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ON REGULAR AND SINGULAR SOLUTIONS FOR EQUATION $u_{xx} + Q(x)u + P(x)u^3 = 0$

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Abstract. The paper is devoted to the equation $u_{xx}+Q(x)u+P(x)u^3 = 0$. The equations of such kind have been used to describe stationary modes in the models of Bose-Einstein condensate. It is known that under some conditions for $P(x)$ and $Q(x)$, the "most part" of solutions for such equations are singular, i.e. tend to infinity at some point of the real axis. In some situations this fact allows us to apply the methods of symbolic dynamics to describe non-singular solutions of this equation and to construct comprehensive classification of these solutions. In the paper we present (i) necessary conditions for existence of singular solutions as well as conditions for their absence; (ii) the results of numerical study of the case when $Q(x)$ is a constant and $P(x)$ is an alternate periodic function. Basing on these results, we formulate a conjecture that all the non-singular solutions of the equation can be coded by bi-infinite sequences of symbols of a countable alphabet.

Keywords: ODE with periodic coefficients, singular solutions, nonlinear Schrodinger equation, stationary modes.

Mathematics Subject Classification: 34L30, 34C11, 35Q55, 37B10

1. Introduction

Starting from 90's of the last century, nonlinear Schrödinger equation with additional spatial non-autonomy is a object of a detailed studying. An interest to this class of equations is motivated mostly by successes in studying Bose-Einstein condensate (BEC). BEC is a matter state initiated under an ultralow temperature; its existence was predicted in 20's of XXth century. In 1995, BEC was obtained experimentally [1]. At that, the dynamics of BEC happened to be well described by a Schrödinger type equation with a non-autonomy being an additional external potential. In the context of BEC theory, it is the *Gross-Pitaevskii equation*. In the spatial one-dimensional case ("cigar-shaped" condensate), Gross-Pitaevskii equation reads as

$$i\Psi_t = -\Psi_{xx} + U(x)\Psi - P(x)|\Psi|^2\Psi. \tag{1}$$

Here $\Psi(x,t)$ is the macroscopic wave function of the condensate, $U(x)$ corresponds to the potential of the trap holding the condensate, and $P(x)$ describes the characteristic length of interatomic interaction. Employing the so-called Feshbach resonance, this length in an experiment can be made variable, in particular, periodic in the space [2]. In this case, one speaks about interaction of condensate with a *non-linear lattice* (the details can be found in review [3]). It should be noted that $P(x)$ can be both sign-definite and sign-indefinite function. In the literature, a special attention was paid to two model cases: $P(x) \equiv 1$ (the case of interatomic attraction) and $P(x) \equiv -1$ (the case of interatomic repulsion).

In the case of an optical trap, potential $U(x)$ is also modeled by a periodic function. In this case one speaks on a *linear lattice* trapping the condensate. The discussion of physical principles

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of trapping the condensate by a laser radiation can be found in work [4], and a review of results on the Gross-Pitaevskii equation with a periodic (linear) potential was provided in work [5].

In studying the problems related with BEC, an important role is played by so-called *stationary modes*. They are associated with the solutions like $\Psi(x,t) = e^{-i\omega t}u(x)$. If $u(x)$ is a real function, it satisfies the ordinary differential equation

$$u_{xx} + Q(x)u + P(x)u^3 = 0, \tag{2}$$

where $Q(x) = \omega - U(x)$. *Localisation condition* for a stationary mode

$$\lim_{x\to\pm\infty} u(x) = 0 \tag{3}$$

is natural from the physical point of view. At the same time, other types of stationary modes are considered in the literature, in particular, spatially periodic and quasi-periodic structures.

Thus, to describe the set of stationary modes, a detailed study of the set of solutions to equation (2) is needed. An attempt of such study was made in [6] in the case $P(x) \equiv -1$ (repulsive interatomic interactions) and with $Q(x)$ being a bounded periodic function. The main idea of work [6] said that the "most part" of the solutions to equation (2) went to infinity at some (finite) point of the real axis. Next, it was found out that under some conditions the set of "remaining" solutions defined on the whole of the real axis could be described by employing the methods of symbolic dynamics. More precisely, there is a one-to-one correspondence between these solutions with bi-infinite sequences of symbols in some alphabet (solution *codes*). Among these codes, one can select those corresponding to localized or periodic modes. In work [6], there was performed a detailed analysis of the case

$$Q(x) = \omega - A\cos 2x,$$

and there was described the domain on the plane of parameters (ω, A), where the solutions defined on the whole of the real axis can be coded by infinite sequences of symbols "0", "+", "−". At that, localized modes correspond to the codes containing just a finite number of non-zero symbols.

The generalization of work [6] for the case $P(x) \neq \text{const}$ looks to be an interesting and topical problem. Here the key role belongs to the study of the set of solutions to (2) going to infinity at a finite point of the real axis, and in additional, the set of solutions continued on the whole real axis. In particular, assuming that $Q(x)$ is a bounded periodic function, we can pose the following questions:

A. Whether application of the methods of work [6] is possible as $P(x) \equiv 1$? In particular, as $P(x) \equiv 1$, whether there exist solutions to equation (2) going to infinity at a finite point of the real axis?

B. What conditions for functions $P(x)$ and $Q(x)$ are needed for equation (2) to have the solutions going to infinity at a finite point $x = x_0$ of the real axis? Whether it is possible to extend the approach of work [6] for the case when $P(x)$ is non-constant?

In the present work we provide partial answers for Questions A and B. Concerning Question A, the negative answer is given by Proposition 1: for each bounded from below function $Q(x)$ having a bounded derivative and each strictly positive function $P(x)$, *all* the solutions to equation (2) are continued over whole real axis. Then, a particular answer for Question B is contained in Propositions 2 and 3. In particular, Proposition 2 says that if $P(x)$ is negative at least at one point of the axis $x = x_0$, equation (2) has C^1-smooth *one-parametric family of solutions going to infinity at this point*. Moreover, in accordance with Proposition 3, if $Q(x) \leqslant Q_0 < 0$ and $P(x) \leqslant P_0 < 0$, *none of solutions to equaion (2) except zero* is continued over whole real axis. If $P(x)$ and $Q(x)$ are periodic functions with the same period and $P(x)$ is sign-changing, a classification of stationary modes similar to [6] seems to be possible. In Section 4 we consider the case when $Q(x)$ is constant and $P(x)$ is π-periodic and changes the sign. The numerical study allows us to assume that in this case a generalization of approach of work [6] is possible. At that, to code possible stationary modes we need an alphabet of countably many symbols.

In what follows we adopt the following terminology. A solution $u(x)$ to equation (2) is called *singular*, if for some finite point $x_0 \in \mathbb{R}$ the relation

$$\lim_{x\to x_0} u(x) = \infty$$

holds true. At that, we say that solution $u(x)$ *collapses* at point x_0. Respectively, solution $u(x)$ to equation (2) collapsing at no points of $\mathbb{R}$ is called *non-singular* or *regular*.

2. Case $P(x) \geqslant P_0 > 0$: absence of singular solutions

A partial answer to Question A is given by the following proposition:

Proposition 1. *Suppose that $Q(x), P(x) \in C^1(\mathbb{R})$ and for each $x \in \mathbb{R}$*

a) $P(x) \geqslant P_0 > 0$, $|P'(x)| \leqslant \widetilde{P}$;

b) $Q(x) \geqslant Q_0$, $|Q'(x)| \leqslant \widetilde{Q}$.

Then the solution to the Cauchy problem for equation (2) with arbitrary initial conditions $u(x_0) = u_0$, $u_x(x_0) = u'_0$ can be continued on the whole real axis $\mathbb{R}$.

Proof. By the existence theorem for solutions to ODE, there exists an interval $[x_0, x_1)$ such that the solution $u(x)$ to the Cauchy problem for equation (2) with initial conditions $u(x_0) = u_0$, $u_x(x_0) = u'_0$ exists and is unique on this interval and $u(x) \in C^2[x_0; x_1)$. Suppose that $[x_0; x_1)$ is the maximal existence interval for the solution, i.e., the solution to the Cauchy problem can not be continued beyond the point $x = x_1$. We multiply the original equation by $4u_x$ and integrate over $[x_0; x)$, $x < x_1$, to obtain

$$2u_x^2(x) + 2Q(x)u^2(x) - 2\int\limits_{x_0}^{x} Q'(\xi)u^2(\xi)d\xi + P(x)u^4(x) - \int\limits_{x_0}^{x} P'(\xi)u^4(\xi)d\xi \tag{4}$$
$$= 2(u'_0)^2 + 2Q(x_0)u_0^2 + P(x_0)u_0^4 \equiv C.$$

Omitting $u_x^2(x) \geqslant 0$ in the left hand side of the identity, and also employing the boundedness from below of the functions $Q(x)$ and $P(x)$, we arrive at the inequality:

$$2Q_0u^2(x) + P_0u^4(x) \leqslant C + 2\int\limits_{x_0}^{x} Q'(\xi)u^2(\xi)d\xi + \int\limits_{x_0}^{x} P'(\xi)u^4(\xi)d\xi. \tag{5}$$

We replace the derivatives of $Q'(\xi)$ and $P'(\xi)$ by its upper bounds: $Q'(\xi) \leqslant \widetilde{Q}$, $P'(\xi) \leqslant \widetilde{P}$, where $\widetilde{Q} \geqslant 0$, $\widetilde{P} \geqslant 0$. We multiply both sides of the inequality by $P_0 > 0$ and get:

$$2Q_0P_0u^2(x) + P_0^2u^4(x) \leqslant P_0C + 2P_0\widetilde{Q}\int\limits_{x_0}^{x} u^2(\xi)d\xi + P_0\widetilde{P}\int\limits_{x_0}^{x} u^4(\xi)d\xi.$$

Denote $v(x) = (P_0u^2(x) + Q_0)^2$, $v(x) \geqslant 0$. Then

$$v(x) \leqslant \widetilde{C} + \frac{\widetilde{P}}{P_0}\int\limits_{x_0}^{x} w(v(\xi))\ d\xi. \tag{6}$$

Here $\widetilde{C} = P_0C + Q_0^2 \geqslant 0$, $\alpha = 2\widetilde{Q}P_0/\widetilde{P} \geqslant 0$, and $w(v)$ is determined by the formula

$$w(v) \equiv \alpha(\sqrt{v} - Q_0) + (\sqrt{v} - Q_0)^2.$$

We introduce the function

$$G(s)=\int\limits_{s_0}^{s}\frac{dv}{w(v)}. \tag{7}$$

Here $s_0>Q_0^2$ is an arbitrary constant, $s\geqslant s_0$. Since $w(v)$ is positive and monotonically decreasing, and the integral

$$\int\limits_{s_0}^{+\infty}\frac{dv}{w(v)}$$

diverges, function $G(s)$ is positive, monotonically increasing and unbounded. It means, that the inverse function $G^{-1}(r)$ is well-defined for $r\geqslant 0$, increases monotonically and is unbounded. The said above allows us to apply Bihari inequality, [7, Th. 2.3.1], to (6):

$$v(x)\leqslant G^{-1}\left(G(\widetilde{C})+\frac{\widetilde{P}}{P_0}\int\limits_{x_0}^{x}d\xi\right)=G^{-1}\left(G(\widetilde{C})+\frac{\widetilde{P}}{P_0}(x-x_0)\right)<\infty. \tag{8}$$

Inequality (8) is valid for each $x\in[x_0;x_1)$. It follows from (8) that function $v(x)$ is bounded on the whole segment $[x_0;x_1)$

$$v(x)\leqslant M=G^{-1}\left(G(\widetilde{C})+\frac{\widetilde{P}}{P_0}(x_1-x_0)\right).$$

We observe that $\widetilde{C}\geqslant Q_0^2$, at that, $\widetilde{C}=Q_0^2$ only as $u_0=u_0'=0$. It means that $G(s)$ is well-defined for $\widetilde{C}$ for each non-zero solution $u(x)$. The boundedness of $v(x)$ yields that solution $u(x)$ is also bounded on the segment $[x_0;x_1)$:

$$|u(x)|\leqslant\sqrt{\frac{\sqrt{M}-Q_0}{P_0}},\ x\in[x_0,x_1). \tag{9}$$

Substituting estimate (9) into identity (4), we obtain the upper estimate for the derivative $u_x(x)$ on the semi-interval $x\in[x_0;x_1)$. Since functions $u(x)$ and $u_x(x)$ are continuous and bounded on $[x_0;x_1)$, the values $u(x_1)$ and $u_x(x_1)$ are finite. Hence, there exists a continuation of the solution to the Cauchy problem with the initial conditions $u(x_0)$, $u_x(x_0)$ on an interval large than $[x_0;x_1)$. It contradicts to the original assumption.

Thus, we have proven the possibility of continuation of the solution to the half-line $x>x_0$. To prove the same for $x<x_0$, it is sufficient to make the change $x\to -x$ and to reproduce the above arguments. □

Comment: The proof of Proposition 1 implies in particular that if Conditions (a) and (b) are satisfied not on the whole real line but on a segment $[x_1;x_2]$, the solution to the Cauchy problem for equation (2) with arbitrary initial data collapses at no points of segment $[x_1;x_2]$.

3. Singular solutions in the case $P(x)$ is sign-indefinite

3.1. Asymptotic expansions. If $P(x)$ is negative at least at one point $x_0\in\mathbb{R}$, formal asymptotic expansions predict the existence of *two one-parametric families of the solutions* to equation (2) collapsing at this point.

Let us construct these asymptotic expansions. We suppose that $P(x_0)=-1$ (this condition can be satisfied by renormalization of an independent variable). We introduce the notation $\eta=x-x_0$ and assume that in the vicinity of the point $x=x_0$, the expansions

$$Q(x)=Q_0+Q_1\eta+Q_2\eta^2+\ldots,\quad P(x)=-1+P_1\eta+P_2\eta^2+\ldots$$

hold true. We have

$$u_{\eta\eta} + (Q_0 + Q_1\eta + Q_2\eta^2 + \ldots)u + (-1 + P_1\eta + P_2\eta^2 + \ldots)u^3 = 0.$$

The solutions to this equation collapsing at the point $x = x_0$, satisfies the condition $u(\eta) \to \pm\infty$, $\eta \to 0$. Suppose that η tends to zero from *the right*, $\eta > 0$. We make the changes $v(\eta) = \eta u(\eta)$, $\eta = e^{-t}$. We obtain

$$v_{tt} + 3v_t + 2v + e^{-2t}Q(t)v + P(t)v^3 = 0, \tag{10}$$

We determine the leading term in the expansion by balancing $2v$ and $-v^3$. Respectively, we have

$$V_0(t) = \pm\sqrt{2}. \tag{11}$$

Let us find the first corrector, $v(t) = \pm\sqrt{2} + V_1(t) + o(V_1(t))$. Substituting the latter expression into equation (10), employing the expansions for functions $Q(t)$ and $P(t)$ and omitting the terms of orders higher than e^{-t}, we obtain

$$V_{1,tt} + 3V_{1,t} - 4V_1 = \mp 2\sqrt{2}e^{-t} \tag{12}$$

that yileds $V_1(t) = \pm\frac{\sqrt{2}}{3}e^{-t}$. The second, third, and fourth correctors V_n, $n = 2, 3, 4$, can be found in the same way. For each of them the associated equation reads as

$$V_{n,tt} + 3V_{n,t} - 4V_n = C_n e^{-nt}. \tag{13}$$

However, once for $n = 2, 3$ the solutions to equation (13) read as $V_n \sim e^{-nt}$, in the case $n = 4$ the exponent in the right hand side coincides with one of the roots of the characteristic polynomial of the operator in the left hand side. In this case the solution to equation (13) should be chosen as $Ce^{-4t} - A_3te^{-4t}$. Here constant C is arbitrary, while A_3 is determined uniquely by the coefficients in the expansions for $P(x)$ and $Q(x)$. If constant C is fixed, at the further steps of this procedure the corresponding equations are uniquely solvable. We observe that the replacement of "+" by "−" in formula (11) leads us to the replacement of the signs for all the coefficients A_n, $n = 0, 1, \ldots$, that is natural due to the invariancy of equation (2) w.r.t. the change $u \to -u$. We obtain

$$\pm v(t) = \sqrt{2} + A_0e^{-t} + A_1e^{-2t} + A_2e^{-3t} + A_3 \cdot (-t) \cdot e^{-4t} + Ce^{-4t} + \ldots.$$

The explicit expressions for A_0 - A_3 are as follows:

$$A_0 = \frac{\sqrt{2}}{3}P_1, \tag{14}$$

$$A_1 = \frac{\sqrt{2}}{3}P_2 + \frac{\sqrt{2}}{6}Q_0 + \frac{2\sqrt{2}}{9}P_1^2; \tag{15}$$

$$A_2 = \frac{2\sqrt{2}}{3}P_2P_1 + \frac{7\sqrt{2}}{27}P_1^3 + \frac{\sqrt{2}}{6}Q_0V_1 + \frac{\sqrt{2}}{4}Q_1 + \frac{\sqrt{2}}{2}P_3; \tag{16}$$

$$A_3 = -\frac{\sqrt{2}}{6}Q_1P_1 - \frac{\sqrt{2}}{5}Q_2 - \frac{32\sqrt{2}}{45}P_2P_1^2 - \frac{3\sqrt{2}}{5}P_3P_1 - \frac{2\sqrt{2}}{15}P_2Q_0$$
$$-\frac{2\sqrt{2}}{15}Q_0P_1^2 - \frac{2\sqrt{2}}{5}P_4 - \frac{28\sqrt{2}}{135}P_1^4 - \frac{4\sqrt{2}}{15}P_2^2. \tag{17}$$

In the case $\eta \to 0$ from the left, $\eta < 0$, to construct similar expansions one should make the changes $v(\eta) = \eta u(\eta)$, $\eta = -e^{-t}$. The formulae for the coefficients A_n happened to be the same as in the case $\eta > 0$.

Finally, for the original solution $u(x)$, as $x \to x_0 \pm 0$ we obtain

$$\pm u(x) = \frac{\sqrt{2}}{\eta} + A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3 \ln|\eta| + C\eta^3 + A_4\eta^4 \ln|\eta| + \ldots, \tag{18}$$

where A_0-A_3 are defined by formulae (14)-(17), and other coefficients A_n, $n > 3$, are expressed via Q_n and P_n and arbitrary constant C.

Summarizing the said above, asymptotic expansion (18) predicts the existence of two one-parametric families of solutions collapsing at point x_0. These families are related by the symmetry $u \to -u$. As $x \to x_0 - 0$, the solutions in one of these families tends to $+\infty$, and to $-\infty$ for the other family.

3.2. Existence of one-parametric families of collapsing solutions. The possibility of constructing asymptotic expansions (18) does not prove the existence of one-parametric families of solutions collapsing at point x_0. At the same time, the following rigorous statement is true.

Proposition 2. *Let Ω be a neighborhood of point x_0, and $Q(x) \in C^2(\Omega)$ and $P(x) \in C^4(\Omega)$. Then there exist two C^1-smooth one-parametric families of solutions to equation (2) corresponding to expansions (18), collapsing at the point $x = x_0$ (while approaching from the left, $x < x_0$) and related by the symmetry $u \to -u$. Each of these families can be parametrized by a free variable $C \in \mathbb{R}$ in expansion (18).*

Proof. By the hypothesis, the identities

$$Q(x) = Q_0 + Q_1\eta + Q_2\eta^2 + Q_3\eta^3 + \widetilde{Q}(\eta)\eta^4$$

$$P(x) = -1 + P_1\eta + P_2\eta^2 + P_3\eta^3 + P_4\eta^4 + \widetilde{P}(\eta)\eta^5$$

hold true, where $\eta = x - x_0$ and $\widetilde{Q}(\eta), \widetilde{P}(\eta) \in C(\Omega)$. To prove the existence of the family corresponding to the sign "+" in (18), we make the change

$$u(x) = \frac{\sqrt{2}}{\eta} + A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3\ln(-\eta) + z(\eta)\eta^3,$$

where $z(\eta)$ is a new unknown function. The coefficients A_0, A_1, A_2 and A_3 are chosen in accordance with formulae (14)-(17), i.e., so that the coefficients at the powers η^{-2}, η^{-1}, η^0 and η vanish. The straightforward check shows that under such choice of A_k, $k = 0, 1, 2, 3$, equation (2) implies the following equation for z

$$z_{\eta\eta} + \frac{6}{\eta}z_\eta + g(\eta, z) = 0, \tag{19}$$

where $g(\eta, z)$ is a third order polynomial in z and $g(\eta, z) \sim \frac{\ln(-\eta)}{\eta}$ as $\eta \to -0$ for a fixed z. The change $\eta = -e^{-t}$ maps the point $\eta = 0$ into $t = +\infty$, and equation (19) is transformed into the equation

$$z_{tt} - 5z_t - f(t, z) = 0, \tag{20}$$

at that, $f(t, z) \sim te^{-t}$ as $t \to +\infty$. Assumptions for $f(t, z)$ allow us to apply the lemma on bounded solutions in Appendix to equation (20). This lemma states that all bounded as $t \to +\infty$ solutions to equation (20), as $t \to +\infty$, tend to some constant C. At that, for each value $C \in \mathbb{R}$ there exists the unique solution approaching asymptotically this constant as $t \to +\infty$. Moreover, these solutions form C^1-smooth family. Returning back to equation (19) and then, to (2), we arrive at the statement of the proposition. The existence of the second family of solutions corresponding to the sign "−" in (18) follows from the invariance of equation (2) under the change $u \to -u$. □

Comment: Similar one-parametric families of collapsing solutions exist to the *right* of the point $x = x_0$.

Once $P(x)$ is negative on some segments of the real axis, the set of non-singular solutions can have rather complicated structure. It was found in [6] that if $P(x) = -1$ and $Q(x)$ satisfies some additional conditions, to each non-singular solution of equation (2), one can associate a bi-infinite sequence of symbols in a finite alphabet. At that, the correspondence between the set of non-singular solutions and the set of symbols is a homeomorphism. In Section 4 we provide the results of numerical study for one more case: $Q(x)\equiv\omega=\text{const}$, $P(x)=\alpha+\cos 2x$, and $|\alpha|<1$. At that, in some cases the structure of the set of non-singular solutions to equation (2) happens to be trivial.

3.3. Case $P(x)\leqslant P_0<0$, $Q(x)\leqslant Q_0<0$.

The next statement happens to be true.

Proposition 3. *Suppose that for $x\in\mathbb{R}$ the conditions $P(x)\leqslant P_0<0$, $Q(x)\leqslant Q_0<0$ hold true. Then all the solutions to equation (2) except the zero one are singular.*

Before we prove Proposition 3, let us prove the following auxiliary statement.

Lemma 1. *All the solutions to the equation*

$$v_{xx}-qv-pv^3=0, \tag{21}$$

where $p,q>0$ are constants, except the zero one are singular.

Proof of Lemma 1. The solution to the Cauchy problem for equation (21) with the initial conditions $v_0=v(x_0)$, $v_0'=v_x(x_0)$ can be written implicitly

$$\pm\int_{v_0}^{v}\frac{d\xi}{\sqrt{C+q\xi^2+\frac{p}{2}\xi^4}}=x-x_0;$$

$$C\equiv(v_0')^2-qv_0^2-\frac{p}{2}v_0^4.$$

The choice of the sign in the left hand side depends on the initial conditions and value of x. The integral in the left hand side of the identity converges as $v\to\infty$, and hence, there exists a value x such that

$$x_{collapse}=x_0+\int_{v_0}^{\infty}\frac{d\xi}{\sqrt{C+q\xi^2+\frac{p}{2}\xi^4}},$$

while approaching each, $v(x)$ tends to infinity. □

Proof of Proposition 3. We employ Comparison Lemma from [8, Appendix C]. Consider the equation

$$v_{xx}+Q_0v+P_0v^3=0.$$

We introduce the notations

$$g(x,\xi)=-Q(x)\xi-P(x)\xi^3,\quad f(x,\xi)=f(\xi)=-Q_0\xi-P_0\xi^3.$$

We apply Comparison Lemma to the pair of equations

$$u_{xx}=g(x,u); \tag{22}$$

$$v_{xx}=f(x,v). \tag{23}$$

In the domain $D_+=\{x\in\mathbb{R},\ \xi\in(0;+\infty)\}$ we have $f(x,\xi)\leqslant g(x,\xi)$. Let $\tilde{u}(x)$ be the solution to the Cauchy problem for equation (22) with initial conditions $u(x_0)=u_0$, $u'(x_0)=u_0'$. We choose the initial conditions for the Cauchy problem for equation (23): $v(x_0)=u(x_0)=u_0$, $v'(x_0)=u'(x_0)=u_0'$. Let $\tilde{v}(x)$ be its solution. Let $u_0>0$, then there can be two cases:

(i) $u_0'\geqslant 0$. Function $\tilde{v}(x)$ increases monotonically; this fact can be easily established by the phase portrait of equation (23). In view of Comparison Lemma, solution $\tilde{u}(x)$ is an upper bound for solution $\tilde{v}(x)$ which is singular. Therefore, solution $\tilde{u}(x)$ is also singular.

(ii) $u_0'<0$. Let us make the change $\tilde{x}=-x$. In this case solution $\tilde{v}(\tilde{x})$ also decreases monotonically, and thanks to Comparison Lemma, $\tilde{u}(\tilde{x})$ is its upper bound, and hence, it is singular.

In the same way, in the domain $D_- = \{x \in \mathbb{R}, \xi \in (-\infty; 0)\}$, the inequality $f(x,\xi) \geqslant g(x,\xi)$ holds true. By similar arguments we prove that in domain D_-, solution $u(x)$ is also singular. □

4. $P(x)$ and $Q(x)$ are periodic functions: numerical study of the set of non-singular solutions

4.1. General scheme of study and some definitions. We proceed to the case when $P(x)$ is π-periodic and take negative values for at least some values x. We introduce some definitions.

Definition 1. *We define* ***Poincaré mapping*** $T\ :\mathbb{R}^2 \to \mathbb{R}^2$ *generated by equation (2) as follows:* [9]

$$T\begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \begin{pmatrix} u(\pi) \\ u_x(\pi) \end{pmatrix},$$

where $u(x)$ is the solution to equation (2) with initial conditions $u(0) = u_0$, $u_x(0) = u_0'$.

Definition 2. *A sequence of points (finite or infinite) $\{p_n\} \in \mathbb{R}^2$ such that $Tp_n = p_{n+1}$ will be called* ***orbit****.*

Definition 3. *We define sets $\mathcal{U}_L^+ \subset \mathbb{R}^2$ and $\mathcal{U}_L^- \subset \mathbb{R}^2$, $L > 0$ as follows: $p = (u_0, u_0') \in \mathcal{U}_L^+$ if and only if the solution to the Cauchy problem for equation (2) with initial data $u(0) = u_0$, $u_x(0) = u_0'$ does not collapse on the segment $[0; L]$. In the same way, $p = (u_0, u_0') \in \mathcal{U}_L^-$ if and only if the solution to the Cauchy problem for equation (2) with initial conditions $u(0) = u_0$, $u_x(0) = u_0'$ does not collapse on the segment $[-L; 0]$.*

We observe that Poincaré mapping T is defined only on set $\mathcal{U}_\pi^+$, while the inverse mapping T^{-1} is defined only on set $\mathcal{U}_\pi^-$, at that, $T\mathcal{U}_\pi^+ = \mathcal{U}_\pi^-$. Consider the sequence of sets

$$\begin{aligned} &\Delta_0 = \mathcal{U}_\pi^+ \cap \mathcal{U}_\pi^-, \\ &\Delta_{n+1}^+ = T\Delta_n^+ \cap \Delta_0, \qquad n = 0, 1, \dots, \\ &\Delta_{n+1}^- = T^{-1}\Delta_n^- \cap \Delta_0, \quad n = 0, 1, \dots \end{aligned}$$

It is obvious that Δ_0 consists of the points which have T-image and T-preimage. Moreover, the statements

$$\begin{aligned} \{p \in \Delta_n^+\} \quad &\Leftrightarrow \quad \{Tp, T^2p, \dots, T^np \in \Delta_0\} \\ \{p \in \Delta_n^-\} \quad &\Leftrightarrow \quad \{T^{-1}p, T^{-2}p, \dots, T^{-n}p \in \Delta_0\} \end{aligned}$$

hold true that implies

$$\begin{aligned} \ldots \subset \Delta_{n+1}^+ \subset \Delta_n^+ \ldots \subset \Delta_1^+ \subset \Delta_0 \\ \ldots \subset \Delta_{n+1}^- \subset \Delta_n^- \ldots \subset \Delta_1^- \subset \Delta_0. \end{aligned}$$

We define the sets

$$\Delta^+ = \bigcap_{n=1}^{\infty} \Delta_n^+, \quad \Delta^- = \bigcap_{n=1}^{\infty} \Delta_n^-.$$

Non-singular solutions to equation (2) correspond to the initial data of the Cauchy problem belonging to the set $\Delta = \Delta^+ \cap \Delta^-$. This set is invariant w.r.t. the action of T. The description of set Δ and of the action of mapping T on Δ allows us to list all non-singular solutions to (2).

It was shown in work [6] that if

a) Set Δ_0 has a finite amount N of connectivity components, $\Delta_0 = \bigcup_{k=1}^N D_k$, at that, each of components D_k is a curvilinear quadrilateral, whose boundaries satisfy special smoothness and monotonicity conditions;
b) All the sets $TD_k \cap D_m$ and $T^{-1}D_k \cap D_m$, $k, m = 1, \dots, N$, are non-empty, at that, the action of T on the curves lying in D_k preserves properly the monotonicity properties;
c) the areas of $\Delta_n^\pm$ tend to zero as $n \to \infty$;

the orbits generated by T in Δ are in one-to-one correspondence with bi-infinite sequences formed by an alphabet of N symbols. As the symbols, we choose $1, \dots, N$. The sequence $\dots \alpha_{-1}, \alpha_0, \alpha_1, \dots$, where $\alpha_k \in \{1, \dots, N\}$, is associated with the unique orbit satisfying

$$\dots, T^{-1}p \in D_{\alpha_{-1}}, \quad p \in D_{\alpha_0}, \quad Tp \in D_{\alpha_1}, \dots$$

Work [6] contains rigorous formulations of conditions (a), (b) and (c). In [6], its verification was made numerically for the case

$$Q(x) = \omega - A\cos 2x, \quad P(x) = -1$$

In what follows we provide the results of numerical study for another case, when $P(x)$ is a sign-changing function.

4.2. Case $P(x)$ is a sign-changing function. Numerical study. We consider the model case

$$Q(x) = \omega, \quad P(x) = \alpha + \cos 2x, \quad \alpha \in (-1, 1).$$

Equation (2) becomes

$$u_{xx} + \omega u + (\alpha + \cos 2x)u^3 = 0. \tag{24}$$

In what follows we provide a brief summary of the results for this case obtained by numerical calculations. The statements are not rigorous and are *conjectures*. A more detailed exposition of numerical study is planned to be presented later.

Sets $\mathcal{U}_\pi^\pm$. We search set $\mathcal{U}_\pi^\pm$ by scanning the plane of initial conditions (u, u_x) with sufficiently small steps in u and u_x. For each point (u_0, u_0') we solve the Cauchy problem with the initial conditions $u(0) = u_0$, $u_x(0) = u_0'$. If on the segment $[0; \pi]$ the absolute value of the solution exceeds a prescribed great number u_{max}, the solution is assumed to be collapsed otherwise the point (u_0, u_0') is supposed to belong to set $\mathcal{U}_\pi^+$. In numerical experiments we first assume that $u_{max} = 10^5$, and the result is checked for $u_{max} = 10^7$. Sets $\mathcal{U}_\pi^+$ obtained for different values u_{max} are in good accordance. The examples of sets $\mathcal{U}_\pi^+$ for various parameters ω and α are presented in Figure 1. Since equation (24) is invertible, sets $\mathcal{U}_\pi^-$ are obtained from sets $\mathcal{U}_\pi^+$ by a reflection along the axis u. The numerical results allow us to assume that sets $\mathcal{U}_\pi^\pm$ are *spirals with infinitely many rotations* around the origin.

Set Δ_0. The structure of set Δ_0 is determined by geometric properties of sets $\mathcal{U}_\pi^\pm$. If sets $\mathcal{U}_\pi^\pm$ are spirals with infinitely many rotations around the origin, Δ_0 consists of infinitely many connectivity components. These components are located symmetrically along the axes u and u_x and can be indexed by indices $\{A_k\}$, $k = \pm 1, \pm 2, \dots$ (for the components along axis u) and $\{B_k\}$, $k = \pm 1, \pm 2, \dots$ (for the components along axis u_x). At the origin there is one more connectivity component denoted by O, cf. Figure 2. In contrast to the case studied [6], set Δ_0 seems to be unbounded. It means that assumption (a) is not satisfied and moreover, it prevents a numerical verification of conjectures (b) and (c) similar to that in [6].

Despite of this circumstance, we employ the general idea of work [6] to describe the set of non-singular solutions. In view of the geometric properties of spirals $\mathcal{U}_\pi^\pm$, we assume that each of the connectivity components A_k and B_k is a curvilinear quadrilateral whose boundaries satisfy the smoothness and monotonicity conditions. Generally speaking, this is not the case for central component O. This component is not necessary a curvilinar quadrilateral with monotonous boundaries but it can be under an appropriate choice of parameters ω and α, cf. Figure 3.

Solutions coding. Suppose that all the connectivity components A_k, B_k, and O are curvilinear quadrilaterals. Then the results of numerical calculations allow us to assume that sets $T^{-1}A_k$, $T^{-1}B_k$, $k = 1, 2, \dots$, and $T^{-1}O$ infinite curvilinear strips located inside $\mathcal{U}_\pi^+$. In the same way, TA_k, TB_k, $k = 1, 2, \dots$, and TO are also infinite curvilinear strips located inside $\mathcal{U}_\pi^-$. Next, T-preimages of the sets

$$T^{-1}Z \cap A_l, \quad T^{-1}Z \cap B_l, \quad T^{-1}Z \cap O, \quad l = \pm 1, \pm 2, \dots,$$
$$Z \in \{O, A_k, B_k, k = \pm 1, \pm 2, \dots\},$$

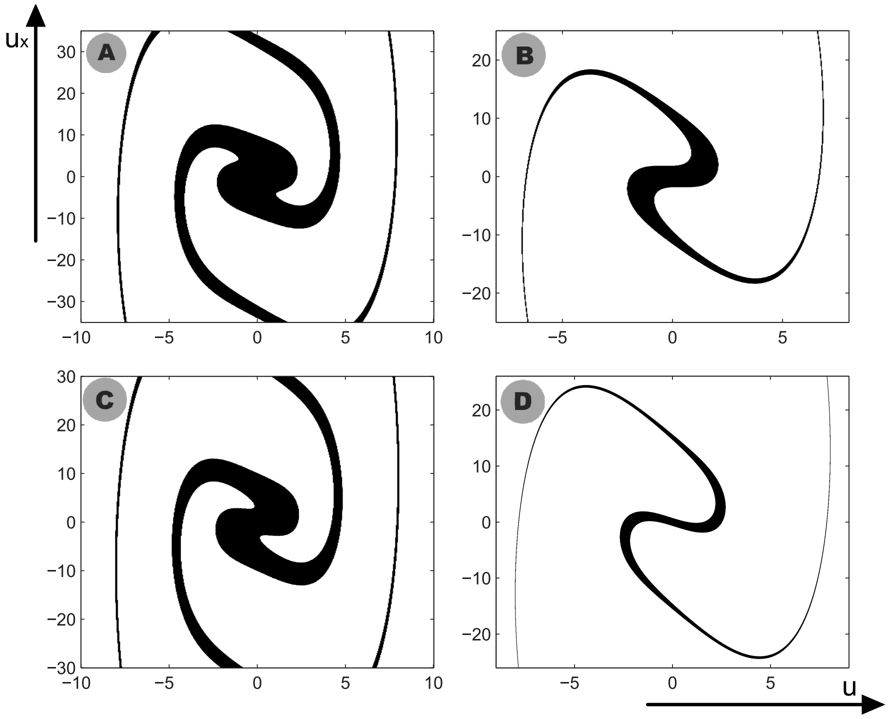

FIGURE 1. Sets $\mathcal{U}_\pi^+$: A) $\omega = 0.5$, $\alpha = 0.5$; B) $\omega = 1$, $\alpha = 0$; C) $\omega = -0.5$, $\alpha = 0.5$; D) $\omega = -1$, $\alpha = -0.1$.

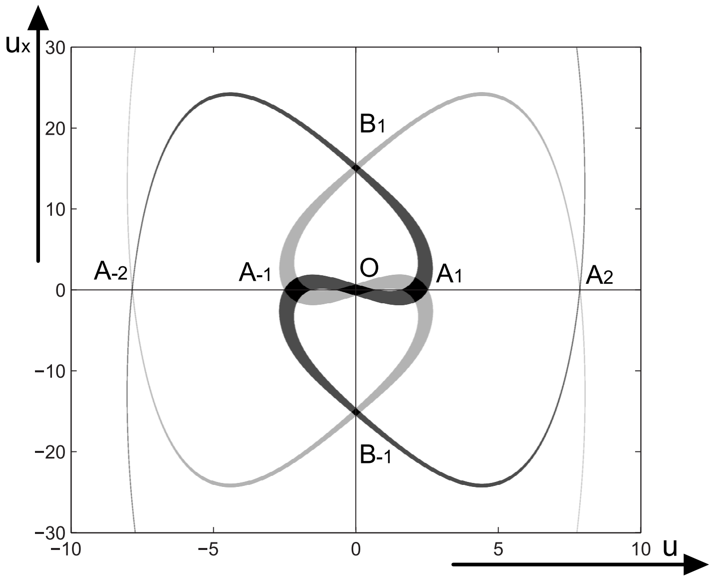

FIGURE 2. Sets $\mathcal{U}_\pi^+$ (dark grey color), $\mathcal{U}_\pi^-$ (light gray color), and their intersection $\mathcal{U}_\pi$ (black color) for equation (24) with parameters $\omega = 1$, $\alpha = -0.1$.

are infinite curvilinear strips lying in $T^{-1}Z$. A similar statement is also true for T-images of sets $TZ \cap A_l$, $TZ \cap B_l$, $TZ \cap O$ $l = \pm 1, \pm 2, \ldots$, which are located inside TZ, $Z \in \{O, A_k, B_k, k = \pm 1, \pm 2, \ldots\}$. Thus, the dynamics of mapping T happens to be similar to the dynamics of the Poincaré mapping described in work [6].

In view of this, by analogy with [6], we assume that *all non-singular solutions to equation (24) are in one-to-one correspondence with bi-infinite sequences of symbols of the form* $\{\ldots Z_{-1}, Z_0, Z_1, \ldots\}$*, where* $Z_m \in \{O, A_k, B_k, k = \pm 1, \pm 2, \ldots\}$. At that, the orbit associated with the code $\{\ldots Z_{-1}, Z_0, Z_1, \ldots\}$ come sequentially connectivity components Z_m, $m =$

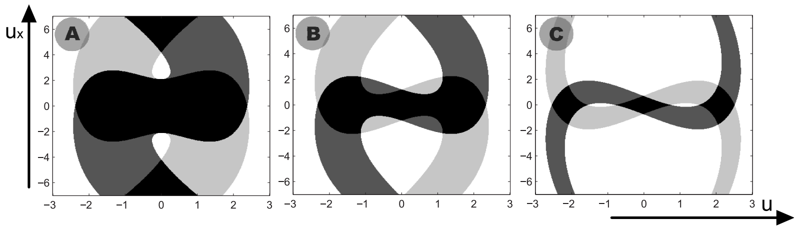

FIGURE 3. Central connectivity component O for various values of the parameters: A) $\omega = -1$, $\alpha = 0.5$; B) $\omega = -1$, $\alpha = 0.3$; C) $\omega = -1$, $\alpha = -0.1$;

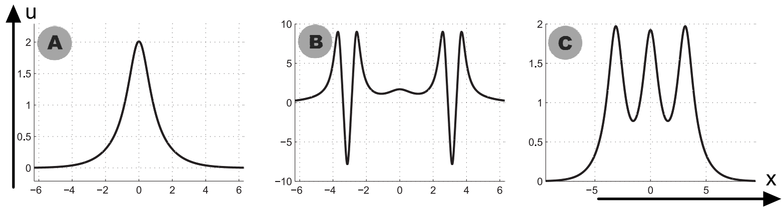

FIGURE 4. Solutions and codes: A)..OA_1O..; B)..$OA_{-2}A_1A_{-2}O$..; C)..$OA_1A_1A_1O$..;

$\ldots, -1, 0, 1, \ldots$ In Figure 4, we provide several localized solutions to equation (24) and there codes.

5. CONCLUSION

In the work we present the results of the study of equation (2) describing the stationary modes for non-autonomous non-linear Schrödinger equation. The main attention is paid to singular solutions going to infinity at some point on the real axis. As work [6] shows, if there are “sufficiently many” singular solutions, this fact can be employed for classification of stationary modes for equation (1).

It is convenient to present the results of the study as Table 1. Apart from rigorous statements, in the work we also present the results of numerical studying of equation (2) for the case when $Q(x) \equiv \omega$ and $P(x) = \alpha + \cos 2x$, $-1 < \alpha < 1$. On the basis of these results we propose a conjecture that in some range of parameters α and ω, non-singular solutions to equation (2) are in one-to-one correspondence with bi-infinite sequences $\{\ldots Z_{-1}, Z_0, Z_1, \ldots\}$, where $Z_m \in \{O, A_k, B_k, k = \pm1, \pm2, \ldots\}$, $m = 0, \pm1, \pm2, \ldots$. Unfortunately, rigorous sufficient conditions for parameters ω and α ensuring such coding are not known by the present time.

Apart the question on rigorous justification of such coding the solutions, there are other interesting subjects for further studies. In particular, in our opinion, the estimate for solution to equation (2) implied by Proposition 1 is not optimal. Under the made assumptions one can likely prove the boundedness of solutions on the whole real line. One more issue important for physical applications concerns the stability of stationary modes in the framework of the original evolution equation (1). In our opinion, the study of the connection between the stability and instability of a stationary mode and its coding is of interest. In particular, to the best of the authors’ knowledge, a systematic study of stability of stationary modes described by (24) has not been made yet.

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P	Q	
$P(x) > 0$	—	All the solutions are continued on the whole real line, singular solutions are absent (Proposition 1).
$P(x) < 0$ at least at one point $x = x_0$	—	There exists a pair of one-parametric family of solutions collapsing at point $x = x_0$ and related by the symmetry $u \to -u$ (Proposition 2)
$P(x) < 0$	$Q(x) < 0$	All the solutions except the zero one are singular (Proposition 3).
$P(x)$ is sign-changing	—	Complicated classes of non-singular solutions are possible. Under some additional restrictions, a coding of non-singular solutions is likely possible by bi-infinite sequences of symbols in some alphabet (numerical study, Section 4).

Table 1. Summary table of the results on existence or absence of singular solutions to equation (2) for periodic and sufficiently smooth $P(x)$ and $Q(x)$

6. Appendix: Lemma on bounded solutions

Lemma on bounded solutions. *Suppose that $f(t, z)$ is a function continuous w.r.t. t and infinitely differentiable w.r.t. z defined for $t \geqslant t_0$ and $|z| < \infty$ and possessing the following properties:*

(i) as $|z| < \rho$, $\rho > 0$, the estimate $|f(t, z)| < \eta_\rho(t)|z|$ holds true, at that, $\eta_\rho(t) \in L_1(t_0; \infty)$;

(ii) for each z_1 and z_2, such that $|z_{1,2}| < \rho$, $\rho > 0$, there exists a function $\tilde{\eta}_\rho(t) \in L_1(t_0; \infty)$ such that $|f(t, z_2) - f(t, z_1)| \leqslant \tilde{\eta}_\rho(t)|z_2 - z_1|$;

(iii) as $|z| < \rho$, $\rho > 0$, the inequality $|f_z(t, z)| < \theta_\rho(t)|z|$ holds true, at that, $\theta_\rho(t) \in L_1(t_0; \infty)$;

(iv) for each z_1 and z_2 such that $|z_{1,2}| < \rho$, $\rho > 0$, there exists a function $\tilde{\theta}_\rho(t) \in L_1(t_0; \infty)$ such that $|f_z(t, z_2) - f_z(t, z_1)| \leqslant \tilde{\theta}_\rho(t)|z_2 - z_1|$.

Then for the equation

$$z_{tt} - \alpha z_t + f(t, z) = 0, \quad \alpha > 0, \tag{25}$$

the following statements hold true:

(A) for each solution $z(t)$ to equation (25) bounded as $t \to \infty$ there exists $C \in \mathbb{R}$ such that $z(t) \to C$ as $t \to \infty$;

(B) for each $C \in \mathbb{R}$ there exists the unique solution $Z(t; C)$ to equation (25) defined on the segment (t_C, ∞) such that

$$Z(t; C) = C + o(1) \quad as \quad t \to +\infty; \tag{26}$$

(C) Family of solutions $Z(t; C)$ is C^1-smooth w.r.t. parameter C.

Proof of Lemma on bounded solutions. Let us prove Statement (A). Employing the variation of constants, we obtain that the solution to equation (25) satisfies the identity

$$z(t)=\varkappa_1+\varkappa_2e^{\alpha t}+\int_{t_0}^{t}e^{\alpha\eta}\left(\int_{\eta}^{\infty}e^{-\alpha\xi}\ f(\xi,z(\xi))\ d\xi\right)\ d\eta.$$

Due to Condition (i), if $z(t)$ is bounded as $t\to\infty$, the interal

$$\int_{t_0}^{\infty}e^{\alpha\eta}\left(\int_{\eta}^{\infty}e^{-\alpha\xi}\ f(\xi,z(\xi))\ d\xi\right)\ d\eta \tag{27}$$

converges. For bounded solution $\varkappa_2=0$, therefore, $z(t)$ tends to a constant as $t\to+\infty$. Statement (A) is proven.

Let us prove Statement (B). We make the change $u(t)=z(t)-C$, where C is an arbitrary number. We write equation (25) as the system

$$y_t=Ay+F(t,y), \tag{28}$$

where

$$y=\begin{pmatrix}u\\ v\end{pmatrix},\ A=\begin{pmatrix}0&1\\0&\alpha\end{pmatrix},\ F(t,y)=\begin{pmatrix}0\\ f(t,u+C)\end{pmatrix}.$$

We use Theorem 9.1 in [10, Ch. XII]. It states that system (28) has a solution vanishing at infinity if the following conditions are satisfied:

(1) function $F(t,y)$ is continuous, at that, $||F(t,y)||\leqslant\lambda(t)$ as $t\in[t_0,\infty)$, $||y||\leqslant\rho$, where $\lambda(t)\in L_1(t_0,\infty)$;

(2) for each $g(t)=\mathrm{col}(g_1(t),g_2(t))$, $g(t)\in L_1(t_0;\infty)$, there exists a solution $y(t)\in L_0^\infty(t_0;\infty)$ to the inhomogeneous system

$$y_t=Ay+g(t); \tag{29}$$

(hereinafter as the norm we mean the Euclidean norm in $\mathbb{R}^2$).

First, by (i), as $|u|\leqslant\rho$ and $t>t_0$, the relation $\|f(t,u,C)\|\leqslant\rho\eta_\rho(t)$ holds true. At that, $\eta_\rho(t)\in L_1(t_0,\infty)$ and hence, Condition (1) of the theorem is satisfied. Second, the general solution to inhomogeneous system of equations (29) reads as

$$u(t)=C_2+\int\limits_{t_0}^{t}\left(g_1(\eta)+e^{\alpha\eta}\left(C_1-\int\limits_{\infty}^{\eta}e^{-\alpha\xi}g_2(\xi)d\xi\right)\right)d\eta;$$
$$v(t)=u_t(t)-g_1(t).$$

Since $g_{1,2}(t)\in L_1[t_0;\infty)$, choosing appropriate constants C_1, C_2, we can obtain the solution vanishing as $t\to\infty$, i.e., Condition (2) is satisfied. Thus, the assumptions of the cited theorem are satisfied for system (28). It means that for each value C, equation (25) has a solution $z(t)$ tending to C as $t\to\infty$.

Let us prove that this solution is unique. Suppose that for the same C there exist two solutions to the equation

$$u_{tt}-\alpha u_t+f(t,u+C)=0 \tag{30}$$

Then their difference $\Delta(t)=u_2(t)-u_1(t)$ satisfies the equation

$$\Delta_{tt}-\alpha\Delta_t+R(t)\Delta=0 \tag{31}$$

and the boundary condition $\Delta\to0$ as $t\to+\infty$. Here

$$R(t)\equiv\frac{f(t,u_2(t)+C)-f(t,u_1(t)+C)}{u_2(t)-u_1(t)}.$$

By Condition (ii), we can apply Theorem 3 in [11, Ch. 3] to equation (31). It states that there exists a homeomorphism between the bounded solutions to equation (31) and to the equation

$$\Delta_{tt} - \alpha\Delta_t = 0. \tag{32}$$

At that, see the remark after this theorem in [11], by the linearity of the perturbation, this homeomorphism is a linear mapping. It means that only the zero solution to (31) satisfies the zero asymptotic condition at infinity, i.e., $u_2(t) \equiv u_1(t)$. Thus, we have proven the existence of the family of solutions $Z(t;C)$ parametrized by $C \in \mathbb{R}$, i.e., Statement (B) is proven.

To prove Statement (C), we observe that the derivative

$$\frac{\partial Z}{\partial C}(t,C) \equiv \Theta(t,C)$$

satisfies equation (30) differentiated w.r.t. C, and $\Theta(t,C) \to 0$ as $t \to \infty$. We have

$$\Theta_{tt} - \alpha\Theta_t + f_z(t,u+C)\Theta + f_z(t,u+C) = 0 \tag{33}$$

Applying again Theorem 11 in [11, Ch. 3] and employing (iii), we conclude that there exists a solution to this equation $\Theta(t,C)$ such that $\Theta(t,C) \to 0$ as $t \to \infty$ and $\Theta(t,C)$ is continuous w.r.t. parameter C. The proof is complete. □

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