac{\pi}{2}y\right)\right| \left(\int\limits_{-\infty}^{+\infty} g_m(t)v(0,y,z)dt\right)|dy||dz|.$$

Therefore, we get

$$k_{T_f\circ K_1\circ T_{g_m}\circ K_2}\mid_\Delta (y,z,y,z) - k_{T_f\circ K_1\circ K_2}\mid_\Delta (y,z,y,z) = h_m(y,z)\left|\frac{dy}{y-1}\right||dz|,$$

where

$$h_m(y,z) = (1-y)\tan\left(\frac{\pi}{2}y\right)\int\limits_{-\infty}^{+\infty} g_m(t)(v(t,y,z)-v(0,y,z))dt.$$

Function h_m is a smooth, 1-periodic in z function on $(0,2)\times\mathbb{R}$, moreover

$$h_m(1,z)=\frac{2}{\pi}\int_{-\infty}^{+\infty} g_m(t)(v(t,1,z)-v(0,1,z))dt.$$

Therefore, by definition, we have

$$\left(k_{T_f\circ K_1\circ T_{g_m}\circ K_2}\mid_\Delta - k_{T_f\circ K_1\circ K_2}\mid_\Delta\right)\mid_{y=1} = h_m(1,z)|dz|$$

and

$$\begin{aligned}T_m :=& \,\text{r-Tr}(T_f\circ K_1\circ T_{g_m}\circ K_2)-\text{r-Tr}(T_f\circ K_1\circ K_2)\\ =& \lim_{\varepsilon\to 0}\left(\int_{1>|y-1|>\varepsilon}\int_0^2 \operatorname{tr} h_m(y,z)|dz|\left|\frac{dy}{y-1}\right| + 2\ln\varepsilon\int_0^2 \operatorname{tr} h_m(1,z)|dz|\right).\end{aligned}$$

There exists a constant C such that, for each $t\in(-1,1)$, $z\in\mathbb{R}$ and $y\in(0,2)$, we have the inequality $|v(t,y,z)-v(0,y,z)|\leqslant Ct$. Hence, we obtain that, for each $z\in\mathbb{R}$ and $y\in(0,2)$, the estimate

$$|h_m(y,z)|\leqslant C\int_{-\infty}^{+\infty} g_m(t)|v(t,y,z)-v(0,y,z)|dt\leqslant \frac{C_3}{m} \tag{50}$$

holds true with some constant $C_3>0$.

A similar estimate can be proved for the partial derivative with respect to y:

$$\left|\frac{\partial h_m(y,z)}{\partial y}\right|\leqslant\frac{C_4}{m},\quad z\in\mathbb{R}, y\in(0,2). \tag{51}$$

Since

$$\int_{1>|y-1|>\varepsilon}\frac{dy}{|y-1|}=-2\ln\varepsilon,$$

the formula for T_m can be rewritten as follows:

$$T_m=\int_0^2\int_0^1 \frac{\operatorname{tr} h_m(y,z)-\operatorname{tr} h_m(1,z)}{|y-1|}|dz||dy|. \tag{52}$$

In view of estimates (50) and (51), it follows immediately from (52) that

$$|T_m|\leqslant\frac{C_5}{m}.$$

Therefore, $\lim_{m\to\infty}\text{r-Tr}(T_f\circ K_1\circ T_{g_m}\circ K_2 - T_f\circ K_1\circ K_2)=\lim_{m\to\infty}T_m=0$. Similarly, one can show that $\text{r-Tr}(T_{g_m}\circ K_2\circ T_f\circ K_1)\to\text{r-Tr}(K_2\circ T_f\circ K_1)$. Hence, as $m\to\infty$, the left hand side of (47) tends to the left hand side of (38). Thus, formula (38) is proved.

B. Computation of integrals

In this section, we consider an arbitrary vector bundle $E^0=\mathbb{R}\times\mathbb{C}^N$ equipped with the flow given by a map $r_t(1,z): E^0_{Z(t,1,z)}\to E^0_{1,z}$. For each $f\in C_0^\infty(\mathbb{R})$, operator $T_f^{S^1,\lambda}$ in $L^2(S^1,E^0\otimes\Omega^{\frac12}_{S^1})$ is defined by (34).

Proposition 7. *For each $\phi\in\mathcal{A}$, the formula*

$$\int_{-\infty}^{+\infty}\operatorname{tr}(T_f^{S^1,\lambda}\circ\phi(D_z-\lambda))d\lambda=\frac{1}{\alpha}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{\alpha}\right)\int_0^1 e^{-\frac12(z-Z(\frac{n}{\alpha},1,z)+n)}\hat{\phi}\left(z-Z(\frac{n}{\alpha},1,z)+n\right)\operatorname{tr} r_{\frac{n}{\alpha}}(1,z)dz$$

holds true.

Proof. Operator $\phi(D_z-\lambda)$ reads as

$$\phi(D_z-\lambda)v(z)=\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{i(z-z_1)\xi}\phi(\xi-\lambda)v(z_1)dz_1d\xi.$$

Therefore, we have

$$T_f^{S^1,\lambda}\circ\phi(D_z-\lambda)v(z)$$
$$=\frac{1}{2\pi}\iiint f(\tau)e^{(i\lambda-\frac{1}{2})(z-Z(\tau,1,z)+\alpha\tau)}e^{i(Z(\tau,1,z)-z_1)\xi}\phi(\xi-\lambda)r_\tau(1,z_1)v(z_1)dz_1d\xi d\tau.$$

The kernel of this operator as an operator on S^1 reads as

$$K_\lambda(z,z_1)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\iint\limits_{\mathbb{R}^2} f(\tau)e^{(i\lambda-\frac{1}{2})(z-Z(\tau,1,z)+\alpha\tau)}\cdot e^{i(Z(\tau,1,z)-z_1-n)\xi}\phi(\xi-\lambda)r_\tau(1,z_1)d\xi d\tau. \tag{53}$$

The integral can be understood as an absolutely convergent double integral. The restriction of K_λ to the diagonal reads as

$$K_\lambda(z,z)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\iint\limits_{\mathbb{R}^2} f(\tau)e^{(i\lambda-\frac{1}{2})(z-Z(\tau,1,z)+\alpha\tau)}e^{i(Z(\tau,1,z)-z-n)\xi}\phi(\xi-\lambda)r_\tau(1,z)d\xi d\tau. \tag{54}$$

For a function $f\in\mathcal{S}(\mathbb{R})$, we denote by $\hat{f}\in\mathcal{S}(\mathbb{R})$ its Fourier transform:

$$\hat{f}(k)=\int_{-\infty}^{+\infty} e^{-ikx}f(x)dx.$$

Making the change of variables $\xi_1=\xi-\lambda, \tau_1=\alpha\tau-n$ in the integral, formula (54) can be rewritten as follows:

$$\begin{aligned}K_\lambda(z,z)=&\frac{1}{2\pi\alpha}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty} f\left(\frac{\tau_1+n}{\alpha}\right)e^{(i\lambda-\frac{1}{2})\tau_1}e^{-\frac{1}{2}(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n)}\\ &\cdot\hat{\phi}\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)r_{\frac{\tau_1+n}{\alpha}}(1,z)d\tau_1.\end{aligned} \tag{55}$$

One can show that, for each $\beta>0$, there exists a constant $C>0$ such that for each $\tau_1\in\mathbb{R}, n\in\mathbb{Z}$ and $z\in\mathbb{R}$, the estimate

$$\left|f\left(\frac{\tau_1+n}{\alpha}\right)e^{-\frac{1}{2}(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n)}\hat{\phi}\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)\right|<Ce^{-\beta(|\tau_1|+|n|)} \tag{56}$$

holds true. Indeed, since f is compactly supported, there exists a constant $K>0$ such that $\operatorname{supp} f\subset[-K,K]$. Therefore, we can assume that

$$\left|\frac{\tau_1+n}{\alpha}\right|<K. \tag{57}$$

By the Paley-Wiener theorem, for each $\beta>0$, there exists $C>0$ such that

$$|\hat{\phi}(z)|<Ce^{-\beta|z|},\quad z\in\mathbb{R}. \tag{58}$$

Since the function $z-Z(\tau,1,z)$ is periodic with period 1, there exists a constant $r>0$ such that, for each τ_1 and n, satisfying (57), and for each z, the estimate

$$\left|z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)\right|<r \tag{59}$$

holds true. Estimates (58) and (59) imply the existence of a constant $C_1>0$ such that, for each τ_1 and n satisfying (57), and for each z, the estimate

$$\left|\hat{\phi}\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)\right|<C_1e^{-\beta(|\tau_1|+|n|)}$$

holds true. It implies immediately estimate (56). This estimate yields that the series in the right hand side of (53) converges.

Estimate (56) allows us to change the order of integration in the formula:

$$\begin{aligned}\operatorname{tr}(T_f^{S^1,\lambda}\circ\phi(D_z-\lambda)) &= \int_0^1 K_\lambda(z,z)dz\\ &=\frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{(i\lambda-\frac12)\tau_1}\left(\int_0^1\sum_{n\in\mathbb{Z}} f\left(\frac{\tau_1+n}{\alpha}\right)e^{-\frac12\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)}\right. \quad (60)\\ &\left.\cdot\,\hat\phi\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)\operatorname{tr} r_{\frac{\tau_1+n}{\alpha}}(1,z)dz\right)d\tau_1.\end{aligned}$$

Observe that the sum over n in the right hand side of (60) consists of finitely many non-vanishing terms, because f is compactly supported.

Let us consider the function

$$\begin{aligned}F(\tau_1) =& e^{-\frac12\tau_1}\int_0^1\sum_{n\in\mathbb{Z}} f\left(\frac{\tau_1+n}{\alpha}\right)e^{-\frac12\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)}\\ &\cdot\,\hat\phi\left(z-Z\left(\frac{\tau_1+n}{\alpha},1,z\right)+n\right)\operatorname{tr} r_{\frac{\tau_1+n}{\alpha}}(1,z)dz.\end{aligned}$$

Formula (60) can be rewritten as:

$$\operatorname{tr}(T_f^{S^1,\lambda}\circ\phi(D_z-\lambda)) = \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{i\lambda\tau_1}F(\tau_1)d\tau_1 = \frac{1}{2\pi\alpha}\hat F(-\lambda).$$

Using the inversion formula for the Fourier transform, we obtain that:

$$\begin{aligned}\int_{-\infty}^{+\infty}\operatorname{tr}(T_f^{S^1,\lambda}\circ\phi(D_z-\lambda))d\lambda &= \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty}\hat F(-\lambda)d\lambda = \frac1\alpha F(0)\\ &= \frac1\alpha\sum_{n\in\mathbb{Z}} f\left(\frac n\alpha\right)\int_0^1 e^{-\frac12\left(z-Z\left(\frac n\alpha,1,z\right)+n\right)}\hat\phi\left(z-Z\left(\frac n\alpha,1,z\right)+n\right)\operatorname{tr} r_{\frac n\alpha}(1,z)dz.\end{aligned}$$

□

Corollary 1. *In the scalar case, we have*

$$\int_{-\infty}^{+\infty}\operatorname{tr}(T_f^{(0),S^1,\lambda}\circ\phi(D_z-\lambda))d\lambda = \frac1\alpha\sum_{n\in\mathbb{Z}} f(\frac n\alpha)\int_0^1 e^{-\frac12(z-Z(\frac n\alpha,1,z)+n)}\hat\phi(z-Z(\frac n\alpha,1,z)+n)dz.$$

Proof. In this case, we have $r_t(1,z)=1$. □

Corollary 2. *In the case* $E^0 = T^*\mathcal{F}\otimes\mathbb{C}\,|_{S^1}\cong T^*\mathbb{R}$, *we have*

$$\int_{-\infty}^{+\infty}\operatorname{tr}(T_f^{(1),S^1,\lambda}\circ\phi(D_z-\lambda))d\lambda = \frac1\alpha\sum_{n\in\mathbb{Z}} f\left(\frac n\alpha\right)\int_0^1 e^{-\frac12(Z(-\frac n\alpha,1,z_1)-z_1+n)}\hat\phi\left(Z\left(-\frac n\alpha,1,z_1\right)-z_1+n\right)dz_1.$$

Proof. In this case, we have $r_t(1,z)=\frac{\partial Z}{\partial z}(t,1,z)$. Therefore, the formula becomes

$$\int_{-\infty}^{+\infty}\operatorname{tr}(T_f^{S^1,\lambda}\circ\phi(D_z-\lambda))d\lambda$$

$$= \frac{1}{\alpha} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\alpha}\right) \int\limits_0^1 e^{-\frac{1}{2}\left(z-Z\left(\frac{n}{\alpha},1,z\right)+n\right)} \hat{\phi}\left(z - Z\left(\frac{n}{\alpha},1,z\right)+n\right) \frac{\partial Z}{\partial z}\left(\frac{n}{\alpha},1,z\right) dz.$$

The change of variable $z_1 = Z\left(\frac{n}{\alpha},1,z\right)$ in the integral in the right hand side of the last identity completes the proof. □

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