ISSN 2304-0122 Ufa Mathematical Journal. Vol. 7. No 2 (2015). P. 17-32.

doi:10.13108/2015-7-2-17 UDC 517.5

ON ABSENCE CONDITIONS OF UNCONDITIONAL BASES OF EXPONENTS

R.A. BASHMAKOV, A.A. MAKHOTA, K.V. TROUNOV

Abstract. In the classical space $L^2(-\pi,\pi)$ there exists the unconditional basis $\{e^{ikt}\}$ (k is integer). In the work we study the existence of unconditional bases in weighted Hilbert spaces $L^2(I,\exp h)$ of the functions square integrable on an interval I in the real axis with the weight $\exp(-h)$, where h is a convex function. We obtain conditions showing that unconditional bases of exponents can exist only in very rare cases.

Keywords: Riesz bases, unconditional bases, series of exponents, Hilbert space, Fourier-Laplace transform

Mathematics Subject Classification: 30D20

Let I be an interval in the real axis, $h(t)$ be a convex function on this interval $L^2(I,\exp h)$ be the space of locally integrable functions on I satisfying the condition

$$\|f\| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\,dt} < \infty.$$

It is a Hilbert space with the scalar product

$$(f,g) = \int_I f(t)\overline{g}(t)e^{-2h(t)}\,dt.$$

Definition 1. *The family $\{e^{\lambda_k t},\ k=1,2,\ldots\}$ is called unconditional basis in space $L^2(I,\exp h)$ if*

1) family $\{e^{\lambda_k t},\ k=1,2,\ldots\}$ is dense in space $L^2(I,\exp h)$;

2) there exist positive constants m, M such that for each finite sequence $a_k\in\mathbb{C}$ the two-sided estimate

$$m\sum_k |a_k|^2\|e^{\lambda_k t}\|^2 \leqslant \|\sum_k a_k e^{\lambda_k t}\|^2 \leqslant M\sum_k |a_k|^2\|e^{\lambda_k t}\|^2. \tag{1}$$

holds true.

We follow the definition in work [2]. As it was mentioned in this work, if system $\{e^{\lambda_k t}\}$ forms an unconditional basis in space $L^2(I,\exp h)$, each function $f\in L^2(I,\exp h)$ is uniquely expanded into absolutely (reodered) convergent series over this system:

$$f(t) = \sum_{k=1}^{\infty} f_k e^{\lambda_k t}, \quad t\in I. \tag{2}$$

In this section we consider the existence of unconditional exponential bases in space $L^2(I,\exp h)$.

The main tool of the study is the Laplace transform.

R.A. Bashmakov, A.A. Makhota, K.V. Trounov, On absence conditions of unconditional bases of exponents.

Submitted April 1, 2015.

As it was shown in work [7], Laplace transform $L: S \longmapsto \widehat{S}$ makes an isomorphism of the space adjoint to $L^2(I, \exp h)$ with Hilbert space $\widehat{L}^2(I, \exp h)$ of functions F analytic in space $J + i\mathbb{R}$, where

$$J = \{x : \ \widetilde{h}(x) = \sup_{t \in I}(xt - h(t)) < \infty\}$$

with the norm

$$\|F\| = \sqrt{\int_0^\infty \int_J \frac{|F(x+iy)|^2}{K(x)}\, d\widetilde{h}'(x) dy},$$

and

$$K(x) = \int_I e^{2xt - 2h(t)}\, dt = \|e^{\lambda t}\|^2, \ \ \lambda = x + iy.$$

Suppose that system $\{e^{\lambda_k t}\}$ forms an unconditional basis in space $L^2(I, \exp h)$. By S_k we denote a linear functional in space $L^2(I, \exp h)$ which maps each function $f \in L^2(I, \exp h)$ into the coefficient f_k in expansion (2):

$$S_k(f) = f_k.$$

If by P we denote $\max(M, \frac{1}{m})$, where M, m are the constants in relation (1), then for each n the two-sided estimate

$$\frac{1}{P}\sum_{k=1}^n |f_k|^2 \|e^{\lambda_k t}\|^2 \leqslant \|\sum_{k=1}^n f_k e^{\lambda_k t}\|^2 \leqslant P \sum_{k=1}^n |f_k|^2 \|e^{\lambda_k t}\|^2$$

holds true. Passing to the limit as $n \to \infty$, we obtain

$$\frac{1}{P}\sum_{k=1}^\infty |f_k|^2 \|e^{\lambda_k t}\|^2 \leqslant \|f\|^2 \leqslant P \sum_{k=1}^\infty |f_k|^2 \|e^{\lambda_k t}\|^2.$$

By the definition of function $K(\lambda) = K(\operatorname{Re}\lambda)$ this relation can be written as

$$\frac{1}{P}\sum_{k=1}^\infty |f_k|^2 K(\lambda_k) \leqslant \|f\|^2 \leqslant P \sum_{k=1}^\infty |f_k|^2 K(\lambda_k). \tag{3}$$

The left inequality implies the boundedness of functional S_k:

$$|S_k(f)| \leqslant \sqrt{\frac{P}{K(\lambda_k)}}\|f\|.$$

Thus, functions $\widehat{S}_k(\lambda)$ lie in space $\widehat{L}^2(I, \exp h)$ and moreover,

$$\widehat{S}_k(\lambda_n) = \begin{cases} 0, & n \neq k, \\ 1, & n = k. \end{cases} \tag{4}$$

We observe that λ_n, $n \neq k$, are simple zeroes of function $\widehat{S}_k(\lambda)$. Indeed, if for some $m \neq k$ the quantity $\widehat{S}_k'(\lambda_m)$ vanished, function $(\lambda_k - \lambda_m)\widehat{S}_k(\lambda)/(\lambda - \lambda_m)$ lying in $\widehat{L}^2(I, \exp h)$ would vanish at points λ_n, $n \neq k$, and would equal to 1 at point λ_k, i.e., at all the points λ_n, $n = 1, 2, \ldots$, it would coincide with function $\widehat{S}_k(\lambda)$. Then by the completeness of the system $e^{\lambda_n t}$ in space $L^2(I, \exp h)$ the system of points λ_n, $n = 1, 2, \ldots$, is the uniqueness set for space $\widehat{L}^2(I, \exp h)$. Hence, functions $(\lambda_k - \lambda_m)\widehat{S}_k(\lambda)/(\lambda - \lambda_m)$ and $\widehat{S}_k(\lambda)$ should coincide identically.

Let

$$L(\lambda) = \widehat{S}_1(\lambda)(\lambda - \lambda_1).$$

This function is analytic in the strip $J + i\mathbb{R}$ with simple zeroes at points λ_n, $n = 1, 2, \ldots$ The functions

$$\frac{L(\lambda)}{L'(\lambda_k)(\lambda-\lambda_k)} = \frac{\widehat{S}_1(\lambda)(\lambda-\lambda_1)}{(\lambda-\lambda_k)(\lambda_k-\lambda_1)\widehat{S}_1'(\lambda_k)}, \quad k \neq 1,$$
$$\frac{L(\lambda)}{L'(\lambda_1)(\lambda-\lambda_1)} = \widehat{S}_1(\lambda)$$

are also elements of space $\widehat{L}^2(I, \exp h)$ and they coincide with function $\widehat{S}_k(\lambda)$ at all the points λ_n, $n = 1, 2, \ldots$. Again by the completeness of system $\{e^{\lambda_n t}\}$ in space $L^2(I, \exp h)$ we have

$$\widehat{S}_k(\lambda) = \frac{L(\lambda)}{L'(\lambda_k)(\lambda-\lambda_k)}, \quad \lambda \in \mathbb{C}. \tag{5}$$

For a fixed $\lambda \in \mathbb{C}$, function $e^{\lambda t}$ lies in space $L^2(I, \exp h)$ and thus, it can be expanded into the series over system $e^{\lambda_k t}$:

$$e^{\lambda t} = \sum_{k=1}^{\infty} c_k(\lambda) e^{\lambda_k t}. \tag{6}$$

We apply functional S_n to this identity. In view of relations (4) we obtain

$$\widehat{S}_n(\lambda) = \sum_{k=1}^{\infty} c_k(\lambda)\widehat{S}_n(\lambda_k) = c_n(\lambda).$$

Together with (5) it yields

$$c_n(\lambda) = \frac{L(\lambda)}{L'(\lambda_n)(\lambda-\lambda_n)}, \quad \lambda \in \mathbb{C}.$$

Representation (6) and condition (3) imply

$$\frac{1}{P}\sum_{k=1}^{\infty} |c_k(\lambda)|^2 K(\lambda_k) \leqslant K(\lambda) \leqslant P\sum_{k=1}^{\infty} |c_k(\lambda)|^2 K(\lambda_k)$$

or

$$\frac{1}{P}K(\lambda) \leqslant \sum_{k=1}^{\infty} \frac{|L(\lambda)|^2 K(\lambda_k)}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} \leqslant PK(\lambda). \tag{7}$$

Thus, we have proven the theorem:

Theorem 1. *If system $\{e^{\lambda_k t}\}$ is an unconditional basis in space $L^2(I, \exp h)$, there exists a function L analytic in the strip $J + i\mathbb{R}$ with simple zeroes at poins λ_k, $k = 1, 2, \ldots$, satisfying relation (7).*

Relation (7) allows us to find out some properties of the distribution of zeroes of function $L(\lambda)$.

We introduce a characteristics $\tau(u, z, p)$ for a convex function $u(x)$.

Let z be a fixed point in the plane. For each positive $r > 0$, by $B(z, r)$ we denote the circle $\{w : |w - z| < r\}$, and for a function f continuous in $\overline{B}(z, r)$ we let

$$\|f\|_r = \max_{w \in \overline{B}(z,r)} |f(w)|.$$

Let $d(f, z, r)$ be the distance from function f to the subspace of harmonic in $B(z, r)$ functions:

$$d(f, z, r) = \inf\{\|f - H\|_r,\ H \text{is harmonic in } B(z, r)\}.$$

If $u(x)$ is a convex function on interval $I \subset \mathbb{R}$, function $u(w) = u(\operatorname{Re} w)$ is continuous in the vertical strip $I + i\mathbb{R}$ in the plane. For a positive number p we let

$$\tau(u, z, p) = \sup\{r : \ d(u, z, r) \leqslant p\}.$$

It is clear that $\tau(u, z, p)$ depends only on $\operatorname{Re} z$. If needed, we redefine function u letting it being equal to $+\infty$ outside interval I. Then $\tau(u, z, p)$ can not exceed the distance from y to the boundary of the domain for function u. Thus, $\tau(u, z, p)$ is the radius of the maximal circle centered at point z, in which function u deviate from the space of harmonic functions in this circle at most by p.

The introduced characteristics $\tau(u, z, p)$ of a convex function $u(x)$ happens to be closely related with the geometric characteristic of convexity $\rho_2(u, y, p)$ introduced in works [4], [7]:

$$\rho_2(u, y, p) = \sup\{t > 0 : \int_{y-t}^{y+t} |u'(\tau) - u'(y)|\, d\tau \leqslant p\}.$$

This quantity $\rho_2 = \rho_2(u, y, p)$ can be determined by the identity

$$\frac{u(y - \rho_2) + u(y + \rho_2)}{2} - u(y) = \frac{p}{2}.$$

We observe that

$$\rho(u, y, p) = \rho_2(u, y, 2p).$$

For an arbitrary continuous function $u(y)$ on the real axis and a positive number r, by $d_1(u, y, r)$ we denote the deviation in the uniform norm for function u on the segment $[y-r; y+r]$ from linear functions:

$$d_1(u, y, r) = \inf\{\max_{t\in[y-r;y+r]} |u(t) - l(t)|, \ l \text{is linear}\}.$$

By $\rho(u, y, p)$ we denote the maximal number r such that on the interval $[y - r; y + r]$ function u deviates from linear functions at most by p:

$$\rho(u, y, p) = \sup\{r : \ d_1(u, y, r) \leqslant p\}.$$

Lemma 1. *1. For each positive p, function $\tau(y, p) = \tau(u, y, p)$ satisfy the estimates*

$$\tau(y, p) \geqslant \rho(y, p) \geqslant \frac{1}{16}\tau(y, p).$$

2. For $q \geqslant p > 0$, the two-sided estimates

$$\tau(y, q) \geqslant \tau(y, p) \geqslant \frac{p}{16q}\tau(y, q)$$

hold true. 3. Function $\tau(y) = \tau(u, y, p)$ satisfies Lifschitz condition: for each x, y in the domain of u

$$|\tau(y) - \tau(x)| \leqslant |y - x|.$$

Proof. 1. We fix a point $z \in \mathbb{C}$ so that $y = \operatorname{Re} z$ lies in the domain of function u. We let $r = \rho(u, y, p)$. Then there exists a linear function l satisfying the condition

$$|u(x) - l(x)| \leqslant p, \quad x \in [y - r; y + r].$$

The function $v(w) = l(\operatorname{Re} w)$ is harmonic and

$$|u(\operatorname{Re} w) - l(\operatorname{Re} w)| \leqslant p, \quad w \in B(z, r).$$

Hence,

$$\tau(y, p) \geqslant r = \rho(y, p).$$

We let $r = \tau(u, y, p)$. In circle $B(z, r)$, there exists a harmonic function H such that $\|u-H\|_r \leqslant p$. We choose a linear function l such that $l(x) \leqslant u(x)$ for each x, $l(y) = u(y)$. The existence of such function is ensured by the convexity of function u. We let $v(w) = l(\operatorname{Re} w)$. Then in circle $B(z, r)$ the inequalities

$$v(w) \leqslant u(w) \leqslant H(w) + p$$

hold true. Therefore,

$$(H(w) + p) - v(w) \geqslant 0.$$

Moreover, since $v(z) = u(\operatorname{Re} z)$, then

$$(H(z) + p) - v(z) = (H(z) + p) - u(\operatorname{Re} z) = (H(z) - u(\operatorname{Re} z)) + p \leqslant 2p.$$

We apply Harnack inequality for non-negative harmonic functions to the function $H(w) + p - v(w)$: in circle $B(z, \frac{r}{2})$ we have the estimate

$$(H(w) + p) - v(w) \leqslant 3\left((H(z) + p) - v(z)\right) \leqslant 6p.$$

Then in the same circle $B(z, \frac{r}{2})$ the estimate

$$|u(\operatorname{Re} w) - v(w)| \leqslant |u(\operatorname{Re} w) - H(w)| + |h(w) + p - v(w)| + p \leqslant 8p$$

is valid. The functions in the left hand side of this inequality depend on $x = \operatorname{Re} w$ only. Thus, we obtain

$$|u(x) - l(x)| \leqslant 8p, \quad x \in \left[y - \frac{r}{2}; y + \frac{r}{2}\right].$$

It follows from this estimate that

$$\rho(y, 8p) \geqslant \frac{r}{2} = \tau(y, p)$$

or

$$\tau(y, p) \leqslant 2\rho(y, 8p).$$

It implies

$$\tau(y, p) \leqslant 16\rho(y, p)$$

2. The second part of Lemma 1 can be obtained on the basis of the properties of function $\rho(y, z, p)$.

3. We choose points y_1, y_2 in the domain of function $u(x)$, and let $r = \tau(u, y_1, p)$. It means that in circle $B(y_1, r)$ there exists a harmonic function $H(z)$ obeying the condition

$$|u(\operatorname{Re} z) - H(z)| \leqslant p.$$

If $|y_1 - y_2| < r$, this inequality is satisfied also in the circle $B(y_2, r - |y_1 - y_2|)$, hence,

$$\tau(u, y_2, p) \geqslant r - |y_1 - y_2| = \tau(u, y_1, p) - |y_1 - y_2|,$$

or

$$\tau(u, y_1, p) - \tau(u, y_2, p) \leqslant |y_1 - y_2|.$$

If $|y_1 - y_2| \geqslant r = \tau(u, y_1, p)$, then

$$\tau(u, y_1, p) - \tau(u, y_2, p) \leqslant |y_1 - y_2|.$$

We swap y_1, y_2:

$$\tau(u, y_2, p) - \tau(u, y_1, p) \leqslant |y_1 - y_2|.$$

Thus,

$$|\tau(u, y_1, p) - \tau(u, y_2, p)| \leqslant |y_1 - y_2|.$$

$\square$

It was shown in work [11] that the quantity $\tau = \tau(u, \lambda, p)$ is well determined by the condition: if $H(z)$ is a harmonic majorant for function $u(z)$ in circle $B(\lambda, \tau)$, then

$$\max_{z \in \overline{B}(\lambda,\tau)} (H(z) - u(z)) = 2p. \tag{8}$$

We determine this quantity for the function $\ln K(\lambda)$ and the number $\ln(5P)$, where P is the constant in relation (7). In what follows we shall denote it simply by $\tau(\lambda)$. So,

$$\inf_{v \in A(B(\lambda,\tau))} \max_{z \in \overline{B}(\lambda,\tau)} |\ln K(z) - v(z)| = \ln(5P),$$

where by $A(B(\lambda, \tau))$ we denote the set of the functions harmonic in circle $B(\lambda, \tau)$.

Theorem 2. *Let $L(\lambda)$ be a function analytic in the strip $J + i\mathbb{R}$, with simple zeroes λ_k, $k = 1, 2, \dots$ for some P satisfying the two-sided estimate*

$$\frac{1}{P} K(\lambda) \leqslant \sum_{k=1}^{\infty} \frac{|L(\lambda)|^2 K(\lambda_k)}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leqslant P K(\lambda).$$

Then

1) Each circle $B(\lambda, 2\tau(\lambda))$ contains at least one zero λ_k of function L.

2) For each n, k, $n \neq k$, the inequality

$$|\lambda_k - \lambda_n| \geqslant \frac{\max(\tau(\lambda_k), \tau(\lambda_n))}{10 P^{\frac{3}{2}}}$$

holds true.

3) For each k, in the circle $B(\lambda_k, \frac{\tau(\lambda_k)}{20P^{\frac{3}{2}}})$, the relation

$$\frac{1}{5^6 P^8} K(\lambda) \leqslant \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leqslant P K(\lambda)$$

is valid.

Proof. 1. We argue by contradiction to prove the first statement. Suppose that for some $\lambda \in \mathbb{C}$, the circle $B = B(\lambda, 2\tau(\lambda))$ contains no zeros of function L. We take a point $z \in B(\lambda, \tau(\lambda))$. Then for each k we have $\tau(\lambda) \leqslant |\lambda_k - \lambda|/2$ and $|\lambda - z| < \tau(\lambda) \leqslant |\lambda_k - \lambda|/2$. Thus,

$$|z - \lambda_k| \geqslant |\lambda_k - \lambda| - |\lambda - z| \geqslant \frac{1}{2}|\lambda - \lambda_k|,$$

$$|z - \lambda_k| \leqslant |\lambda_k - \lambda| + |\lambda - z| \leqslant \frac{3}{2}|\lambda - \lambda_k|.$$

It yields

$$\frac{1}{2} \leqslant \frac{|z - \lambda_k|}{|\lambda - \lambda_k|} \leqslant \frac{3}{2} < 2.$$

This relation implies the two-sided estimate valid for $z \in B(\lambda, \tau(\lambda))$:

$$\frac{1}{4} C(\lambda)|L(z)|^2 \leqslant \sum_{k=1}^{\infty} \frac{|L(z)|^2 K(\lambda_k)}{|L'(\lambda_k)|^2 |z - \lambda_k|^2} \leqslant 4C(\lambda)|L(z)|^2,$$

where $C(\lambda)$ stands for the number

$$C(\lambda) = \sum_{k=1}^{\infty} \frac{K(\lambda_k)}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2}.$$

Thanks to relation (7) satisfied by the assumption of the theorem for function $L(\lambda)$, we get

$$\frac{1}{4P} C(\lambda)|L(z)|^2 \leqslant K(z) \leqslant 4PC(\lambda)|L(z)|^2, \; z \in B(\lambda, \tau).$$

We find the logarithm of this relation:

$$|\ln K(z) - \ln(C(\lambda)|L(z)|^2)| \leqslant \ln(4P) < \ln(5P),\ \ z \in B(\lambda, \tau).$$

Since by the assumption the circle $B(\lambda, 2\rho(\lambda))$ contains no zeroes of function L, function $u(z) = \ln(C(\lambda)|L(z)|^2)$ is harmonic in the circle $B(\lambda\tau(\lambda))$ and is continuous in its closure. Then the latter estimate contradicts the definition of quantity $\tau(\lambda)$.

2. We fix two different numbers k, n. By relation (7), the function

$$F(\lambda) = \frac{\sqrt{K(\lambda_n)}}{L'(\lambda_n)(\lambda_n - \lambda)} L(\lambda)$$

satisfies the upper estimate

$$|F(\lambda)| \leqslant \sqrt{PK(\lambda)}.$$

And due to the definition of quantity $\tau(\lambda_k)$, in circle $B(\lambda_k, \tau(\lambda_k))$ there exists a harmonic function $u_k(\lambda)$ satisfying the estimate

$$|\ln K(\lambda) - u_k(\lambda)| \leqslant \ln(5P). \tag{9}$$

In particular,

$$\sqrt{K(\lambda)} \leqslant \sqrt{5P} e^{\frac{u_k(\lambda)}{2}}.$$

Let $g_k(\lambda)$ be a function analytic in the circle $B(\lambda_k, \tau(\lambda_k))$ such that $\operatorname{Re} g_k(\lambda) = u_k(\lambda)/2$. Then the function

$$f(z) = F(\tau(\lambda_k)z + \lambda_k)e^{-g_k(\tau(\lambda_k)z+\lambda_k)}$$

is analytic in the unit circle $B(0,1)$ and satisfies the upper estimate

$$|f(z)| \leqslant \sqrt{5}P,$$

and $f(0) = 0$. By Schwarz lemma, we have the upper estimate

$$|f(z)| \leqslant \sqrt{5}P|z|,$$

thus,

$$|f'(0)| \leqslant \sqrt{5}P.$$

Calculating $f'(0)$, we obtain

$$|F'(\lambda_k)| \leqslant \sqrt{5}P \frac{e^{\frac{u_k(\lambda_k)}{2}}}{\tau(\lambda_k)}.$$

Together with relation (9) it implies

$$|F'(\lambda_k)| \leqslant 5P^{\frac{3}{2}} \frac{\sqrt{K(\lambda_k)}}{\tau(\lambda_k)}.$$

We calculate the value of $F'(\lambda_k)$ by the definition to obtain

$$\frac{|L'(\lambda_k)|\sqrt{K(\lambda_n)}}{|L'(\lambda_n)||\lambda_k - \lambda_n|} \leqslant 5P^{\frac{3}{2}} \frac{\sqrt{K(\lambda_k)}}{\tau(\lambda_k)}.$$

Indices k, n are arbitrary and we can swap them:

$$\frac{|L'(\lambda_n)|\sqrt{K(\lambda_k)}}{|L'(\lambda_k)||\lambda_n - \lambda_k|} \leqslant 5P^{\frac{3}{2}} \frac{\sqrt{K(\lambda_n)}}{\tau(\lambda_n)}.$$

Multiplying two latter estimates, we get:

$$\frac{1}{|\lambda_k - \lambda_n|^2} \leqslant \frac{25P^3}{\tau(\lambda_k)\tau(\lambda_n)},$$

or

$$|\lambda_k-\lambda_n|^2 \geqslant \frac{\tau(\lambda_k)\tau(\lambda_n)}{25P^3}. \qquad (10)$$

Let $\tau(\lambda_k) \geqslant \tau(\lambda_n)$ and assume that the inequality in Statement 2 is not valid, i.e.,

$$|\lambda_k-\lambda_n| < \frac{\tau(\lambda_k)}{10P^{\frac{3}{2}}}. \qquad (11)$$

The circle

$$B' = \{|\lambda-\lambda_n| \leqslant \frac{10P^{\frac{3}{2}}-1}{10P^{\frac{3}{2}}}\tau(\lambda_k)\}$$

lies in the circle $B(\lambda_k, \tau(\lambda_k))$ in which there exists a harmonic function $u_k(z)$ with the estimate

$$|\ln K(z) - u_k(z)| \leqslant \ln(5P).$$

Then

$$\tau(\lambda_n) \geqslant \frac{10P^{\frac{3}{2}}-1}{10P^{\frac{3}{2}}}\tau(\lambda_k).$$

This estimate and (10) lead us to the inequality

$$|\lambda_k-\lambda_n|^2 \geqslant \frac{1}{25P^3}\tau(\lambda_k)\tau(\lambda_n) \geqslant \frac{1}{25P^3}\frac{10P^{\frac{3}{2}}-1}{10P^{\frac{3}{2}}}\tau(\lambda_k)^2.$$

Since $P>1$, then

$$\frac{10P^{\frac{3}{2}}-1}{10P^{\frac{3}{2}}} > \frac{9}{10} > \frac{1}{4}$$

and

$$|\lambda_k-\lambda_n|^2 > \frac{1}{100P^3}\tau^2(\lambda_k)$$

or

$$|\lambda_k-\lambda_n| > \frac{1}{10P^{\frac{3}{2}}}\tau(\lambda_k)$$

that contradicts assumption (11).

3. We fix an index k. The right inequality in Statement 3 follows directly from the assumptions of the theorem. By the definition of $\tau(\lambda_k)$, in circle $B(\lambda_k, \tau(\lambda_k))$ there exists a harmonic function $u_k(\lambda)$ such that

$$-\ln(5P) \leqslant \ln K(\lambda) - u_k(\lambda) \leqslant \ln(5P). \qquad (12)$$

By the assumptions of the theorem,

$$K(\lambda) \geqslant \frac{1}{P}\sum_{n=1}^{\infty}\frac{K(\lambda_n)|L(\lambda)|^2}{|L'(\lambda_n)|^2|\lambda-\lambda_n|^2} \geqslant \frac{K(\lambda_k)|L(\lambda)|^2}{P|L'(\lambda_k)|^2|\lambda-\lambda_k|^2}$$

or

$$\ln K(\lambda) \geqslant \ln\frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} - \ln P.$$

Therefore, for $\lambda \in B(\lambda_k, \tau(\lambda_k))$ we have

$$\ln\frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} - \ln P - u_k(\lambda) - \ln(5P) \leqslant 0,$$

i.e.,

$$u_k(\lambda) + 2\ln P + \ln 5 - \ln\frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} \geqslant 0.$$

By Statement 2, circle $B\left(\lambda_k, \frac{1}{10P^{\frac{3}{2}}}\tau(\lambda_k)\right)$ contains no zeroes of function $L(\lambda)$ except λ_k. Therefore, the function

$$v_k(\lambda) = -\ln \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2}$$

is harmonic in this circle. And the function $u_k(\lambda)+v_k(\lambda)+\ln(5P^2)$ is harmonic and non-negative in it. By Harnack inequality, in the circle $B(\lambda_k, \frac{\tau(\lambda_k)}{20P^{\frac{3}{2}}})$, the estimate

$$u_k(\lambda)+v_k(\lambda)+\ln(5P^2) \leqslant 3(u_k(\lambda_k)+v_k(\lambda_k)+\ln(5P^2)) = 3(u_k(\lambda_k)-\ln K(\lambda_k)+\ln(5P^2))$$

holds true. By the left inequality in (12) we have $u_k(\lambda_k) \leqslant \ln K(\lambda_k)+\ln(5P)$ and therefore

$$u_k(\lambda)+v_k(\lambda) \leqslant 3\ln(5P)+2\ln(5P^2) = \ln 5^5P^7.$$

The right inequality in (12) implies $u_k(\lambda) \geqslant \ln K(\lambda)-\ln(5P)$ and hence,

$$-v_k(\lambda) \geqslant \ln K(\lambda)-\ln(5P)-\ln 5^5P^7 = \ln K(\lambda)-\ln(5^6P^8).$$

Thus,

$$\frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} \geqslant \frac{1}{5^6P^8}K(\lambda).$$

The proof is complete. □

Theorem 3. *Let λ_k, $k=1,2,\ldots$, be the zeroes of function $L(\lambda)$ satisfying the assumption of the previous theorem. Then in each bounded set B containing at least two of points λ_k, $k=1,2,\ldots$, there exists a point λ_n such that*

$$\sum_{\lambda_k\in B, k\neq n} \frac{1}{|\lambda_k-\lambda_n|^2} \leqslant \frac{(5P)^{12}}{\tau^2(\lambda_n)}. \tag{13}$$

Proof. By relation (7), for each λ the estimate

$$\sum_{\lambda_k\in B} \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2} \leqslant PK(\lambda) \tag{14}$$

holds true. There exists an index n such that

$$\frac{K(\lambda_n)}{|L'(\lambda_n)|^2} = \min_{\lambda_k\in B}\left(\frac{K(\lambda_k)}{|L'(\lambda_k)|^2}\right).$$

By Statement 3 of Theorem 2, for points λ lying on the boundary of circle $B\left(\lambda_n, \frac{1}{20P^{\frac{3}{2}}}\tau(\lambda_n)\right)$ the estimate

$$\frac{1}{5^6P^8}K(\lambda) \leqslant 20^2P^3\frac{K(\lambda_n)|L(\lambda)|^2}{|L'(\lambda_n)|^2\tau^2(\lambda_n)}$$

holds true or

$$\frac{K(\lambda)}{|L(\lambda)|^2} \leqslant 4^25^8P^{11}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)}.$$

Together with estimate (14) it implies

$$4^25^8P^{11}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)} \geqslant \frac{1}{P}\sum_{\lambda_k\in B}\frac{K(\lambda_k)}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2}.$$

In view of the choice of index n, for points λ on the boundary of $B\left(\lambda_n, \frac{1}{20P^{\frac{3}{2}}}\tau(\lambda_n)\right)$ we have

$$4^25^8P^{11}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)} \geqslant \frac{1}{P}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2}\sum_{\lambda_k\in B}\frac{1}{|\lambda-\lambda_k|^2}$$

or

$$\sum_{\lambda_k\in B}\frac{1}{|\lambda-\lambda_k|^2}\leqslant\frac{4^2 5^8 P^{12}}{\tau^2(\lambda_n)}. \tag{15}$$

By Statement 2 of Theorem 2, for the mentioned points λ with $k\neq n$ the estimate

$$|\lambda-\lambda_k|\leqslant|\lambda-\lambda_n|+|\lambda_n-\lambda_k|\leqslant\frac{3}{2}|\lambda_n-\lambda_k|$$

holds true. Therefore, it follows from (15) that

$$\sum_{\lambda_k\in B,k\neq n}\frac{1}{|\lambda_n-\lambda_k|^2}<\frac{(5P)^{12}}{\tau^2(\lambda_n)}.$$

The proof is complete. □

On the basis of Theorem 3 one can show that the existence of Riesz basis of exponentials is rather an exception than a rule.

Theorem 4. *Let I be an arbitrary interval in $\mathbb{R}$, $h(t)$ be a convex function on this interval,*

$$K(\lambda)=\int_I e^{2\operatorname{Re}\lambda t-2h(t)}\,dt,\qquad J=\{x:\ K(x)<\infty\}.$$

Suppose that for some $p>0$ there exists a sequence of segments $[a_m;b_m]$ and positive numbers τ_m, $m=1,2,\ldots$, such that

1) for some positive number δ and for each $x\in[a_m;b_m]$

$$\delta\tau_m\leqslant\tau(\ln K(z),x,p)\leqslant\tau_m,\ \ m=1,2,\ldots,$$

2) the relation

$$\lim_{m\to\infty}\frac{b_m-a_m}{\tau_m}=\infty$$

holds true.

Then there is no Riesz basis of exponentials in space $L^2(I,\exp h)$.

Proof. First of all we observe that if the assumptions of the theorem hold true for some p, by Statement 2 of Lemma 1, these assumptions hold true for each $p>0$.

Suppose that the system $e^{\lambda_k t}$ forms a Riesz basis in space $L^2(I,\exp h)$. By Theorem 1, there exists an entire function with simple zeroes at points λ_k obeying relation (7). In what follows we assume that in the assumption of Theore 4, as p we choose the number $\ln(5P)$, where P is the constant in relation (7). For the sake of brief notation, we denote $\tau(\ln K(z),\lambda,\ln(5P))$ by $\tau(\lambda)$. By Theorem 3, the set of points λ_k possesses property (13). We choose an arbitrary index m. Let

$$\tau_m=\sup_{\lambda\in[a_m,b_m]}(\tau(\lambda)),$$

s_m be the maximal natural number such that

$$a_m+4s_m\tau_m\leqslant b_m.$$

Then

$$a_m+4(s_m+1)\tau_m\geqslant b_m-a_m,$$

therefore,

$$\lim_{m\to\infty}\frac{(a_m+s_m\tau_m)-a_m}{\tau_m}=\infty.$$

To simplify the notations, in what follows we suppose that $a_m + s_m\tau_m = b_m$. For a fixed index m, we consider the system P formed by square with side $4\tau_m$:

$$P_{ql} = \{z:\ a_m + 2l\tau_m \leqslant \operatorname{Re} z \leqslant a_m + 2(l+1)\tau_m, \quad 2q\tau_m \leqslant \operatorname{Im} z \leqslant 2(q+1)\tau_m\},$$
$$l = 0, 1, \dots, s_m - 1, \quad q \in \mathbb{Z}.$$

Two squares in these system will be called adjacent if they have a common vertex. Let Q_1, Q_2 be two non-adjacent squares in this system and $z_1, w_1 \in Q_1$, $z_2, w_2 \in Q_2$. Then

$$|z_1 - z_2| \leqslant 4|w_1 - w_2|. \tag{16}$$

Indeed, since the squares are not adjacent, it follows that $|w_1 - w_2| \geqslant 4\Delta\tau_m$ or

$$\tau_m \leqslant \frac{1}{4}|w_1 - w_2|.$$

Therefore,

$$|z_1 - z_2| \leqslant |z_1 - w_1| + |w_1 - w_2| + |w_2 - z_2| \leqslant 8\sqrt{2}\tau_m + |w_2 - z_2| \leqslant 4|w_2 - z_2|.$$

We denote the center of the square P_{ql} by ζ_{ql}. Each square P_{ql} contains the circle $B(\zeta_{ql}, 2\tau_m)$, which, in its turn, under the assumption of the theorem, contains the circle $B(\zeta_{ql}, 2\tau(\zeta_{ql})$. By Statement 1 of Theorem 2, this circle contains at least one of the exponents λ_k. We take a sufficiently great N and by B_N we denote the union of squares P_{ql} over all q and l, $|l| \leqslant N$. We apply Theorem 5 to system λ_k and to set B_N. There exists an index n such that the relation

$$\sum_{\lambda_k \in B_N, k \neq n} \frac{1}{|\lambda_n - \lambda_k|^2} < \frac{(5P)^{12}}{\tau^2(\lambda_n)}$$

holds true. By Condition 1) of the theorem it implies the estimate

$$\sum_{\lambda_k \in B_N, k \neq n} \frac{1}{|\lambda_n - \lambda_k|^2} < \frac{(5P)^{12}}{\delta^2\tau_m^2}. \tag{17}$$

Let Q_0 be a square in system P containing point λ_n, and point λ_k lies in square Q (in our system) not adjacent with Q_0. We take arbitrary point $\lambda \in Q$ and employ relation (16):

$$|\lambda_k - \lambda_n| \leqslant 4|\lambda - \lambda_n|$$

or

$$\frac{1}{16|\lambda - \lambda_n|^2} \leqslant \frac{1}{|\lambda_k - \lambda_n|^2}.$$

We integrate this inequality w.r.t. λ over square Q:

$$\frac{1}{16\tau_m^2}\int_Q \frac{1}{16|\lambda - \lambda_n|^2}\, dv(\lambda) \leqslant \frac{1}{|\lambda_k - \lambda_n|^2}.$$

By B'_N we denote set B_N without the squares adjacent with Q_0. Since each of the squares contains at least one point in the system of exponents, the latter inequality and relation (23) yield

$$\int_{B'_N} \frac{1}{|\lambda - \lambda_n|^2} dv(\lambda) \leqslant \frac{256(5P)^{12}}{\delta^2}.$$

Let $Q_0 = P_{sj}$ and for the definiteness we assume that $j < 0$, $s \leqslant \frac{s_m}{2}$ and we let

$$\widetilde{B}_N = \{\lambda:\ a_m + (s+2)\tau_m \leqslant \operatorname{Re}\lambda \leqslant b_m,\ (j+2)\tau_m \leqslant \operatorname{Im}\lambda \leqslant (N+1)\tau_m\}.$$

Then $\widetilde{B}_N \subset B'_N$ and hence, the inequality

$$\int_{\widetilde{B}_N} \frac{1}{|\lambda - \lambda_n|^2} dv(\lambda) \leqslant \frac{256(5P)^{12}}{\delta^2}$$

holds true. We employ the change of variables $\lambda-(a_m+(s+2)\tau_m+i(j+2)\tau_m=\tau_m w$ and by w_0 we denote the image of point λ_n under this change:

$$\int_0^{s_m-s-1}\int_0^{N+1}\frac{1}{|w-w_0|^2}dv(\lambda)\leqslant\frac{256(5P)^{12}}{\delta^2}.$$

At that, for point w_0 we have $-2\leqslant \operatorname{Re} w_0,\ \operatorname{Im} w_0\leqslant -1$. Therefore, we can assume that $w_0=-2-2i$, it lessens the left hand side in the latter inequality and the inequality is kept. Indices m, N have influence only on the integration limits, this is why in the latter inequality we can pass to the limit as $m,N\to\infty$. Since

$$s_m=\frac{b_m-a_m}{\tau_m}\to\infty,$$

and we assume that $s\leqslant\frac{s_m}{2}$, then $s_m-s-2\geqslant\frac{s_m}{2}-2\to\infty$ and we obtain

$$\int_0^{\infty}\int_0^{\infty}\frac{1}{(x+2)^2+(y+2)^2}dxdy\leqslant\frac{256(5P)^{12}}{\delta^2}.$$

The integral in the left hand side diverges and we arrive at the contradiction. The proof is complete. □

This theorem requires to calculate function $K(x)$ that is not always simple. It happens that it is sufficient to calculate function $\widetilde{h}$.

Theorem 4 (a). *Let I be an arbitrary interval in $\mathbb{R}$, $h(t)$ be a convex function on this interval,*

$$\widetilde{h}(x)=\sup_{t\in I}(xt-h(t)).$$

Suppose that for some $p>0$ there exist a sequence of segments $[a_m;b_m]$ and positive numbers t_m, $m=1,2,\dots$, such that

1) for some positive number δ and for each $x\in[a_m;b_m]$

$$\delta t_m\leqslant\tau(2\widetilde{h},x,p)\leqslant t_m,\ \ m=1,2,\dots,$$

2) the relation

$$\lim_{m\to\infty}\frac{b_m-a_m}{t_m}=\infty$$

holds true.

Then there is Riesz basis of exponentials in space $L^2(I,\exp h)$.

Proof. In accordance with the results of works [3], [9], [10], for some constants c, $C>0$ depending only on number p the relation

$$c\frac{e^{2\widetilde{h}(x)}}{\rho_1(2\widetilde{h},x,p)}\leqslant K(x)\leqslant C\frac{e^{2\widetilde{h}(x)}}{\rho_1(2\widetilde{h},x,p)}$$

holds true. It follows that for some other constants c, $C>0$ the estimate

$$c\frac{e^{2\widetilde{h}(x)}}{\tau(2\widetilde{h},x,p)}\leqslant K(x)\leqslant C\frac{e^{2\widetilde{h}(x)}}{\tau(2\widetilde{h},x,p)}$$

holds true. Under the assumptions of the theorem we get

$$c\leqslant K(x)e^{-2\widetilde{h}(x)}t_m)\leqslant\frac{C}{\delta}.$$

We let $C'=\max(|\ln c|,|\ln C-\ln\delta|)$. Then

$$|\ln K(x)-(2\widetilde{h}(x)-\ln t_m)|\leqslant C',\quad x\in[a_m;b_m].$$

It is obvious that $\tau(2\widetilde{h}, x, p) = \tau(2\widetilde{h} - t_m, x, p)$. Let $a'_m = a_m + t_m$, $b'_m = b_m - t_m$. In the interval $[a'_m; b'_m]$, we apply Statement 4 of Lemma 1 to the functions $u_1(x) = \ln K(x)$, $u_2(x) = 2\widetilde{h}(x) - \ln t_m$. Then under the assumptions of the theorem

$$\frac{\delta p}{p + C'} t_m \leqslant \frac{p}{p + C'} \tau_2(x) \leqslant \tau_1(x) \leqslant \frac{p + C'}{p} \tau_2(x) \leqslant \frac{p + C'}{p} \tau_m.$$

We let $t'_m = \frac{p+C'}{p} t_m$, $\delta' = \frac{\delta p^2}{(p+C')^2}$. The latter inequalities imply

$$\delta' t'_m \leqslant \tau(\ln K, x, p) \leqslant t'_m, \quad x \in [a'_m; b'_m],$$

and

$$\lim_{m\to\infty} \frac{b'_m - a'_m}{t'_m} = \infty.$$

Thus, the assumptions of Theorems 4 are satisfied and the proof of Theorem 4(a) is complete. □

In the formulation of the latter theorem we have employed the quantity $\tau(\lambda)$, which is not always easy to calculate. Let us prove a lemma simplifying the calculation of quantity $\tau(\lambda)$ for particular examples.

Lemma 2. *Let $u(x)$ be a twice differentiable convex function on some interval $I \subset \mathbb{R}$. Suppose that for some point $y \in I$ and some constants $A, B, C > 0$ the relation*

$$A \leqslant \frac{u''(x)}{(u''(y)} \leqslant B, \quad as\ |x - y| \leqslant C\sqrt{\frac{1}{u''(y)}}$$

holds true. Then

$$\min\left(C, \frac{p}{BC}\right)\sqrt{\frac{1}{u''(y)}} \leqslant \tau(u, y, p) \leqslant 32 \max\left(C, \frac{p}{AC}\right)\sqrt{\frac{1}{u''(y)}}.$$

Proof. Since

$$u'(x) - u'(y) = u''(x^*)(x - y),$$

where x^* is a point between x, y, under the assumptoins of the theorem we have

$$Au''(y)|x - y| \leqslant |u'(x) - u'(y)| \leqslant Bu''(y)|x - y|, \quad \text{if } |x - y| \leqslant C\sqrt{\frac{1}{u''(y)}}.$$

Therefore, for each $r \in [0; C\sqrt{1/u''(y)}]$ we have

$$\int_{y-r}^{y+r} |u'(x) - u'(y)| \leqslant Bu''(y) \int_{y-r}^{y+r} |x - y| = Bu''(y)r^2 \leqslant BC^2,$$

$$\int_{y-r}^{y+r} |u'(x) - u'(y)| \geqslant Au''(y) \int_{y-r}^{y+r} |x - y| = Au''(y)r^2.$$

The first inequality imply the estimate

$$\rho_2(u, y, BC^2) \geqslant C\sqrt{\frac{1}{u''(y)}}.$$

We observe that

$$\rho_2(u, y, p) \geqslant \begin{cases} C\sqrt{\frac{1}{u''(y)}}, & \text{if } p \geqslant BC^2, \\ \frac{p}{BC}\sqrt{\frac{1}{u''(y)}}, & \text{if } p \leqslant BC^2. \end{cases}$$

Hence,

$$\rho_2(u,y,p) \geqslant \min\left(C, \frac{p}{BC}\right)\sqrt{\frac{1}{u''(y)}}. \qquad (18)$$

On the other hand, as $r = C\sqrt{1/u''(y)}$, we have

$$\int_{y-r}^{y+r} |u'(x) - u'(y)|\, dx \geqslant Au''(y)r^2 = AC^2,$$

and this is why

$$\rho_2(u,y,AC^2) \leqslant C\sqrt{\frac{1}{u''(y)}}.$$

We observe that

$$\rho_2(u,y,p) \leqslant \begin{cases} C\sqrt{\frac{1}{u''(y)}}, & \text{if } p \leqslant AC^2, \\ \frac{p}{AC}\sqrt{\frac{1}{u''(y)}}, & \text{if } p \geqslant AC^2. \end{cases} \qquad (19)$$

Thus,

$$\rho_2(u,y,p) \leqslant \max\left(C, \frac{p}{AC}\right)\sqrt{\frac{1}{u''(y)}}.$$

Together with estimate (18) it yields

$$\min\left(C\frac{p}{BC}\right)\sqrt{\frac{1}{u''(y)}} \leqslant \rho_2(u,y,p) \leqslant \rho_2(u,y,2p) = \rho(u,y,p),$$

$$\rho(u,y,p) = \rho_2(u,y,2p) \leqslant 2\rho_2(u,y,p) \leqslant 2\max\left(C, \frac{p}{AC}\right)\sqrt{\frac{1}{u''(y)}}.$$

Then we employ Statement 1 of Lemma 1 and arrive at the statement of Lemma 2. The proof is complete. □

Now we are in position to formulate a useful particular case of Theorem 4(a).

Theorem 4 (b)**.** *Let I be an arbitrary interval on $\mathbb{R}$, $h(t)$ be a convex function on this interval,*

$$\widetilde{h}(x) = \sup_{t\in I}(xt - h(t)).$$

Suppose that for some $p > 0$ there exists a sequence of segments $[a_m; b_m]$ and positive numbers t_m, $m = 1, 2, \dots$, such that

1) for some positive number δ and each $x \in [a_m; b_m]$

$$\delta t_m \leqslant \sqrt{\frac{1}{\widetilde{h}''(x)}} \leqslant t_m, \ m = 1, 2, \dots,$$

2) the relation

$$\lim_{m\to\infty} \frac{b_m - a_m}{t_m} = \infty$$

holds true.

Then there is not Riesz basis of exponentials in space $L^2(I, \exp h)$.

Examples.

1. Let $I = \mathbb{R}$ and $h(t) = A|t|^\alpha$, where $\alpha \geqslant 1$.

1a. If $\alpha > 1$, then

$$\widetilde{h}(x) = \left(1 - \frac{1}{\alpha}\right)\left(\frac{1}{A\alpha}\right)^{\frac{1}{\alpha-1}} |x|^{\frac{\alpha}{\alpha-1}}, \quad x \in \mathbb{R},$$

i.e., the Young conjugate reads as $B|x|^\beta$, where $\beta > 1$ is determined by the restriction $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then for $x \neq 0$

$$\sqrt{\frac{1}{\widetilde{h}''(x)}} = \sqrt{\frac{1}{B\beta(\beta-1)}}|x|^{-\frac{\beta}{2}+1},$$

and Condition 1 of Theorem 4(b) is satisfied, for instance, for the sequence of segments $[n; 2n]$. Thus, there are no Riesz bases of exponentials in spaces $L^2(\mathbb{R}, e^{A|t|^\alpha})$.

1b. If $\alpha = 1$, i.e., $h(t) = A|t|$, then

$$\widetilde{h}(x) = \begin{cases} 0, & |x| \leqslant A, \\ +\infty, & |x| > A, \end{cases}$$

and $\rho(\widetilde{h}, x, 1) = 1 - |x|$. Therefore, the assumption of Theorem 4() can not be satisfied and we can not state the existence of Riesz bases in space $L^2(\mathbb{R}, e^{A|t|})$ on the basis of Theorem 4.

2. Let $I = [-1; 1]$ and $h(t) = \frac{A}{(1-|t|)^\alpha}$, where $A > 0, \quad \alpha > 0$. Then

$$\widetilde{h}(x) = |x| - B|x|^{\frac{\alpha}{\alpha+1}}, \quad B = \frac{A(\alpha+1)}{(A\alpha)^{\frac{\alpha}{\alpha+1}}},$$

and

$$\sqrt{\frac{1}{\widetilde{h}''(x)}} = \frac{\sqrt{\alpha}}{\sqrt{B}(\alpha+1)}|x|^{\frac{\alpha-2}{2(\alpha+1)}}$$

and again Condition 1 of Theorem 4(b) holds true, for instance, for the sequence of segments $[n; 2n]$. Thus, there are no Riesz bases of exponentials in spaces $L^2(\mathbb{R}, \exp\frac{A}{(1-|t|)^\alpha})$.

2a. We take $A = 0$ in Example 2. Then

$$h(t) = 0, \ |t| \leqslant 1,$$

i.e., $L^2(I, e^{h(t)}) = L^2[-1; 1]$ and

$$\widetilde{h}(x) = |x|, \ x \in \mathbb{R}.$$

Therefore,

$$\rho(\widetilde{h}, x, 1) \geqslant |x| + 1.$$

Suppose that there exists a sequence of segments $[a_m; b_m]$ satisfying the assumptions of Theorem 4. Suppose that $b_m > 0$, then for sufficiently large m

$$b_m - a_m \geqslant 2\tau_m \geqslant 2\rho(\widetilde{h}, b_m, 1) \geqslant 2b_m + 2,$$

and thus,

$$a_m \leqslant -b_m - 2 < 0$$

and $0 \in [a_m; b_m]$. Then the estimate $\rho(\widetilde{h}, 0, 1) \geqslant \delta\tau_m$ should be satisfied. Since $\rho(\widetilde{h}, 0, 1) = 1$, then $\delta \leqslant \frac{1}{\tau_m}$. However, $\tau_m \to \infty$ as $m \to \infty$, thus, $\delta = 0$. We obtain the contradiction and Theorem 4 is not applicable in the present case.

BIBLIOGRAPHY

1. N.K. Bari. *On bases in Hilbert space* // Doklady Akad. Nauk. **54**, 383-386 (1946). (in Russian).
2. N.K. Nikolskii, B.S. Pavlov, S.V. Khuschev. *Unconditional bases of exponentials and reproducing kernels. I* // Preprint LOMI, 8-80 (1980). (in Russian).
3. R.A. Bashmakov, K.P. Isaev. *Asymptotic behavior of Laplace integrals* // Vestnik Bashkirskogo Univer. 4, 3-6 (2006). (in Russian).
4. V.I. Lutsenko. *Unconditional bases of exponentials in Smirnov spaces* // PhD thesis in physics and mathematics, Institute of Mathematics CC USC RAS (1992). (in Russian).
5. K.P. Isaev, R.S. Yulmukhametov. *Unconditional bases of reproducting kernels in Hilbert spaces of entire functions* // Ufimskij Matem. Zhurn. **5**:3, 67-77 (2013). [Ufa Math. J. **5**:3, 67-76 (2013).]
6. R.A. Bashmakov. *Exponentials systems in weighted Hilbert spaces on* $\mathbb{R}$ // PhD thesis in physics and mathematics, Institute of Mathematics CC USC RAS (2006). (in Russian.)
7. V.I. Lutsenko, R.S. Yulmukhametov. *Generalization of the Paley-Wiener theorem in weighted spaces* // Matem. Zametki.[Math. Notes. **48**:5, 1131-1136 (1990).]
8. K.P. Isaev. *Riesz bases of exponentials Riesz bases in Bergman spaces on convex polygons* // Ufimskij Matem. Zhurn. **2**:1, 60-71 (2010). (in Russian.)
9. V.V. Napalkov, R.A. Bashmakov, R.S. Yulmukhametov. *Asymptotic behavior of Laplace integrals and geometric characteristic of convex funtions* // Doklady Akad. Nauk. **413**:1, 20-22 (2007). [Dokl. Math. **75**:2, 190-192 (2007).]
10. R.A. Bashmakov, K.P. Isaev, R.S. Yulmukhametov. *On geometric characteristics of convex functions and Laplace integrlas* // Ufimskij Matem. Zhurn. **2**:1, 3-16 (2010). (in Russian).
11. R.S. Yulmukhametov. *Asymptotic approximation of subharmonic functions* // Sibir. Matem. Zhurn. **26**:4, 159-175 (1985). (in Russian.)
12. R.A. Bashmakov, A.A. Putintseva, R.S. Yulmukhametov. *Entire functions of sine type and their applications* // Alg. Anal. **22**:5, 49-68 (2010). [St. Petersbg. Math. J. **22**:5, 737-750 (2011).]

Rustem Abdraufovich Bashmakov,
Bashkir State University,
Zaki Validi str., 32,
450076, Ufa, Russia
E-mail: Bashmakov_Rustem@mail.ru

Alla Alexandrovna Makhota,
Bashkir State University,
Zaki Validi str., 32,
450076, Ufa, Russia
E-mail: allarum@mail.ru

Kirill Vladimirovich Trounov,
Bashkir State University,
Zaki Validi str., 32,
450076, Ufa, Russia
E-mail: trounovkv@mail.ru