EQUIVALENCE OF SECOND-ORDER ODES
TO EQUATIONS OF FIRST PAINLEVÉ EQUATION TYPE

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Abstract.   We consider equivalence problem for equations of a degenerate type, which involve, for example, the first Painlevé equation. In terms of algebraic and differential invariants of the family of equations with the cubic nonlinearity in the first-order derivative, we obtain the necessary condition of equivalence to some equations of this type with a known solution. We prove a criterion of equivalence to the first Painlevé equation under point transformations.

Keywords: first Painlevé equation, equivalence, invariant.

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1.   INTRODUCTION

The problem on equivalence of second order ordinary differential equations (ODEs)

$$\frac{d^2y}{dx^2} = S(x,y)\left(\frac{dy}{dx}\right)^3 + 3R(x,y)\left(\frac{dy}{dx}\right)^2 + 3Q(x,y)\frac{dy}{dx} + P(x,y) \tag{1}$$

under point change of variables

$$z = \xi(x,y), \qquad w = \eta(x,y), \qquad \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0 \tag{2}$$

attracts a lot of attention since the end of XIXth century [1]–[10]. Class of equations (1) is closed w.r.t. transformations (2). It contains 50 equations [11, Ch. 14] obtained in the classification of second order ODEs having no moving critical points except for the poles [12, 13] including six Painlevé equations. The problem of equivalence of a second order ODE to the first Painlevé equation

$$\frac{d^2w}{dz^2} = 6w^2 + z \tag{3}$$

under point transformations was studied in [10] and [14]–[19]. In [20], there was considered a problem of equivalence of the first Painlevé equation to the generalized Emden-Fowler equation under (non-local) Sundman transformation.

It was proven in [21] that all Painlevé equations can be reduced to the equation

$$\frac{d^2y}{dx^2} = f(x,y) \tag{4}$$

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by some change (2); canonical form (4) was obtained for third, fourth, fifth, and sixth Painlevé equation. It was also shown in [21] that only special change of variables

$$\tilde{x} = k \int \phi^2(x)dx + x_0, \quad \tilde{y} = \phi(x)y + \chi(x), \quad k, x_0 = \text{const}, \quad k, \phi(x) \neq 0 \tag{5}$$

does not lead equation (4) outside this class.

In the most part of works devoted to the equivalence problem for equations (1) [1]–[5, 8] there was studied only the general (non-degenerate) case for equation (1) described by the relation $J_0 \neq 0$ (the formula for J_0 is provided in the next section). For all the Painlevé equations we have $J_0 = 0$ and thus, they belong to degenerate types of equation (1). In [1] there were also studied degenerate cases of equation (1) but their complete classification was not made. Such classification was made in [6, 7] by the methods of differential geometry. Later, in [9], it was made by exterior Cartan forms, and later in [10] by employing Lie's infinitesimal approach. At that, the result applicable to each equation (1) was provided only in works [7, 10]. In [6] the formulae for the invariants of equation (1) were obtained in a special coordinate system and to apply them, one should first make a preliminary change of variables (i.e., requires making several interations). The result of [9] for degenerate types of equation (1) including, in particular, Painlevé equations needs that equation (1) should be first reduced to (4).

Equation (1) of first type in [10] correspond to the general case determined in [6, 7]. Equations of ninth type in [10] coincide with the case of maximal degeneration in [6, 7]. For other seven types of equation (1) introduced in [10], it is to be studied independently to which cases of intermediate degeneration in original classification [6, 7] these types correspond. In the main case the formulae of connection between the invariants of equation (1) constructed in [7] and [10] are provided in [10, Sect. 1]. Similar formulae in the degenerate case which includes five of six Painlevé equations are given in [22, Sect. 7]. The correspondence between the invariants in [10] and those applied in [1, 21, 23] was established in [22, Sect. 3] and [24].

In the present work the equivalence conditions were obtained by employing algebraic (depending on x, y) invariants of equations (1) constructed in [10]. Equivalent equations have the same sets of (absolute) invariants. According to the classification in [10], equation (3) belongs to the sixth type of equation (1). Other five types of Painlevé equations belong to the fourth type and the equivalence conditions for them were studied in [10, 19, 22, 23, 24]. The present paper is devoted to the equations of sixth type, i.e., to the most degenerate equations (1) possessing invariants. In Section 2 we describe invariants of such equations and we show that by some change of variables (2) all of them are reduced either to

$$\frac{d^2w}{dz^2} + \frac{1}{4w^3} + F(z) = 0, \qquad F(z) \neq 0, \tag{6}$$

or to

$$\frac{d^2w}{dz^2} = 6w^2 + f(z). \tag{7}$$

In Sections 3, 4 we calculate the invariants for these equations and obtain necessary conditions of equivalence to equations (4) of sixth type for which in [25] either the general solution was given or the order was reduced. Namely, in Theorems 2–6 we describe five non-equivalent equations of sixth type admitting point symmetries. Four of these equations are integrable and in one of them the order can be reduced. Transformation (2) relating equivalent equations is constructed by employing both algebraic and differential (depending on x, y, $y' = dy/dx$) invariants of equations (1). It was shown in [26] that in solving the equivalence problem of a second order ODE the maximal amount of differentiations required for constructing the invariants should be made for equation (7) with $(f^{-1/4})'' \neq 0$.

As papers [14]–[19] show, the criterion of equivalence to equation (3) can be obtained by means of various approaches. In Section 5 it is reproduced in Theorem 7. At the example of the first Painlevé equation we show how to prove the sufficiency for conditions of equivalence to equation (1), once among its algebraic invariants there are invariants I_j, I_k such that

$$\frac{\partial(I_j, I_k)}{\partial(x, y)} \neq 0. \tag{8}$$

Relation (8) expresses the functional independence of invariants I_j and I_k. The approach based on the functional independence of the invariants it known and employed for a long time (see, for instance, [5, P. 7], [17], [18] or formulae (7) in [19]). In [22] this approach was employed in the proof of a criterion of equivalence to the second Painlevé equation and to equation XXXIV in [11] which can be reduced to the second Painlevé equation by a differential substitution. For the other Painlevé equations the sufficiency of equivalence conditions was established if applying of these conditions to ODE (4), up to transformation (5) gives the canonical form of the corresponding Painlevé equation in [21]. In the case of third and fourth Painlevé equations, this approach was employed in works [24] and [22], respectively.

In Section 6 we provide examples of applying differential invariants in constructing transformation (2) relating two equivalent equations with no algebraic invariants satisfying condition (8). We consider the case when all the algebraic invariants of an equation are constant as well as the case when all of them depend on one variable. It is also shown that the generalized Emden-Fowler equation found in [20] is related with the first Painlevé equation by a point change of variables (5).

2. INVARIANTS FOR EQUATIONS OF SIXTH TYPE

Here we partially reproduce the classification of second order ODEs in [10, Thms. 2, 7] concerning the equations of sixth type.

Theorem 1. *Equation (1) of sixth type are characterized by the relations*

$$J_0 = 0, \quad \beta_1 \neq 0, \quad j_0 = 0, \quad j_1 = 0, \quad j_2 = 0, \quad j_3 \neq 0, \quad \Gamma_0 j_3 \neq 5\beta_1. \tag{9}$$

The basis of its differential invariants is

$$
\begin{aligned}
I_0 &= \frac{1}{j_3^{3/2}} \left(\frac{15e_0}{I_1 - 5} - \frac{3y'}{\beta_1(\beta_1 + y'\beta_2)} \right), \\
I_1 &= \frac{\Gamma_0 j_3}{\beta_1}, \qquad I_2 = \frac{1}{5j_3^2} \left((5 - I_1)\Lambda + \frac{4\Gamma_0 e_0^2}{\beta_1} \right).
\end{aligned}
\tag{10}
$$

An arbitrary algebraic invariant of equation (1) can be obtained by applying functional-algebraic operations and operators of invariant differentiation

$$\mathcal{D}_1 = \frac{1}{j_3}(\beta_2\partial_x - \beta_1\partial_y), \quad \mathcal{D}_2 = \frac{15e_0}{(I_1 - 5)\sqrt{j_3}}(\beta_2\partial_x - \beta_1\partial_y) - \frac{3}{\beta_1\sqrt{j_3}}\partial_x \tag{11}$$

to I_1, I_2. Algebraic invariants for all the equations of sixth type satisfy trivial relations

$$
\begin{aligned}
& 5I_{11} + 3I_1(5 - I_1) = 0, \quad 5I_{12} - 2(2I_1 + 15)I_2 + (5 - I_1)\left(\frac{46}{45}I_1 + 33\right) = 0, \\
& I_{21} = 0, \qquad \mathcal{D}_1(\mathcal{D}_2^n I_2) = \left(\frac{1}{10}(n + 8)I_1 + \frac{3}{2}(n + 4)\right)\mathcal{D}_2^n I_2, \quad n \in \mathbb{N}.
\end{aligned}
\tag{12}
$$

In this theorem we employ the notations

$$I_{ij} = \mathcal{D}_i I_j, \quad I_{kij} = \mathcal{D}_k(\mathcal{D}_i I_j), \quad I_{lkij} = \mathcal{D}_l(\mathcal{D}_k(\mathcal{D}_i I_j)), \quad i, j, k, l = 1, 2, \tag{13}$$

for the derivatives of the invariants obtained by applying invariant differentiations (11) to I_1, I_2. The quantities involved in (9)–(11)

$$\Gamma_0 = 3\beta_2\gamma_{10} + \beta_1(\gamma_{20} - 4\gamma_{11}), \qquad \Gamma_1 = \beta_2(4\gamma_{20} - \gamma_{11}) - 3\beta_1\gamma_{21},$$

$$J_0 = \frac{1}{3}(\beta_2\Gamma_0 - \beta_1\Gamma_1), \qquad j_0 = \frac{3}{\beta_1}\left(\frac{\beta_2}{\beta_1}\delta_{10} - \delta_{11}\right) + \frac{6\gamma_{10}}{\beta_1^2}\left(\gamma_{11} - \frac{\beta_2}{\beta_1}\gamma_{10}\right),$$

$$j_1 = \frac{5}{6}\left(2\beta_2\delta_{20} - \beta_1\delta_{30} - \frac{\beta_2^2}{\beta_1}\delta_{10}\right) + \left(\gamma_{20} + \gamma_{11} - 2\frac{\beta_2}{\beta_1}\gamma_{10}\right)\left(\gamma_{20} - \frac{2}{3}\gamma_{11} - \frac{\beta_2}{3\beta_1}\gamma_{10}\right),$$

$$j_2 = \frac{1}{\beta_1}\left(\delta_{20} - \frac{\beta_2}{\beta_1}\delta_{10}\right) + \frac{\gamma_{10}}{5\beta_1^2}\left(7\frac{\beta_2}{\beta_1}\gamma_{10} - 6\gamma_{20} - \gamma_{11}\right), \quad j_3 = \frac{3}{5}\left(\frac{\delta_{10}}{\beta_1^3} - \frac{6\gamma_{10}^2}{5\beta_1^4}\right),$$

$$e_0 = \frac{\epsilon_{10}}{5\beta_1^4} - \frac{21\gamma_{10}\delta_{10}}{25\beta_1^5} + \frac{84\gamma_{10}^3}{125\beta_1^6}, \qquad \Lambda = \frac{\lambda_{10}}{5\beta_1^5} + \frac{4}{25\beta_1^6}(7\delta_{10}^2 - 8\gamma_{10}\epsilon_{10})$$

are calculated by means of relative invariants

$$\beta_1 = \alpha_{1x} - \alpha_{0y} + R\alpha_0 - 2Q\alpha_1 + P\alpha_2, \quad \beta_2 = \alpha_{2x} - \alpha_{1y} + S\alpha_0 - 2R\alpha_1 + Q\alpha_2,$$

$$\gamma_{10} = \beta_{1x} - Q\beta_1 + P\beta_2, \qquad \delta_{10} = \gamma_{10x} - 2Q\gamma_{10} + P(\gamma_{20} + \gamma_{11}) - 5\alpha_0\beta_1,$$

$$\gamma_{11} = \beta_{2x} - R\beta_1 + Q\beta_2, \qquad \delta_{11} = \gamma_{11x} - R\gamma_{10} + P\gamma_{21} - \alpha_1\beta_1 - 4\alpha_0\beta_2,$$

$$\gamma_{20} = \beta_{1y} - R\beta_1 + Q\beta_2, \qquad \delta_{20} = \gamma_{20x} - R\gamma_{10} + P\gamma_{21} - 4\alpha_1\beta_1 - \alpha_0\beta_2,$$

$$\gamma_{21} = \beta_{2y} - S\beta_1 + R\beta_2, \qquad \delta_{21} = \gamma_{21x} - R(\gamma_{20} + \gamma_{11}) + 2Q\gamma_{21} - 5\alpha_1\beta_2,$$

$$\delta_{30} = \gamma_{20y} - S\gamma_{10} + Q\gamma_{21} - 4\alpha_2\beta_1 - \alpha_1\beta_2,$$

$$\epsilon_{10} = \delta_{10x} - 3Q\delta_{10} + P(2\delta_{20} + \delta_{11}) - 12\alpha_0\gamma_{10},$$

$$\lambda_{10} = \epsilon_{10x} - 4Q\epsilon_{10} + P(3\epsilon_{20} + \epsilon_{11}) - 21\alpha_0\delta_{10},$$

$$\alpha_0 = Q_x - P_y + 2PR - 2Q^2, \qquad \alpha_1 = R_x - Q_y + PS - QR,$$

$$\alpha_2 = S_x - R_y + 2QS - 2R^2.$$

In contrast to absolute invariants I_0, I_1, I_2, relative invariants are invariants not for the whole group of equivalence transformations of equation (1), but for some subgroup. A detailed description how to construct the relative and absolute invariants for equations (1) by means of Lie's infinitesimal approach can be found in [10]. We note that relations (12) and their differential consequences are satisfied for each equation of sixth type. Such trivial relations should be excluded from the necessary conditions of equivalence, since they do not reflect essential properties of a studied equation distinguishing it from other equations of the same type. At that, they play an essential role in the proof of sufficiency of equivalence conditions allowing us to express a part of invariants in terms of some "basic" invariants. This possibility is demonstrated in Section 5 in the proof of Theorem 7.

Remark. *Equation (1) with $\beta_1 = 0$, $\beta_2 \neq 0$ is reduced to equation with $\beta_1 \neq 0$ by the hodograph transformation.*

It was shown in [21] that each ODE (1) with relative invariants $J_0 = 0$, $j_0 = 0$ is reduced to (4) by some change of variables (2). Condition $j_1 = 0$ for such equation reads as

$$\left(\frac{\partial^3 f}{\partial y^3}\right)^2 - \frac{5}{6}\frac{\partial^2 f}{\partial y^2}\frac{\partial^4 f}{\partial y^4} = 0.$$

It implies that the right hand side of (4) is equal either to

$$f(x, y) = b_0(x) + b_1(x)y + \frac{b_4(x)}{(y + b_3(x))^3}, \tag{14}$$

or to

$$f(x, y) = b_0(x) + b_1(x)y + b_2(x)y^2. \tag{15}$$

Imposing other conditions in (9) for function $f(x, y)$ leads us to the conditions $b_4 = \text{const} \neq 0$, $b_0 - b_1 b_3 + b_3'' \neq 0$ for function (14) and to condition $b_2 \neq 0$ for function (15). Under such restrictions, equation (4), (14) is reduced to (6) by an appropriate transformation (5), while equation (4), (15) is reduced to (7). The families of equations (6), (7) are non-equivalent since for the former we have $I_1 \neq 0$, while the latter satisfies $I_1 = 0$. Necessary conditions of equivalence to equations of sixth type with $I_1 \neq 0$ and $I_1 = 0$ are obtained in the subsequent sections by means of invariants I_0, I_1, I_2, $\mathcal{D}_2^n I_2$, $n \in \mathbb{N}$. It is easy to see that other derivatives of invariants (13) can be expressed by trivial relations (12) and their differential consequences in terms of invariants I_1, I_2, $\mathcal{D}_2^n I_2$, $n \in \mathbb{N}$.

3. EQUATIONS OF SIXTH TYPE WITH $I_1 \neq 0$

Equation (6) has the following invariants (here $w' = dw/dz$)

$$I_1 = 5(1 + w^3 F),$$

$$I_2 = \frac{w^{10}(4F'^2 - 3FF'')}{27(1 + w^3 F)^2} - \frac{w^3 F}{36(1 + w^3 F)^2}(184 w^6 F^2 + 389 w^3 F + 196),$$

$$I_{22} = -\frac{w^{12}(9F^2 F''' - 45 FF'F'' + 40 F'^3)}{27\sqrt{3}F(1 + w^3 F)^{5/2}}, \qquad I_0 = -\frac{w(wF' + 3w'F)}{\sqrt{3}F(1 + w^3 F)^{3/2}}.$$

Comparing invariants I_1, I_2, I_{22}, we see that the algebraic invariants

$$\hat{J}_1 = \frac{3^9 \cdot 5^4}{(I_1 - 5)^{10}}\left(I_1^2 I_2 + (I_1 - 5)\left(\frac{46}{45}I_1^2 + \frac{7}{12}I_1 - \frac{5}{4}\right)\right)^3, \quad \hat{J}_2 = \frac{3^7 \cdot 5^3 I_1^5 I_{22}^2}{(I_1 - 5)^8} \tag{16}$$

for equation (1) of sixth type with $I_1 \neq 0$ depend on one variable only. In particular, for ODE (6) they are equal

$$\hat{J}_1 = -\frac{(3FF'' - 4F'^2)^3}{F^{10}}, \qquad \hat{J}_2 = \frac{(9F^2 F''' - 45 FF'F'' + 40 F'^3)^2}{F^{10}}.$$

The condition $\hat{J}_2 = 0$ coincides with the existence of a point symmetry. Namely, ODE (6) admits the symmetry operator X in the cases

1) $F(z) = \text{const}$, $X = \partial_z$;

2) $F(z) = \frac{c}{z^{3/2}}$, $c = \text{const} \neq 0$, $X = 2z\partial_z + w\partial_w$.

In the case $I_1 \neq 0$, in obtaining necessary conditions of equivalence, it is reasonable to employ basis invariants I_0, I_1 and invariants (16) instead of invariants I_0, I_1, I_2, I_{22}. If $\hat{J}_1$, $\hat{J}_2 \neq \text{const}$ for a given equation, then excluding the common variable in the expressions for $\hat{J}_1$, $\hat{J}_2$ gives a relation for $\hat{J}_1$, $\hat{J}_2$ being an invariant characteristic for this equation. Two equations with different relations for $\hat{J}_1$, $\hat{J}_2$ can not be equivalent.

In [25, Sect. 2.4.2] some equations (4) of sixth type with $I_1 \neq 0$ were integrated. All of them are reduced to two ODEs. The necessary conditions of equivalence to these equations are formulated in the following statements.

Theorem 2. *If equation (1) is equivalent to ODE*

$$\frac{d^2 w}{dz^2} = \frac{A_1}{w^3} + A_2, \qquad A_1, A_2 = \text{const} \neq 0, \tag{17}$$

it is an equation of sixth type, its invariants satisfy the conditions

$$I_1 \neq 0, \qquad \hat{J}_1 = 0, \qquad \hat{J}_2 = 0. \tag{18}$$

If equations (1), (17) are equivalent, change of variables (2) relating them is determined by the identities

$$I_1 = 5 + \frac{5A_2 w^3}{4A_1}, \qquad I_0 = -\frac{4\sqrt{3}A_1 w w'}{(-(4A_1 + A_2 w^3))^{3/2}}. \tag{19}$$

Theorem 3. *If equation (1) is equivalent to ODE*

$$\frac{d^2 w}{dz^2} = \frac{A_1}{w^3} + \frac{A_2}{z^{3/2}}, \qquad A_1, A_2 = \text{const} \neq 0, \tag{20}$$

it is an equation of sixth type, its invariants satisfy the conditions

$$I_1 \neq 0, \qquad \hat{J}_1 = \text{const} \neq 0, \qquad \hat{J}_2 = 0. \tag{21}$$

If equations (1), (20) are equivalent, change of variables (2) relating them and the relation between the parameters of the equations are determined by the identities

$$\hat{J}_1 = \frac{729A_1}{16A_2^4}, \qquad I_1 = 5 + \frac{5A_2 w^3}{4A_1 z^{3/2}}, \qquad I_0 = \frac{2\sqrt{3}A_1 z^{5/4} w (w - 2zw')}{(-(4A_1 z^{3/2} + A_2 w^3))^{3/2}}. \tag{22}$$

If equations (1) and (17) (or (20)) are equivalent, their invariants coincide, and thus, they satisfy the same relations. This is why the proof of the above theorem consists in straightforward calculating of invariants I_0, I_1 and (16) for equations (17) and (20).

4. EQUATIONS OF SIXTH TYPE WITH $I_1 = 0$

Equation (7) have the following invariants

$$I_1 = 0, \quad I_2 = \frac{11}{2} + \frac{f}{108w^2}, \quad \mathcal{D}_2^n I_2 = \frac{f^{(n)}}{(-2)^n 108 w^{2+n/2}}, \quad n \in \mathbb{N}, \quad I_0 = -\frac{w'}{6w^{3/2}}.$$

We see that the algebraic invariants

$$\tilde{J}_1 = \frac{I_{22}^4}{27(2I_2 - 11)^5}, \qquad \tilde{J}_2 = \frac{I_{222}^2}{27(2I_2 - 11)^3} \tag{23}$$

depend on a single variable for equation (1) of sixth type $I_1 = 0$, $I_2 \neq 11/2$. Thus, while obtaining necessary equivalence conditions for such equations of sixth type, it is useful to employ invariants (23) together with basis invariants (10). In particular, for ODE (7) with $f(z) \neq 0$ they read as

$$\tilde{J}_1 = \frac{(f')^4}{128 f^5}, \qquad \tilde{J}_2 = \frac{(f'')^2}{32 f^3}.$$

Equation (7) admits symmetry operators if $2(2I_2 - 11)I_{222} - 5I_{22}^2 = 0$, i.e., in the cases
  1) $f(z) = 0$, $X_1 = \partial_z$, $X_2 = z\partial_z - 2w\partial_w$;
  2) $f(z) = \text{const} \neq 0$, $X = \partial_z$;
  3) $f(z) = \frac{c}{z^4}$, $c = \text{const} \neq 0$, $X = z\partial_z - 2w\partial_w$.
  Equations (4) of sixth type with $I_1 = 0$ considered in [25, Sect. 2.3.1, 2.4.2, 2.9.1] are reduced to equation (3), two ODES for which in [25, Sect. 2.3.1] the general solution was provided, and to an equation admitting the order reducing. The criterion of equivalence to the first Painlevé equation is proven in the next section, while necessary conditions of equivalence to other three equations in [25] are formulated in the following theorems.

Theorem 4. *If equation (1) is equivalent to ODE*

$$\frac{d^2w}{dz^2} = Aw^2, \qquad A = \text{const} \neq 0, \tag{24}$$

it is an equation of sixth type and its invariants satisfy the conditions

$$I_1 = 0, \qquad I_2 = \frac{11}{2}. \tag{25}$$

If equations (1), (24) are equivalent, change of variables (2) relating them is determined by the identity

$$I_0 = -\frac{w'}{\sqrt{6A}w^{3/2}}. \tag{26}$$

Theorem 5. *If equation (1) is equivalent to ODE*

$$\frac{d^2w}{dz^2} = \frac{Aw^2}{z^{5/2}}, \qquad A = \text{const} \neq 0, \tag{27}$$

it is an equation of sixth type and its invariants satisfy the conditions

$$I_1 = 0, \qquad I_2 \neq \frac{11}{2}, \qquad \tilde{J}_1 = 0, \qquad \tilde{J}_2 = 0. \tag{28}$$

If equations (1), (27) are equivalent, change of variable (2) relating them is determined by the identities

$$I_2 = \frac{11}{2} - \frac{z}{18(\sqrt{z} + 8Aw)^2}, \qquad I_0 = \frac{8Az^{1/4}(w - 2zw')}{\sqrt{3}(\sqrt{z} + 8Aw)^{3/2}}. \tag{29}$$

Theorem 6. *If equation (1) is equivalent to ODE*

$$\frac{d^2w}{dz^2} = \frac{3}{25}\left(\frac{w^2}{z^3} + \frac{2cw}{z^2}\right), \qquad c = \text{const} \neq \pm 1, \tag{30}$$

it is an equation of sixth type and its invariants satisfy the conditions

$$I_1 = 0, \quad I_2 \neq \frac{11}{2}, \quad \tilde{J}_1 = \text{const} \neq 0, \quad \tilde{J}_2 = \text{const} \neq 0, \quad \frac{\tilde{J}_2}{\tilde{J}_1} = \frac{25}{4}. \tag{31}$$

If equations (1), (30) are equivalent, change of variable (2) relating them is determined by the identities

$$\tilde{J}_1 = \frac{4}{3(1 - c^2)}, \qquad I_2 = \frac{11}{2} + \frac{(1 - c^2)z^2}{18((c+1)z + w)^2},$$
$$I_0 = \frac{\sqrt{z}(3w - 2(c+1)z - 5zw')}{3\sqrt{2}((c+1)z + w)^{3/2}}. \tag{32}$$

5. CRITERION OF EQUIVALENCE TO FIRST PAINLEVÉ EQUATION

An invariant characteristic (necessary equivalence conditions) in terms of invariants (10), (13) for equation (3) was obtained in [10]. Here we prove the sufficiency of these conditions.

Theorem 7. *Equation (1) is equivalent to the first Painlevé equation if and only if it is an equation of sixth type and its invariants satisfy the conditions*

$$I_1 = 1, \qquad I_{222} = 0, \qquad \frac{\partial(I_2, I_{22})}{\partial(x, y)} \neq 0. \tag{33}$$

Change of variables (2) reducing it to equation (3) is determined by the identities

$$w^5 = \frac{1}{6^6 I_{22}^2}, \qquad \frac{z}{w^2} = 54(2I_2 - 11). \tag{34}$$

Proof. To the prove the necessity, let us find the relations to be satisfied by the invariants of equation (3) and therefore, the same relations are to be satisfied by the invariants of equation (1) equivalent to ODE (3). Basis algebraic invariants (10), operators (11) and the invariants $\mathcal{D}_2^n I_2$, $n = 1, 2$, for ODE (3) are equal to

$$I_1 = 0, \qquad I_2 = \frac{11}{2} + \frac{z}{108w^2}, \qquad \mathcal{D}_1 = -3w\partial_w, \qquad \mathcal{D}_2 = -\frac{1}{2\sqrt{w}}\partial_z, \tag{35}$$

$$I_{22} = -\frac{1}{216w^{5/2}}, \qquad I_{222} = 0.$$

Invariants (35) satisfy first two identities in (33) and the condition $\partial(I_2, I_{22})/\partial(z, w) \neq 0$, which is invariant w.r.t. a non-degenerate change of variables. The expressions for I_2, I_{22} can be resolved w.r.t. w^5, zw^{-2} that gives (34).

To probe the sufficiency, let us show that once conditions (33) for equation (1) are satisfied, the change of variables determined by (34) transforms it into equation (3). Differentiating (34) twice, for the derivatives of w w.r.t. z we obtain the expressions

$$w\frac{dw}{dz} = \frac{\Omega}{54(2(2I_2 - 11)\Omega - 5I_{22})},$$

$$w^3\frac{d^2w}{dz^2} = \frac{4(11 - 2I_2)\Omega^3 + 5I_{22}\Omega^2 + 25I_{22}^2 d\Omega/dI_2}{18^3(2(2I_2 - 11)\Omega - 5I_{22})^3}, \tag{36}$$

where

$$\Omega = \frac{dI_{22}}{dI_2} = \frac{\partial_x I_{22} + y'\partial_y I_{22}}{\partial_x I_2 + y'\partial_y I_2}, \qquad \frac{d\Omega}{dI_2} = \frac{\partial_x\Omega + y'\partial_y\Omega + y''\partial_{y'}\Omega}{\partial_x I_2 + y'\partial_y I_2}. \tag{37}$$

Equation (3) can be represented as

$$w^3\frac{d^2w}{dz^2} = w^5\left(6 + \frac{z}{w^2}\right).$$

By substituting (34), (36) we transform it into

$$50I_{22}^4\frac{d\Omega}{dI_2} + 8(11 - 2I_2)I_{22}^2\Omega^3 + 10I_{22}^3\Omega^2 + 3(49 - 9I_2)(2(2I_2 - 11)\Omega - 5I_{22})^3 = 0. \tag{38}$$

By the identities $I_{1j} = \mathcal{D}_1 I_j$, $I_{2j} = \mathcal{D}_2 I_j$, where $I_j = I_2$ and $I_j = I_{22}$, while $\mathcal{D}_1$, $\mathcal{D}_2$ are determined in (11), we find the derivatives

$$\partial_x I_j = \frac{1}{3}(M_2 I_{1j} - M_1 I_{2j}), \qquad\qquad \partial_y I_j = \frac{1}{3}(M_4 I_{1j} - M_3 I_{2j}), \tag{39}$$

$$M_1 = \beta_1\sqrt{j_3}, \quad M_2 = \frac{15\beta_1 e_0}{j_3(I_1 - 5)}, \qquad M_3 = \frac{\beta_2}{\beta_1}M_1, \quad M_4 = \frac{\beta_2}{\beta_1}M_2 - \frac{3\beta_1}{M_1^2}. \tag{40}$$

Second expression (37) depends also on the derivatives

$$\partial_x^2 I_j = \frac{1}{9}\left(M_2^2 I_{11j} - M_1 M_2(I_{12j} + I_{21j}) + M_1^2 I_{22j}\right) + \frac{1}{3}\left(M_{2x}I_{1j} - M_{1x}I_{2j}\right),$$

$$\partial_x\partial_y I_j = \frac{1}{9}\left(M_2(M_4 I_{11j} - M_3 I_{21j}) + M_1(M_3 I_{22j} - M_4 I_{12j})\right)$$

$$\qquad\qquad + \frac{1}{3}\left(M_{2y}I_{1j} - M_{1y}I_{2j}\right), \tag{41}$$

$$\partial_y^2 I_j = \frac{1}{9}\left(M_4^2 I_{11j} - M_3 M_4(I_{12j} + I_{21j}) + M_3^2 I_{22j}\right) + \frac{1}{3}\left(M_{4y}I_{1j} - M_{3y}I_{2j}\right).$$

The derivatives of the first two functions in (40) are equal to

$$M_{1x} = \left(\frac{I_1 - 5}{10} M_2 + \frac{2\gamma_{10}}{5\beta_1} - P\frac{\beta_2}{\beta_1} + Q\right) M_1,$$

$$M_{1y} = \left(\frac{I_1 - 5}{10} M_4 + \frac{2\gamma_{20} - 3\gamma_{11}}{5\beta_1} + \frac{3\beta_2\gamma_{10}}{5\beta_1^2} - Q\frac{\beta_2}{\beta_1} + R\right) M_1,$$

$$M_{2x} = (15 - 2I_1)\frac{M_2^2}{15} + \frac{5M_1^2(49(5 - I_1) - 45I_2)}{3(I_1 - 5)^2} + \left(\frac{2\gamma_{10}}{5\beta_1} + Q\right) M_2 - PM_4,$$

$$M_{2y} = (15 - 2I_1)\frac{M_2 M_4}{15} + \frac{5M_1 M_3}{3(I_1 - 5)^2}(49(5 - I_1) - 45I_2)$$
$$+ \left(\frac{\gamma_{20} + \gamma_{11}}{5\beta_1} - \frac{\beta_2\gamma_{10}}{5\beta_1^2} + R\right) M_2 + \left(\frac{\gamma_{10}}{5\beta_1} - Q\right) M_4,$$

while the derivatives M_{3y}, M_{4y} are calculated by means of the expressions for M_{1y}, M_{2y} and $\beta_{1y} = \gamma_{20} + R\beta_1 - Q\beta_2$, $\beta_{2y} = \gamma_{21} + S\beta_1 - R\beta_2$.

Substituting quantities (37), where the derivatives of invariants I_2, I_{22} are calculated by means of (39), (41), and y'' is replaced by means of (1), we cast (38) into the form

$$\Lambda_0(M_2 + y'M_4)^3 + 3\Lambda_1(M_1 + y'M_3)(M_2 + y'M_4)^2$$
$$+ 3\Lambda_2(M_1 + y'M_3)^2(M_2 + y'M_4) + \Lambda_3(M_1 + y'M_3)^3 = 0 \tag{42}$$

with coefficients Λ_i being functions of the invariants for equation (1). Identity (42) is the condition ensuring that equations (1) and (3) are related by the change of variables determined by (34). It remains to show that Λ_i vanishes if the invariants of equation (1) satisfy conditions (33).

By trivial identities (12) and their consequences, we can express invariants I_{12} and

$$I_{112} = \frac{4}{25}(7I_1^2 + 45I_1 + 225)I_2 + \frac{I_1 - 5}{225}(92I_1^2 + 2217I_1 + 8910),$$

$$I_{122} = \frac{3}{10}(3I_1 + 25)I_{22}, \quad I_{1122} = \frac{9}{20}(3I_1^2 + 24I_1 + 125)I_{22},$$

$$I_{212} = \frac{2}{5}(2I_1 + 15)I_{22}, \quad I_{2122} = \frac{3}{10}(3I_1 + 25)I_{222}, \quad I_{1222} = (I_1 + 9)I_{222}$$

in terms of invariants I_1, I_2, I_{22}, I_{222}. Substituting these expressions into Λ_i, we obtain

$$\Lambda_0 = 2I_1 I_{22}^3[4I_1^2(9I_2 - 49)K_0^3 - 81I_1(33I_1 + 875)I_{22}^2 K_0$$
$$+ 243(3I_1 + 25)(23I_1^2 + 8460I_1 + 56125)I_{22}^2],$$

$$\Lambda_1 = 30I_1 I_{22}^2[6I_1(9I_2 - 49)K_0^2 K_1 - 243(9I_1 + 175)I_{22}^2 K_1$$
$$+ (13K_2 - 10(17I_1 + 105)K_0 - 375(343I_1 + 2245))I_{22}^2 I_{222}],$$

$$\Lambda_2 = 225I_{22}[18I_1(9I_2 - 49)K_0 K_1^2 - 162(3I_1 + 25)I_{22}^2 I_{222} K_1$$
$$- 45(431I_1 + 3525)I_{22}^4 I_{222} + 2I_{22}^2(I_{222}^2 + 45I_{22} I_{2222})K_2],$$

$$\Lambda_3 = 15^3[27(9I_2 - 49)K_1^3 + 36I_{22}^2 I_{222}^2 K_1 + 90I_{22}^4(46I_{222}^2 - 45I_{22} I_{2222})$$
$$+ 5I_{22}^4(I_1 - 5)^{-2}(49(I_1 - 5) + 45I_2)(135(3I_1 + 25)I_{22}^2 - 2I_{222} K_2)],$$

where $K_0 = 9I_2 + 23I_1 + 1073$, $K_1 = 2(2I_2 - 11)I_{222} - 5I_{22}^2$, $K_2 = 45(2I_1 + 15)(2I_2 - 11) + I_1(46I_1 + 2245)$. It can be shown that these quantities vanish if and only if $I_1 = 0$, $I_{222} = 0$, $I_{2222} = 0$. The proof is complete. $\qquad\square$

6. Examples of equivalent equations

Apart from equations (17), (20), (24), (27), in [25] there were integrated other ODEs (4) of sixth type equivalent to these four equations. In two subsequent examples we show how to establish the equivalence of equation by means of invariants and to find the relating change of variables (2). In the third example we establish that the generalized Sundman transform obtained in [20], up to a point change of variables can be regarded as autotransformation of the first Painlevé equation.

Example 1. In [25], there was provided the general solution to ODE

$$\frac{d^2y}{dx^2} = \frac{A_1}{y^3} + \frac{A_2}{x^3}, \qquad A_1, A_2 = \text{const} \neq 0, \tag{43}$$

having invariants

$$I_1 = 5 + \frac{5A_2y^3}{4A_1x^3}, \quad I_0 = \frac{4\sqrt{3}A_1x^{7/2}y(y - xy')}{(-(4A_1x^3 + A_2y^3))^{3/2}}, \quad \hat{J}_1 = 0, \quad \hat{J}_2 = 0, \tag{44}$$

satisfying conditions (18) of Theorem 2. Equating invariants I_1 defined by (19), (44), we obtain the transformation relating equations (43) and (17):

$$z = \xi(x, y), \qquad w = \frac{y}{x}, \qquad \frac{dw}{dz} = \frac{xy' - y}{x^2(\xi_x + y'\xi_y)}. \tag{45}$$

Equating invariants I_0 and substituting (45), we obtain the equations $\xi_x = x^{-2}$, $\xi_y = 0$ to determine function ξ. The solution is $\xi = -1/x$. It is easy to check that the change of variables $z = -1/x$, $w = y/x$ transforms (43) into equation (17).

Example 2. In [25] there was integrated the equation

$$\frac{d^2y}{dx^2} = \frac{Ay^2}{x^2} - \frac{6y}{25x^2}, \qquad A = \text{const} \neq 0, \tag{46}$$

whose invariants are

$$I_1 = 0, \qquad I_2 = \frac{11}{2}, \qquad I_0 = \frac{2y - 5xy'}{5\sqrt{6A}y^{3/2}}. \tag{47}$$

Comparing (47) with (25), (28), (31), we conclude that ODE (46) satisfies the hypothesis of Theorem 4 and is equivalent to equation (24). Change (2) relating (46) and (24) is found by the condition of equality of invariants I_0 for these equations. Substituting

$$z = \xi(x, y), \qquad w = \eta(x, y), \qquad \frac{dw}{dz} = \frac{\eta_x + y'\eta_y}{\xi_x + y'\xi_y}$$

into this identity and equating the coefficients at the like powers of y', for ξ, η we obtain the system of equations

$$5\sqrt{y}\eta_x + 2\eta^{3/2}\xi_x = 0, \qquad 5y^{3/2}\eta_y + \eta^{3/2}(2y\xi_y - 5x\xi_x) = 0, \qquad \xi_y = 0.$$

One of its solutions is $\xi = 5x^{1/5}$, $\eta = yx^{-2/5}$ and thus, ODE (46) is transformed to (24) by the change of variables $z = 5x^{1/5}$, $w = yx^{-2/5}$.

Example 3. It was shown in [20] that the generalized Emden-Fowler equation

$$\frac{d^2y}{dx^2} = \frac{c_0y^2}{(x - k)^5} + \frac{c_1}{(x - k)^4} + \frac{c_2}{(x - k)^3}, \quad c_0, c_1, c_2, k = \text{const}, \quad c_0, c_1 \neq 0 \tag{48}$$

is related with the first Painlevé equation

$$\frac{d^2W}{dZ^2} = 6W^2 - \frac{625}{6}Z$$

by the generalized Sundman transform

$$Z = -\frac{(c_0 c_1)^{1/5}}{5^{4/5}}\left(\frac{1}{x-k} + \frac{c_2}{c_1}\right), \quad U(Z) = \frac{(5c_0 c_1)^{4/5}}{6c_1}\int \frac{y(x)}{(x-k)^3}dx, \tag{49}$$

where $dU(Z)/dZ = W(Z)$. By the dilatation $Z = -5^{-4/5}6^{1/5}z$, $W = 5^{8/5}6^{-2/5}w$, the Painlevé equation is reduced to the standard form (3). ODE (48) is an equation of sixth type. Its algebraic invariants

$$I_1 = 0, \quad I_2 = \frac{11}{2} + \frac{(x-k)(c_1 + c_2(x-k))}{18c_0 y^2}, \quad I_{22} = \frac{c_1(x-k)^{5/2}}{(6c_0)^{3/2}y^{5/2}}, \quad I_{222} = 0$$

satisfy conditions (33) in Theorem 7. Therefore, equation (48) is equivalent to the first Painlevé equation. The corresponding transformation

$$z = \frac{(c_0 c_1)^{1/5}}{6^{1/5}}\left(\frac{1}{x-k} + \frac{c_2}{c_1}\right), \qquad w = \frac{c_0^{3/5}y}{6^{3/5}c_1^{2/5}(x-k)} \tag{50}$$

is determined by relations (34), which in the present case read as

$$w^5 = \frac{c_0^3 y^5}{216c_1^2(x-k)^5}, \qquad \frac{z}{w^2} = \frac{6(x-k)(c_1 + c_2(x-k))}{c_0 y^2}.$$

Thus, equation (48) is related with the first Painlevé equation not only by nonlocal transformation (49), but also by a simpler change of variables (50).

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