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ANALOGUE OF BITSADZE-SAMARSKII PROBLEM FOR A CLASS OF PARABOLIC-HYPERBOLIC EQUATIONS OF SECOND KIND

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Abstract. In this work we prove the unique solvability of a Bitsadze-Samarskii problem for a degenerate parabolic-hyperbolic equation of second kind, when on the first and second part of characteristics Bitsadze-Samarskii condition is imposed.

Keywords: degenerate parabolic-hyperbolic equation, equation of second kind, non-local problems, Bitsadze-Samarsky condition , unique solvability, maximum principle, Fredholm integral equation.

Mathematics Subject Classification: 35L75, 35M10, 35L35, 34B53

1. Introduction

Local and nonlocal boundary value problems for equations of elliptic-hyperbolic and parabolic-hyperbolic types of first kind in various domains were studied in works by A.V. Bitsadze [1], M.S. Salakhitdinov [2], T.D. Dzhuraev [3], E.I. Moiseev [4], A.M. Nakhushev [5], T.Sh. Kal'menov [6], K.B. Sabitov [7], M.S. Salakhitdinov and B.I. Islomov [8], G.C. Wen [9] and their pupils. Later it was found that these problems appear in studying various problems of mathemtical biology, forecasting of soil moisture, solving problems in the physics of plasma, and in the mathematical modelling of laser emission.

It was pointed out in works by M.V. Keldysh [10] and A.V. Bitsadze [1] that there is an essential influence of lower terms in the equations for the formulations of boundary value problems for degenerate elliptic and hyperbolic equations. As mixed type equations of second kind, one usually calls equations whose degenerating line is an envelope for a family of characteristics, i.e., it is itself a characteristics.

Starting from 1953, after the publication of famous paper by I.L. Karol [11] there emerged an interest to studying boundary value problems for the mixed type equations of second kind. Analogues of Tricomi problem for an elliptic-hyperbolic equation of second kind in a domain a part of whose boundary is the degeneration line were considered in works [12]–[18]. In work [19]–[20] the Dirichlet problem for mixed type equations of second kind in a rectangular domain was studied.

Boundary value problems for parabolic-hyperbolic equations of second kind with no degeneration in the parabolic part were studied in works [21]–[22]. However, few works were devoted to mixed parabolic-hyperbolic type equations of second kind with a degeneration in the parabolic part; we mention [23], [24].

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In works [25], [26] there was studied a generalized Tricomi problem for elliptic-hyperbolic equation of first kind in the case where boundary condition on the first part of characteristics was imposed locally while on the second part and on a parallel characteristics a Bitsadze-Samarskii type condition was imposed. Such problems for mixed parabolic-hyperbolic and elliptic-hyperbolic equations of second kind were studied too little; here we mention work [27].

In this paper we study a boundary value problem for a degenerate parabolic-hyperbolic equation of second kind in the case when on the first part of the characteristics a non-local boundary condition is imposed while on the second part and on a parallel characteristics a Bitsadze-Samarskii type condition is introduced.

2. Formulation of the problem

We consider the equation

$$0=\begin{cases} u_{xx}-|x|^p u_y, & p>0 \quad \text{in} \quad D_j,\\ u_{xx}-(-y)^m u_{yy}, & 0<m<1 \quad \text{in} \quad D_3, \end{cases} \tag{1}$$

where D_j is the domain bounded by the segments OA_j, A_jB_j, RB_j, OR in the lines $y=0$, $x=(-1)^{j+1}$, $y=1$, $x=0$, respectively, for $y>0$. Hereinafter $j=1$ as $x\geqslant 0$, $j=2$ as $x\leqslant 0$; D_3 is a characteristic triangle bounded by the characteristics $A_1C:\ x+(1-2\beta)(-y)^{1/(1-2\beta)}=1$, $A_2C:\ x-(1-2\beta)(-y)^{1/(1-2\beta)}=-1$, $A_1A_2:\ y=0,\ -1<x<1$ of equation (1) as $y<0$. Here $2\beta=m/(m-2)$, at that

$$-1<2\beta<0. \tag{2}$$

We introduce the notations:

$$\begin{aligned}
&J_1=\{(x,y):0<x<1,\ y=0\},\\
&J_2=\{(x,y):-1<x<0,\ y=0\},\quad J_3=\{(x,y):x=0,\ \ 0<y<1\}\\
&OC_1:\ x-(1-2\beta)(-y)^{1/(1-2\beta)}=0,\quad OC_2:\ x+(1-2\beta)(-y)^{1/(1-2\beta)}=0,\\
&C\left(0;\ -(2/(2-m))^{2/(2-m)}\right),\quad O(0,0)\in A_1A_2,\quad C_1\in A_1C,\quad C_2\in A_2C,\\
&D=D_1\cup D_2\cup D_3\cup J_1\cup J_2\cup J_3,\quad D_4=D_1\cup D_2\cup J_3,
\end{aligned}$$

$$\Theta_j(x)=\left(\frac{x-1}{2};\ -\left[\frac{x+1}{2(1-2\beta)}\right]^{1-2\beta}\right),\quad (j=1,2), \tag{3_j}$$

$$\Theta^*(x)=\left(\frac{x}{2};\quad -\left[\frac{x}{2(1-2\beta)}\right]^{1-2\beta}\right), \tag{3}$$

$\Theta_1(x)$ and $\Theta_2(x)$ are the intersection points of characteristics A_2C with the characteristics starting at point $M_2(x,0)$, $x\in[-1;0]$ and $M_1(x,0)$, $x\in[0;1]$, respectively, and $\Theta^*(x)$ are the intersection points of characteristics OC_1 with the characteristics starting at point $M_1(x,0)$, $x\in[0;1]$.

By D_{31}, D_{32} and D_{33} we denote respectively the characteristic triangles OC_1A_1, A_2C_2O and quadrilateral OC_1CC_2.

BS problem. In domain D, find a function $u(x,y)$ with the properties:

1)$u(x,y)\in C(\overline{D})$;

2)$u(x,y)$ is a regular solution to equation (1) on sets $D_j\cup B_jR$ $(j=1,2)$;

3)$u(x,y)$ is a generalized solution to equation (1)in class R_2 [28] in domain $D_3\setminus(OC_1\cup OC_2)$;

4) $u(x,y)$ satisfies the boundary conditions:

$$u(x,y)\big|_{A_jB_j} = \varphi_j(y), \quad 0 \leqslant y \leqslant 1, \quad (j=1,2), \tag{4_j}$$

$$\frac{d}{dx} u\left[\Theta_1(x)\right] + a(x)u(x,0) = b(x), \quad (x,0) \in \overline{J}_2, \tag{5}$$

$$u\left[\Theta_2(x)\right] = \mu u\left[\Theta^*(x)\right] + \rho(x), \quad (x,0) \in \overline{J}_1; \tag{6}$$

5) $u_y \in C\left(D_1 \cup J_1\right) \cap C\left(D_2 \cup J_2\right) \cap C\left(D_3 \cup J_1 \cup J_2\right)$ and $u_x \in C\left(D_1 \cup J_3\right) \cap \cap C\left(D_2 \cup J_3\right)$; on the intervals $J_1 \cup J_2$ and J_3 we respectively have conjugation conditions

$$\lim_{y \to -0} \frac{\partial u\left(x,y\right)}{\partial y} = p_j(x) \lim_{y \to +0} \frac{\partial u\left(x,y\right)}{\partial y} + q_j(x), \quad (x,0) \in J_j, \tag{7_j}$$

and

$$\lim_{x \to -0} \frac{\partial u\left(x,y\right)}{\partial x} = \lim_{x \to +0} \frac{\partial u\left(x,y\right)}{\partial x}, \quad (0,y) \in J_3, \tag{7_3}$$

where $a(x)$, $b(x)$, $\rho(x)$, $\varphi_j(y)$, $p_j(x)$, $q_j(x)$ $(j=1,2)$ are given functions, at that,

$$b(-1)=0, \quad b(0)=0, \quad a(0) \neq 0, \quad \rho'(0)=0, \tag{8}$$

$$\mu = const > 0, \quad a(x) \leqslant 0, \quad a'(x) \leqslant 0, \quad \forall\, x \in [-1,1]\,, p_j(x) < 0, \quad \forall\, x \in \bar{J}_j, \tag{9}$$

$$\varphi_1(y),\ \varphi_2(y) \in C(\bar{J}_3) \cap C^1(J_3), \tag{10}$$

$$a(x),\ b(x) \in C^2\left(\bar{J}_2\right), \quad \rho(x) \in C^2\left(\bar{J}_1\right), \ \ p_j(x),\ \ q_j(x) \in C\left(\bar{J}_j\right) \cap C^2\left(J_j\right). \tag{11}$$

3. Main functional relations

We introduce the notations

$$u(x,0) = \tau_j(x), (x,0) \in \bar{J}_j, \ \ u_y\left(x, \pm 0\right) = \nu_j^{\pm}(x), (x,0) \in J_j, (j=1,2), \tag{12_j}$$

$$u(0,y) = \tau_3\left(y\right), \quad (0,y) \in \bar{J}_3, \ \ u_x\left(\pm 0, y\right) = \nu_3\left(y\right), \quad (0,y) \in J_3. \tag{12_3}$$

A generalized solution in class R_2 to the Cauchy problem with condition (12_1) for equation (1) in domain D_{31} is given by the formula [11], [28, Eq. (27.5)]:

$$u\left(\xi,\eta\right) = \int\limits_0^{\xi} (\eta-t)^{-\beta}(\xi-t)^{-\beta} T_1\left(t\right) dt + \int\limits_{\xi}^{\eta} (\eta-t)^{-\beta}(t-\xi)^{-\beta} N_1\left(t\right) dt, \tag{13_1}$$

where

$$N_1(t) = T_1(t)/2\cos\pi\beta - \gamma_1 \nu_1^-(t), \tag{14_1}$$

$$\gamma_1 = \left[2\left(1-2\beta\right)\right]^{2\beta-1}\Gamma\left(2-2\beta\right)/\Gamma^2\left(1-\beta\right),$$

$$\xi = x - (1-2\beta)\left(-y\right)^{1/(1-2\beta)}, \quad \eta = x + (1-2\beta)\left(-y\right)^{1/(1-2\beta)}, \tag{15}$$

$$\tau_1(x) = \int\limits_0^{x} (x-t)^{-2\beta} T_1\left(t\right) dt, \qquad (x,0) \in J_1, \tag{16_1}$$

functions $T_1(x)$ and $\nu_1^-(x)$ are continuous in $(0,1)$ and are integrable on $[0,1]$, while $\tau_1(x)$ has a zero of order at least -2β as $x \to 0$.

The generalized solution in class R_2 to the Cauchy problem with condition (12_2) for equation (1) in domain D_{3j} $(j=2,3)$ is given by the formula:

$$u\left(\xi,\eta\right) = \int\limits_{-1}^{\xi} (\eta-t)^{-\beta}(\xi-t)^{-\beta} T_2\left(t\right) dt + \int\limits_{\xi}^{\eta} (\eta-t)^{-\beta}(t-\xi)^{-\beta} N_2\left(t\right) dt, \tag{13_2}$$

where

$$N_2(t) = T_2(t)/2\cos\pi\beta - \gamma_1\nu_2^-(t), \qquad (14_2)$$

while ξ, η are determined by (15);

$$\tau_2(x) = \int_{-1}^{x} (x-t)^{-2\beta} T_2(t)\,dt, \qquad (x,0)\in J_2, \qquad (16_2)$$

functions $T_2(x)$ and $\nu_2^-(x)$ are continuous in $(-1,0)$ and are integrable on $[-1,0]$, while $\tau_2(x)$ has a zero of order at least -2β as $x\to -1$.

Letting $\xi=-1$ and $\eta = x$ in (13_2), in view of (3_1), (5), (14_2) and

$$u[\Theta_1(x)] = \int_{-1}^{x} (x-t)^{-\beta}(1+t)^{-\beta} N_2(t)\,dt, \quad (x,0)\in \bar{J}_2, \qquad (17_1)$$

we obtain the functional relation for $T_2(x)$ and $\nu_2^-(x)$ transferred from domain D_{32} in J_2:

$$\gamma_1\nu_2^-(x) = \frac{(1+x)^{\beta}}{\Gamma(1-\beta)} D_{-1x}^{-\beta} a(x)\tau_2(x) + \frac{1}{2\cos\pi\beta}T_2(x) - \frac{(1+x)^{\beta}}{\Gamma(1-\beta)} D_{-1x}^{-\beta} b(x), \quad (x,0)\in J_2, \quad (18)$$

where $D_{cx}^{\alpha}[\cdot]$ is the integral-differentiation operator of fractional order α [28], while $\tau_2(x)$ is determined by (16_2).

In the same way, letting $\xi=-1$, $\eta=x$ and $\xi=0$, $\eta=x$ respectively in (13_2) and (13_1), in view of (3_2), (3), after some calculations we get

$$u[\Theta_2(x)] = \int_{-1}^{x} (x-t)^{-\beta}(1+t)^{-\beta} N_2(t)\,dt, \qquad (x,0)\in \bar{J}_2, \qquad (17_2)$$

$$u[\Theta^*(x)] = \int_{0}^{x} (x-t)^{-\beta} t^{-\beta} N_1(t)\,dt, \qquad (x,0)\in \bar{J}_1. \qquad (19)$$

Differentiating (6) w.r.t. x and applying then operator $D_{0x}^{-\beta}[\cdot]$, we have

$$D_{0x}^{-\beta}\frac{d}{dx}u[\Theta_2(x)] = \mu D_{0x}^{-\beta}\frac{d}{dx}u[\Theta^*(x)] + D_{0x}^{-\beta}\rho'(x). \qquad (20)$$

Substituting (17_2), (19) into (20) and taking into consideration (5), (6), (14_1), (14_2), (18) and the identities $D_{-1x}^{-\beta}D_{-1x}^{\beta}(1+x)^{-\beta}N_2(x) = (1+x)^{-\beta}N_2(x)$, $D_{0x}^{-\beta}D_{0x}^{\beta}x^{-\beta}N_1(x) = x^{-\beta}N_1(x)$, we obtain the functional relation for $T_1(x)$ and $\nu_1^-(x)$ transferred from domain D_{31} in J_1:

$$\gamma_1\nu_1^-(x) = \frac{T_1(x)}{2\cos\pi\beta} + \frac{x^{\beta}}{\mu\Gamma(1-\beta)} D_{0x}^{-\beta} a(x)\tau_2(x) + F_1(x), \quad (x,0)\in J_1, \qquad (21)$$

where $\tau_2(x)$ is determined by (16_2),

$$F_1(x) = \frac{x^{\beta}}{\mu\Gamma(1-\beta)}\left\{ D_{0x}^{-\beta}\rho'(x) - D_{0x}^{-\beta}b(x)\right\}. \qquad (22)$$

By condition $\rho'(0)=0$, $b(0)=0$, $a(0)\neq 0$, in view of (17_1), (17_2), (19), it follows from (5) and (6) that $u(0,0)=0$. Therefore, by the conditions of BS problem we have $\tau_1(0)=\tau_2(0)=\tau_3(0)=0$.

Passing to the limit as $y \to +0$ in equation (1) for $x > 0$, $y > 0$ and $x < 0$, $y > 0$, in view of (4_1), (4_2), (8), (16_1), $(12_j)(j = \overline{1,3})$, we respectively have

$$\tau_1''(x) = x^p \nu_1^+(x), \qquad (x,0) \in J_1, \tag{23_1}$$

$$\tau_1(0) = \tau_3(0) = 0, \qquad \tau_1(1) = \varphi_1(0) \tag{24_1}$$

and

$$\tau_2''(x) = (-x)^p \nu_2^+(x), \qquad (x,0) \in J_2, \tag{23_2}$$

$$\tau_2(-1) = \varphi_2(0), \qquad \tau_2(0) = \tau_3(0) = 0. \tag{24_2}$$

Solving problem (23_j) and (24_j), we obtain functional relation for $\tau_j(x)$ and $\nu_j^+(x)$ transferred from domain D_j inJ_j $(j = 1,2)$:

$$\tau_1(x) = \int\limits_0^1 G_1(x,t)\, t^p \nu_1^+(t) dt + \Phi_1(x), \qquad (x,0) \in \bar{J}_1, \tag{25_1}$$

$$\tau_2(x) = \int\limits_{-1}^0 G_2(x,t)\, (-t)^p \nu_2^+(t) dt + \Phi_2(x), \qquad (x,0) \in \bar{J}_2, \tag{25_2}$$

where

$$\Phi_1(x) = x\varphi_1(0), \tag{26_1}$$

$$\Phi_2(x) = -x\, \varphi_2(0), \tag{26_2}$$

$$G_1(x,t) = \begin{cases} x(t-1), & 0 \leqslant x \leqslant t, \\ t(x-1), & t \leqslant x \leqslant 1, \end{cases} \tag{27_1}$$

$$G_2(x,t) = \begin{cases} t(1+x), & -1 \leqslant x \leqslant t, \\ x(1+t), & t \leqslant x \leqslant 0. \end{cases} \tag{27_2}$$

4. Uniqueness of solution to BS problem

Theorem 1. *If conditions (2), (8), (9) hold true, a solution to problem BS in domain D is unique.*

The following two lemmata play important role in the proof of Theorem 1.

Lemma 1. *If conditions (2), (8) hold true, solution*

$$u(x,y) \in C\left(\bar{D}_4\right) \cap C^{2,1}_{x,y}\left(D_1 \cup RB_1 \cup D_2 \cup B_2R\right), \quad u_x(x,y) \in C(D_4 \cup B_1B_2)$$

to equation (1) as $y > 0$ attains its positive maximum and negative minimum in closed domain $\overline{D}_4$ only on $\overline{A_1B_1} \cup \overline{A_2B_2} \cup J_1 \cup J_2$.

Proof. By the maximum principle for parabolic equations [29], [30], a solution $u(x,y)$ to equation (1) for $y > 0$ can not attain its positive maximum and negative minimum inside domains D_1 and D_2.

Let us show that solution $u(x,y)$ to equation (1) as $y > 0$ does not attain its positive maximum (negative minimum) on $\overline{J}_3$.

Assume the opposite, i.e., at some point $(0, y_0)$ in interval J_3 function $u(x,y)$ attains its positive maximum (negative minimum). Then by the maximum principle [31], [32], in domain D_1 we have

$$u_x(+0, y_0) < 0 \quad (> 0). \tag{28}$$

On the other hand, in domain D_2 we obtain

$$u_x(-0, y_0) > 0 \quad (< 0).$$

In view of (7_3), this inequality contradicts (28). Therefore, $u(x, y)$ does not attain its positive maximum (negative minimum) on interval J_3.

By condition (8), it follows from (5) and (6) that $u(x, y) = 0$. Thus, $u(x, y)$ does not attain its extremum at the point $O(0, 0)$.

Employing Lemmata 1.1 and 1.2 [33, Ch. 2, Sect. 2.3], one can show that at the point $(0, 1)$ there is no positive maximum (negative minimum).

Hence, $u(x, y)$ does not attain its positive maximum (negative minimum) on the interval $\bar{J}_3$. The proof is complete. □

Lemma 2. *Let $\tau_2(x) \in C\,[-1, 0] \cap C^{(1,k)}\,(-1, 0)$, where $k > -2\beta$, attain its maximal positive (minimal negative) at the point $x = x_0$ ($x_0 \in (-1, 0)$). Then at the point $x = x_0$ function*

$$T_2(x) \equiv \frac{1}{\Gamma\,(1 - 2\beta)} D_{-1x}^{1-2\beta} \tau_2(x) = \frac{\sin 2\pi\beta}{2\pi\beta} \frac{d^2}{dx^2} \int\limits_{-1}^{x} \tau_2\,(t)\,(x - t)^{2\beta}\,dt \tag{29}$$

can be represented as

$$T_2\,(x_0) = \frac{\sin 2\pi\beta}{\pi} \left[(1 + x_0)^{2\beta - 1}\ \ \tau_2\,(x_0) + (1 - 2\beta) \int\limits_{-1}^{x_0} \frac{\tau_2\,(x_0) - \tau_2\,(t)}{(x_0 - t)^{2 - 2\beta}} d\,t \right], \tag{30}$$

at that,

$$T_2\,(x_0) < 0 \quad (T_2\,(x_0) > 0)\,, \quad (x_0, 0) \in J_2. \tag{31}$$

This lemma can be proved by means of Theorem 1 in [14] and Lemma 27.1 in [28].

As above, we can prove Lemma 2 for the case $x_0 \in (0, 1)$. Hence, the identity

$$\begin{aligned} T_1\,(x_0) \equiv & \frac{1}{\Gamma(1 - 2\beta)} D_{0x_0}^{1-2\beta} \tau_1\,(x_0) = \frac{\sin 2\pi\beta}{\pi} \Big[x_0^{2\beta - 1} \tau_1\,(x_0) \\ & + (1 - 2\beta) \int\limits_{0}^{x_0} \left[\tau_1\,(x_0) - \tau_1\,(t)\right] (x_0 - t)^{2\beta - 2}\,dt \Big] \end{aligned} \tag{32}$$

holds true, as well as the inequality

$$T_1\,(x_0) < 0\,(T_1\,(x_0) > 0)\,, \tag{33}$$

where $(x_0, 0) \in J_1$ is a point of positive maximum (negative minimum) of function $\tau_1(x) \in C\left(\bar{J}_1\right) \cap C^{(1,k)}\,(J_1)\,.$

Lemma 3 (Analogue of A.V. Bitsadze maximum principle)**.** *If conditions (2),(8),(9) hold true, the solution $u\,(x, y)$ to BS problem as $\rho(x) \equiv 0$, $q_1(x) \equiv 0$, $q_2(x) \equiv 0$, $b(x) \equiv 0$, attains its positive maximum and negative minimum in closed domain $\bar{D}_4$ just on $\overline{A_1B_1} \cup \overline{A_2B_2}$.*

Proof. By Lemma 1, for $y > 0$, solution $u(x, y)$ to equation (1) attains its positive maximum and negative minimum in closed domain $\bar{D}_4$ just on $\overline{A_1B_1} \cup \overline{A_2B_2} \cup J_1 \cup J_2$.

Let us show that the solution $u(x, y)$ to equation (1) does not attain its positive maximum (negative minimum) on intervals J_j ($j = 1, 2$) and at point $O(0, 0)$. We assume the opposite. Let $u(x, y)$ attain its positive maximum (negative minimum) at some point $Q\,(x_0, 0) \in J_2$.

Then as $b(x) \equiv 0$ identity (18) becomes

$$\gamma_1 \nu_2^-(x) = \frac{1}{2\cos\pi\beta} T_2(x) + \frac{(1 + x)^\beta}{\Gamma\,(1 - \beta)} D_{-1x}^{-\beta} a(x) \tau_2(x), \qquad (x, 0) \in J_2. \tag{34}$$

By (7_2), the definition and the maximum principle for fractional order integral-differentiation operator [28, Eqs. (4.1), (4.6)], and (34) we obtain

$$
\begin{aligned}
\gamma_1 p_2(x)\nu_2^+(x) = & \frac{1}{2\cos\pi\beta}T_2(x) + \frac{(1+x)^\beta}{\Gamma(1-\beta)\Gamma(1+\beta)}\Bigg[(1+x)^\beta\, a(x)\tau_2(x) \\
& - \beta\int\limits_{-1}^{x}\frac{\tau_2(x)-\tau_2(t)}{x-t}a(t)(x-t)^\beta dt - \beta\tau_2(x)\int\limits_{-1}^{x}\frac{a(x)-a(t)}{x-t}(x-t)^\beta dt\Bigg].
\end{aligned} \tag{35}
$$

Taking into consideration (2), (9), (11), Lemma 2, (35) at the point $Q(x_0,0)$ of positive maximum (negative minimum) we obtain

$$\nu_2^+(x_0) > 0\left(\nu_2^+(x_0) < 0\right). \tag{36}$$

On the other hand, since $\tau_2''(x_0) \leqslant 0\,[\tau_2''(x_0) \geqslant 0]$, by (23_2) we obtain $\nu_2^+(x_0) \leqslant 0\ \left[\nu_2^+(x_0) \geqslant 0\right]$. This inequality contradicts (36).

Hence, $u(x,y)$ does not attain its positive maximum (negative minimum) on interval J_2.

In the same way, employing (2), (7_1), (9), (11), (23_1), (33), by (21) for $b(x)\equiv 0$, $\rho(x)\equiv 0$, $q_1(x)\equiv 0$, we get that$u(x,y)$ does not attain its positive maximum (negative minimum) on interval J_1.

It follows from (8) that $u(x,y)$ does not attain its extremum at the point $O(0,0)$. The proof is complete. □

Proof of Theorem 1.1. Let $\varphi_1(y)\equiv\varphi_2(y)\equiv b(x)\equiv\rho(x)\equiv q_1(x)\equiv q_2(x)\equiv 0$, then by Lemma 3 and (4_1), (4_2) we get

$$u(x,y)\equiv 0 \quad \text{in} \quad \overline{D}_4.$$

It follows that

$$u(x,0)\equiv 0, (x,0)\in\bar{J}_j, \quad u_y(x,\pm 0)\equiv 0, (x,0)\in J_j, (j=1,2). \tag{37}$$

Taking into consideration (14_1), (14_2), (29), (32), (37), by solution to Cauchy problem (13_j) for equation (1) in domains $D_{3j}(j=\overline{1,3})$, we obtain $u(x,y)\equiv 0$ in $\overline{D}_3$. Hence, $u(x,y)\equiv 0$ in domain $\bar{D}$. Thus, solution to BS problem is unique. The proof is complete. □

5. Existence of solution to BS problem

Theorem 2. *If conditions (2), (8), (10), and (11) hold true, then BS problem is solvable in domain D.*

We proceed to the proof of this theorem. By (16_1), the identity $\tau_1(0)=0$, (21) we have

$$\frac{1}{\Gamma(1-2\beta)}D_{0x}^{-2\beta}\tau_1'(x) = 2\cos\pi\beta\left[\gamma_1\nu_1^-(x) - \frac{x^\beta}{\mu\Gamma(1-\beta)}D_{0x}^{-\beta}a(x)\tau_2(x) - F_1(x)\right], (x,0)\in J_1. \tag{38}$$

By (18), (16_2) we get an integral equation for $T_2(x)$:

$$T_2(x) + \int\limits_{-1}^{x}K_2(x,t)T_2(t)\,dt = F_2(x), \quad (x,0)\in J_2, \tag{39}$$

where

$$K_2(x,t) = \frac{2\cos\pi\beta\,(1+x)^{\beta}}{\Gamma(1-\beta)\,\Gamma(1+\beta)} \cdot \left\{ \int\limits_t^x a'(z)\,(x-z)^{\beta}\,(z-t)^{-2\beta}\,dz - 2\beta\int\limits_t^x a(z)\,(x-z)^{\beta}\,(z-t)^{-2\beta-1}\,dz \right\}, \tag{40}$$

$$F_2(x) = 2\cos\pi\beta\left[\gamma_1\nu^-(x) + \frac{(1+x)^{\beta}}{\Gamma(1-\beta)}\,D_{-1x}^{-\beta}b(x)\right]. \tag{41}$$

In view of (2), (11), it follows from (40) that kernel $K_2(x,t)$ satisfies the estimate

$$|K_2(x,t)| \leqslant const\,(1+x)^{\beta}. \tag{42}$$

It follows from (11), (13_2), (41) that the right hand side of equation (39) is continuous in the interval $(-1,0)$ and is integrable on $[-1,0]$.

In view of (2), (42), equation (39) is a second kind Volterra integral equation with a weak singularity. The formulation of BS problem, (42), and properties of function $F_2(x)$ yield that a solution to equation (39) should be sought in the class of functions continuous in $(-1,0)$ and integrable on $[-1,0]$.

In accordance with the theory of integral Volterra equations [34], integral equation (39) is uniquely solvable and its solution is given by the formula

$$T_2(x) = F_2(x) - \int\limits_{-1}^{x} K_2^*(x,t)F_2(t)dt, \quad (x,0)\in J_2, \tag{43}$$

where $K_2^*(x,t)$ is the resolvent of kernel $K_2(x,t)$.

By (16_2), (43), and the identity $\tau_2(-1)=0$ we obtain

$$\frac{1}{\Gamma(1-2\beta)}D_{-1x}^{-2\beta}\tau_2'(x)\gamma_3\nu_2^-(x) - \gamma_3\int\limits_{-1}^{x} K_2^*(x,t)\nu_2^-(t)\,dt + F_3(x), \;\; (x,0)\in J_2, \tag{44}$$

where $\gamma_3 = 2\gamma_1\cos\pi\beta$,

$$F_3(x) = \frac{2\cos\pi\beta}{\Gamma(1-\beta)}\left[(1+x)^{\beta}\,D_{-1x}^{-\beta}b(x) - \int\limits_{-1}^{x} K_2^*(x,t)\,(1+t)^{\beta}\,D_{-1\,t}^{-\beta}b\,(t)\,dt\,. \right. \tag{45}$$

Excluding $\tau_1(x)$ and $\tau_2(x)$ for relations (38), (25_1), (44), (25_2), by (7_1) and (7_2), we get

$$\begin{aligned}\frac{1}{\Gamma(1-2\beta)}D_{0x}^{-2\beta}\left[\int_0^1 G'_{1x}(x,t)\,t^p\nu_1^+(t)dt+\Phi_1'(x)\right]&=\gamma_3\left[p_1(x)\nu_1^+(x)+q_1(x)\right]\\ &-\frac{2\cos\pi\beta x^\beta}{\mu\Gamma(1-\beta)\Gamma(1+\beta)}\frac{d}{dx}\int_0^x (x-t)^\beta a(t)dt\int_{-1}^0 G_2(t,z)(-z)^p\nu_2^+(z)dz\\ &+F_4(x),\quad (x,0)\in J_1,\end{aligned}\tag{46}$$

$$\begin{aligned}\frac{1}{\Gamma(1-2\beta)}D_{-1x}^{-2\beta}\left[\int_{-1}^0 G'_{2x}(x,t)\,(-t)^p\nu_2^+(t)dt+\Phi_2'(x)\right]&=\gamma_3\left[p_2(x)\nu_2^+(x)+q_2(x)\right]\\ &-\gamma_3\int_{-1}^x K_2^*(x,t)\ \left[p_2(t)\nu_2^+(t)+q_2(t)\right]dt+F_3(x),\quad (x,0)\in J_2,\end{aligned}\tag{47}$$

where

$$F_4(x)=-2\cos\pi\beta\left[\frac{x^\beta}{\mu\Gamma(1-\beta)}D_{0x}^{-\beta}a(x)\Phi_2(x)-F_1(x)\right].\tag{48}$$

We introduce the notations

$$R_1(x)=\frac{1}{\Gamma(1-2\beta)}D_{0x}^{-2\beta}\int_0^1 G'_{1x}(x,t)\,t^p\nu_1^+(t)dt,\qquad 0<x<1,\tag{46_1}$$

$$R_2(x)=\frac{1}{\Gamma(1-2\beta)}D_{-1x}^{-2\beta}\int_{-1}^0 G'_{2x}(x,t)\,(-t)^p\nu_2^+(t)dt,\qquad -1<x<0.\tag{47_1}$$

Then by (27_1), (27_2), and the definition of fractional order integral-differential operator [28, Eqs. (4.1), (4.6)], as well by the property of beta-functions [28, Eq. (1.7)], (46_1), (47_1), we have

$$R_1(x)=\gamma_4\left[\int_0^x P_{11}(x,t)\nu_1^+(t)\,dt+\int_x^1 P_{12}(x,t)\nu_1^+(t)\,dt\right]=\int_0^1 P_1(x,t)\,\nu_1^+(t)dt,\tag{46_2}$$

$$R_2(x)=\gamma_4\left[\int_{-1}^x P_{21}(x,t)\nu_2^+(t)\,dt+\int_x^0 P_{22}(x,t)\nu_2^+(t)\,dt\right]=\int_{-1}^0 P_2(x,t)\,\nu_2^+(t)dt,\tag{47_2}$$

where $\gamma_4=1/\Gamma(1-2\beta)\Gamma(1+2\beta)$,

$$P_1(x,t)=\begin{cases}\gamma_4P_{11}(x,t), & 0\leqslant t\leqslant x,\\ \gamma_4P_{12}(x,t) & x\leqslant t\leqslant 1,\end{cases}\tag{46_3}$$

$$P_2(x,t)=\begin{cases}\gamma_4P_{21}(x,t), & -1\leqslant t\leqslant x,\\ \gamma_4P_{22}(x,t), & x\leqslant t\leqslant 0,\end{cases}\tag{47_3}$$

$$P_{11}(x,t)=(x-t)^{2\beta}t^p+x^{2\beta}t^p(t-1),\quad P_{12}(x,t)=x^{2\beta}t^p(t-1),$$

$$P_{21}(x,t)=(x-t)^{2\beta}(-t)^{p+1}+(x-t)^{2\beta}(1+t)(-t)^p-(x+1)^{2\beta}(-t)^p,$$

$$P_{22}(x,t)=-(x+1)^{2\beta}(-t)^{p+1}.$$

Substituting (46_2) and (47_2) into (46) and (47), respectively, we find

$$\nu_2^+(x) - \int_{-1}^{0} P(x,t)\,\nu_2^+(t)\,dt = F_5(x), \quad -1 < x < 0, \tag{49}$$

$$\nu_1^+(x) - \int_{0}^{1} Q(x,t)\,\nu_1^+(t)\,dt = F_6(x) + \int_{-1}^{0} K_1(x,t)\nu_2^+(t)dt, \quad 0 < x < 1, \tag{50}$$

where

$$P(x,t) = \begin{cases} [P_2(x,t) + \gamma_3 K_2^*(x,t)p(t)]/\gamma_3 p_2(x), & -1 \leqslant t \leqslant x, \\ P_2(x,t)/\gamma_3 p_2(x), & x \leqslant t \leqslant 0, \end{cases} \tag{51}$$

$$Q(x,t) = P_1(x,t)/\gamma_3 p_1(x), \tag{52}$$

$$F_5(x) = \frac{D_{-1x}^{-2\beta}\Phi_2'(x)}{\Gamma(1-2\beta)\,\gamma_3 p_2(x)} - \frac{\gamma_3 q_2(x) + F_3(x)}{\gamma_3 p_2(x)} + \int_{-1}^{x} \frac{K_2^*(x,t)q_2(t)}{p_2(x)}dt\,, \tag{53}$$

$$F_6(x) = \frac{D_{0x}^{-2\beta}\Phi_1'(x)}{\Gamma(1-2\beta)\,\gamma_3 p_1(x)} - \frac{F_4(x) + \gamma_3 q_1(x)}{\gamma_3 p_1(x)}, \tag{54}$$

$$K_1(x,t) = \frac{2\cos\pi\beta x^\beta(-t)^p}{\gamma_1\mu\Gamma(1-\beta)\Gamma(1+\beta)p_1(x)}\frac{d}{dx}\int_0^x a(z)(x-z)^\beta G_2(z,t)dz. \tag{55}$$

By (2), (11), (40), (42), (46_3), (47_3), (51), (52) we obtain the estimates

$$|P(x,t)| \leqslant \begin{cases} const\,(x-t)^{2\beta}\,, & -1 \leqslant t \leqslant x, \\ const\,(1+t)^{2\beta}\,, & x \leqslant t \leqslant 0, \end{cases} \tag{56}$$

$$|Q(x,t)| \leqslant \begin{cases} const\,(x-t)^{2\beta}\,, & 0 \leqslant t \leqslant x, \\ const\;x^{2\beta}, & x \leqslant t \leqslant 1. \end{cases} \tag{57}$$

It follows from (56) and (57) that equations (49) and (50) are integral Fredholm equations of second kind with a weak singularity.

By (2), (8), (10), (11), (53), (54), (22), (26_1), (26_2), (40), (41), (42), (45), (48) we conclude that

1) Function $F_5(x)$ is continuous in $(-1,0)$ and integrable on $[-1,0]$. At that, as $x \to -1$ function $F_5(x)$ tends to infinity slower than a power with exponent -2β.

2) Function $F_6(x)$ is continuous in $(0,1)$ and is integrable on $[0,1]$. At that, as $x \to 0$, function $F_6(x)$ tends to infinity slower than a power with exponent -2β.

The solvability of integral Fredholm equation of second kind (49) and (50) (by the equivalence to BS problem) follows from the uniqueness of solution to BS problem. It is given by the formula [34]:

$$\nu_2^+(x) = \int_{-1}^{0} P^*(x,t)\,F_5(t)\,dt + F_5(x), \quad -1 < x < 0 \tag{58}$$

and $\nu_2^+(x) \in C(-1,0) \cap L_1[-1,0]$. As $x \to -1$, function $\nu_2^+(x)$ tends to infinity slower than the power with exponent -2β;

$$\nu_1^+(x) = \int_{0}^{1} Q^*(x,t)\,F_7(t)\,dt + F_7(x), \quad 0 < x < 1, \tag{59}$$

and $\nu_1^+(x) \in C(0,1) \cap L_1[0,1]$. At that, as $x \to 0$, function $\nu_1^+(x)$ tends to infinity slower than the power with exponent -2β. Here

$$F_7(x) = F_6(x) - \int_{-1}^{0} K_1(x,t) \left[\int_{-1}^{0} P^*(t,z) F_5(z)\, dz + F_5(t) \right] dt,$$

while $P^*(x,t)$ and $Q^*(x,t)$ are the resolvents for kernels $P(x,t)$ and $Q(x,t)$, respectively.

Substituting (58) and (59) into (25_1) and (25_2) respectively, we determine $\tau_2(x)$ and $\tau_1(x)$ in the classes

$$\tau_2(x) \in C^1[-1,0] \cap L_1[-1,0] \quad \text{and} \quad \tau_1(x) \in C^1[0,1] \cap L_1[0,1]. \tag{60}$$

Let us solve the following problem.

AD problem. Find solution $u(x,y) \in C(\bar{D}_4) \cap C^{2;1}_{x,y}(D_1 \cup RB_1 \cup D_2 \cup B_2R)$ to equation (1) satisfying conditions (4_1), (4_2), (7_3), and

$$u(x,0) = \tau_1(x), (x,0) \in \bar{J}_1, \tag{61_1}$$

$$u(x,0) = \tau_2(x), (x,0) \in \bar{J}_2, \tag{62_2}$$

where $\tau_j(x)$ $(j=1,2)$ are given functions satisfying condition (60), at that, $\tau_1(0) = \tau_2(0)$.

Theorem 3. *If conditions (2), (10), (60) hold true, AD problem is uniquely solvable in domain D_4.*

Proof. The solution to the Dirichlet problem for equation (1) in $D_1 \cup RB_1$ with conditions (4_1), (61_1), and $u(0,y) = \tau_3(y)$, $(0,y) \in \bar{J}_3$ read as [29], [31]:

$$u(x,y) = \int_0^1 G_0(x,t,y,\alpha) t^p \tau_1(t)dt + \frac{\partial}{\partial y}\int_0^y G^{(1)}(x,t,y,\alpha)\tau_3(t)dt + \frac{\partial}{\partial y}\int_0^y G^{(2)}(x,t,y,\alpha)\varphi_1(t)dt. \tag{62}$$

It belongs to the class $C(\overline{D}_1) \cap C^{2,1}_{x,y}(D_1 \cup RB_1)$ if (10), (60) hold true and $\tau_3(y) \in C(\overline{J}_3) \cap C^1(J_3)$. Here $G^{(j)}(x,t,y,\alpha)$, $(j=1,2)$ are introduced by the formulae

$$G^{(1)}(x,t,y,\alpha) = \frac{(1-\alpha)^{2(\alpha-1)} - x}{(1-\alpha)^{2(\alpha-1)}} - \int_0^1 \frac{(1-\alpha)^{2(\alpha-1)} - t}{(1-\alpha)^{2(\alpha-1)}} G_0(x,t,y,\alpha) t^p dt, \tag{63}$$

$$G^{(2)}(x,t,y,\alpha) = (1-\alpha)^{2(1-\alpha)} x - \int_0^1 G_0(x,t,y,\alpha)(1-\alpha)^{2(1-\alpha)} t^p dt, \tag{64}$$

and $G_0(x,\xi,y,\alpha)$ is the Green function of the Dirichlet problem for equation (1) in domain D_1,

$$G_0(x,\xi,y,\alpha) = \sum_{s=0}^{\infty} e^{-\frac{\lambda_s^2 y}{4}} (1-\alpha)\sqrt{x\xi}\, \frac{J_{1-\alpha}\left(\lambda_s(1-\alpha)x^{\frac{1}{2(1-\alpha)}}\right) J_{1-\alpha}\left(\lambda_s(1-\alpha)\xi^{\frac{1}{2(1-\alpha)}}\right)}{J^2_{2-\alpha}(\lambda_s)},$$

where $J_\chi(z) = \sum\limits_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\chi+1)} \left(\frac{z}{2}\right)^{\chi+2s}$ is the Bessel function of first kind, $\alpha = \frac{p+1}{p+2}$, at that,

$$\frac{1}{2} < \alpha < 1, \tag{65}$$

and λ_s are positive roots to the equation $J_{1-\alpha}(\lambda_s) = 0$, $s = 0,1,2,\ldots$

Differentiation (62) w.r.t. x and passing to the limit as $x \to +0$, by (12_3) we find

$$\nu_3(y) = \frac{\partial}{\partial y} \int_0^y Z(y-t)\tau_3(t)dt + F_8(y), \quad (0,y) \in J_3, \tag{66}$$

where

$$Z(y-t) = (1-\alpha)^{2\alpha-1} \lim_{x\to+0} \frac{\partial G^{(1)}(x,y-t,\alpha)}{\partial x} = -(1-\alpha) - \frac{2^{2\alpha}}{\Gamma^2(1-\alpha)} \sum_{s=0}^{\infty} \frac{\lambda_s^{-2\alpha} e^{-\frac{\lambda_s^2(y-t)}{4}}}{J_{2-\alpha}^2(\lambda_s)},$$

$$F_8(y) = \lim_{x\to+0} \frac{\partial}{\partial x} \left\{ \int_0^1 G_0(x,t,y,\alpha)t^p\tau_1(t)dt + \frac{\partial}{\partial y} \int_0^y G^{(2)}(x,t,y,\alpha)\varphi_1(t)dt. \right. \tag{67}$$

Thanks to the properties of function $J_\chi(z)$, function $Z(y-t)$ is represented as [21], [31]:

$$Z(y-t) = -\frac{1}{\Gamma(1-\alpha)}(y-t)^{\alpha-1} + B(y-t), \tag{68}$$

where $B(y-t)$ is a continuously differentiable function for $y \geqslant t$.

Substituting (68) into (66), we obtain the function relation for $\tau_3(y)$ and $\nu_3(y)$ transferred from domain D_1 to J_3:

$$\nu_3(y) = -\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial y} \int_0^y (y-t)^{\alpha-1}\tau_3(t)dt + \frac{\partial}{\partial y} \int_0^y B(y-t)\tau_1(t)dt + F_8(y). \tag{69}$$

It is easy to observe that the solution to the Dirichlet problem for equation (1) in $D_2 \cup RB_2$ with conditions (4_2), (61_2), and $u(0,y) = \tau_3(y), \ (0,y) \in \bar{J}_3$ is given by the formula

$$\begin{aligned} u(x,y) = &\int_{-1}^0 G_0(-x,t,y,\alpha)(-t)^p\tau_2(t)dt - \frac{\partial}{\partial y} \int_0^y G^{(1)}(-x,t,y,\alpha)\tau_3(t)dt \\ &- \frac{\partial}{\partial y} \int_0^y G^{(2)}(-x,t,y,\alpha)\varphi_2(t)dt. \end{aligned} \tag{70}$$

As above, differentiating (70) w.r.t. x and passing to the limit as $x \to -0$, in view of (12_3), (68) we obtain the functional relation for , $\tau_3(y)$ and $\nu_3(y)$, transferred from domain D_2 in J_3:

$$\nu_3(y) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial y} \int_0^y (y-t)^{\alpha-1}\tau_3(t)dt - \frac{\partial}{\partial y} \int_0^y B(y-t)\tau_3(t)dt + F_9(y), \tag{71}$$

where

$$F_9(y) = \lim_{x\to-0} \frac{\partial}{\partial x} \left\{ \int_{-1}^0 G_0(-x,t,y,\alpha)(-t)^p\tau_2(t)dt - \frac{\partial}{\partial y} \int_0^y G^{(}2)(-x,t,y,\alpha)\varphi_2(t)dt \right\}. \tag{72}$$

By (7_3), (69), and (71) we obtain the identity for $\tau_3(y)$:

$$\frac{2}{\Gamma(1-\alpha)} \frac{\partial}{\partial y} \int_0^y (y-t)^{\alpha-1}\tau_3(t)dt - 2\frac{\partial}{\partial y} \int_0^y B(y-t)\tau_3(t)dt = F_8(y) - F_9(y).$$

By the definition of fractional order integral-differential operator [28, (4.1), (4.6)] it follows that

$$\frac{2\Gamma(\alpha)}{\Gamma(1-\alpha)} D_{0y}^{1-\alpha}\tau_1(y) - 2B(0)\tau_3(y) - 2\int\limits_0^y B'_y(y-t)\tau_3(t)dt = F_8(y) - F_9(y). \tag{73}$$

Applying operator $D_{0y}^{\alpha-1}$ to identity (73), by the identity $\tau_3(0) = 0$ and

$$D_{0y}^{\alpha-1} D_{0y}^{1-\alpha}\tau_3(y) = \tau_3(y), \tag{74}$$

we arrive at the integral equation for $\tau_3(y)$:

$$\tau_3(y) = \int\limits_0^y M(y,t)\tau_3(t)dt + F_{10}(y), \quad (0,y) \in \bar{J}_3, \tag{75}$$

where

$$M(y,t) = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \left\{ B(0)(y-t)^{-\alpha} + \int\limits_t^y B_z(z-t)(y-z)^{-\alpha}dz \right\},$$

$$F_{10}(y) = \frac{\Gamma(1-\alpha)}{2\Gamma(\alpha)} D_{0y}^{\alpha-1}\left[F_8(y) - F_9(y)\right]. \tag{76}$$

Kernel $M(y,t) \in C([0,1]\times[0,1] \setminus \{(y,t) : y = t\})$ for y close to t obeys the estimate

$$|M(y,t)| \leqslant const(y-t)^{-\alpha}. \tag{77}$$

Let us study function $F_{10}(y,t)$. By employing properties of function $G_0(x,\xi,y-\eta,\alpha)$, $J_\chi(z)$ and (2), (10), (60), it was proved in works [21] and [33] that functions F_i $(i = 8,9)$ belong to the class

$$F_i(y) \in C\left(\overline{J}\right) \cap C^1(J), \quad (i = 8,9). \tag{78}$$

By (76), (77), (78), and the definition of fractional order integral-differential operator [28, (4.1)] we obtain the estimate

$$|F_{10}(y)| \leqslant \frac{1}{2\Gamma(\alpha)} \int\limits_0^y (y-t)^{-\alpha}\left[|F_8(t)| + |F_9(t)|\right]dt \leqslant const \int\limits_0^y (y-t)^{-\alpha}dt \leqslant const\, y^{1-\alpha}.$$

Taking into consideration (65) and the inequality $0 \leqslant y \leqslant 1$, we have

$$|F_{10}(y)| < const. \tag{79}$$

By (10), (60), (65), (79), (76) it follows that

$$F_{10}(y) \in C\left(\overline{J}_3\right) \cap C^1(J_3). \tag{80}$$

Thus, by (77), (79), (80) equation (75) is a second kind Volterra integral equation with a weak singularity.

In accordance with the theory of integral Volterra equations [34], we conclude that integral equation (75) is uniquely solvable in the class $C\left(\bar{J}_3\right) \cap C^1(J_3)$ and its solution is given by the formula

$$\tau_3(y) = F_{10}(y) + \int\limits_0^y M^*(y,t)F_{10}(t)dt, \quad (0,y) \in \bar{J}_3, \tag{81}$$

where $M^*(y,t)$ is the resolvent of kernel $M(y,t)$.

Therefore, AD problem is uniquely solvable since it is equivalent to Volterra equation of second kind (75).

Substituting $\tau_3(y)$ from (81) into (62) and (70), we recover the solution to AD problem as the solution to the Dirichlet problem in domains D_1 and D_2 respectively. □

Thus, the solution to BS problem can be found in domains D_1 and D_2 as the solution to the Dirichlet problem for equation (1) (see (62), (70)), while in domain D_3 it can be found as as the solution to the Cauchy problem for equation (1) (see . (13_j) $(j = 1, 2)$). The proof of Theorem 2 is complete.

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