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IDENTIFICATION OF A POLYNOMIAL IN NONSEPARATED BOUNDARY CONDITIONS IN THE CASE OF A MULTIPLE ZERO EIGENVALUE

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Abstract. In the work we discuss the problem of recovering the coefficients of a polynomial in spectral problems with nonseparated boundary conditions by one multiple zero eigenvalue and n nonzero eigenvalues. A uniqueness theorem is proved.

Keywords: eigenvalues, boundary conditions, characteristic determinant.

Mathematics Subject Classification: 35L75, 65A18, 34K29

1. Introduction

In solving applied problems of mathematical physics, there appear spectral problem involving polynomially a spectral parameter in boundary conditions [1]–[4], as well as problems with an operator in boundary conditions [5]. In the associated inverse problems, by known spectra one has to recover the unknown coefficients in equations and boundary conditions [6]–[13]. In [14], a polynomial is separated boundary conditions was recovered by a finite set of different eigenvalues. In [15], a polynomial of degree m in nonseparated boundary conditions was recovered by $m+1$ different eigenvalues. However, the information on multiplicities of the eigenvalues was not employed in [15]. In the present paper we make use a multiplicity of zero eigenvalue. To recover the polynomial in this case we employ less number of eigenvalues ($< m$).

2. Formulation of the problem

We consider the following spectral problem:

$$y'' + p_1(x,\lambda)y' + p_2(x,\lambda)y = 0, \tag{1}$$

$$U_i(y) = a_{i1}(\lambda)y'(0) + a_{i2}(\lambda)y(0) + a_{i3}(\lambda)y'(1) + a_{i4}(\lambda)y(1) = 0, \tag{2}$$

where λ is a spectral parameter; $i = 1, 2$; $x \in [0,1]$; $p_1(x,\lambda)$, $p_2(x,\lambda)$ are continuously differentiable functions w.r.t. x and λ; a_{ij} $(i = 1, 2,\ j = 1, 2, 3, 4)$ are continuously differentiable functions w.r.t. λ and

$$\sum_{j=1}^{4} |a_{ij}(\lambda)| \neq 0, \qquad \text{as } i = 1, 2 \text{ and for each } \lambda. \tag{3}$$

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In the present paper we solve an inverse problem. Suppose that one of the functions $a_{2j}(\lambda)$ ($j = 1, 2, 3, 4$), which we denote by $a_{2p}(\lambda)$, is a polynomial:

$$a_{2p}(\lambda) = \sum_{s=0}^{m} a_{2ps}\lambda^s.$$

We know $n + 1$ eigenvalues λ_0, λ_1, ..., λ_n of problem (1), (2). One of them $\lambda_0 = 0$ has the multiplicity r_0, at that, $m = n + r_0 - 1$ is the degree of polynomial $a_{2p}(\lambda)$. We need to recover polynomial $a_{2p}(\lambda)$.

3. Uniqueness theorem

We denote by $y_1(x, \lambda)$ and $y_2(x, \lambda)$ linearly independent solutions to differential equation (1) satisfying the conditions

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1. \tag{4}$$

Eigenvalues λ_k are roots of the characteristic determinant [16]

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix} = \sum_{j=1}^{4} a_{2j}(\lambda)\, A_{2j}(\lambda), \tag{5}$$

where

$$\begin{aligned}
&A_{21}(\lambda) = a_{12}(\lambda) + a_{13}(\lambda)\, y_1'(1, \lambda) + a_{14}(\lambda) y_1(1, \lambda),\\
&A_{22}(\lambda) = -a_{11}(\lambda) - a_{13}(\lambda) y_2'(1, \lambda) - a_{14}(\lambda) y_2(1, \lambda),\\
&A_{23}(\lambda) = a_{12}(\lambda) y_2'(1, \lambda) + a_{14}(\lambda)\, W(1, \lambda) - a_{11}(\lambda) y_1'(1, \lambda),\\
&A_{24}(\lambda) = a_{12}(\lambda) y_2(1, \lambda) - a_{11}(\lambda) y_1(1, \lambda) - a_{13}(\lambda)\, W(1, \lambda),\\
&W(1, \lambda) = y_1(1, \lambda)\, y_2'(1, \lambda) - y_1'(1, \lambda)\, y_2(1, \lambda), \text{ as } k = 0, 1, \ldots, n.
\end{aligned}$$

If $p_1(x, \lambda) \equiv 0$, then by (4) and by the Liouville formula for the Wronskian [17, Ch. V, Subsect. 17.1] we obtain that $W(1, \lambda_k) = 1$.

Function $A_{2p}(\lambda)$ introduced in (5) with p chosen above is expressed via known coefficients a_{2j} and known functions $y_1(x, \lambda)$ and $y_2(x, \lambda)$.

Theorem. *Polynomial $a_{2p}(\lambda)$ of degree m in boundary condition (2) is uniquely recovered by zero eigenvalue $\lambda_0 = 0$ of multiplicity r_0 and by $n = m - r_0 + 1$ non-zero mutually different eigenvalues λ_1, λ_2, ..., λ_n if $A_{2p}(\lambda_k) \neq 0$, $k = 0, 1, \ldots, n$*

Proof. Let us show that as $A_{2p}(\lambda_k) \neq 0$, polynomial $a_{2p}(\lambda)$ is uniquely recovered, while for $A_{2p}(\lambda_k) = 0$ the unique recovering is impossible.

Let $A_{2p}(\lambda_k) \neq 0$. It follows from identities $\Delta(\lambda_k) = 0$ and (4) that

$$a_{2p}(\lambda_k) = -\sum_{j=1,\, j\neq p}^{4} a_{2j}(\lambda_k)\, \frac{A_{2j}(\lambda_k)}{A_{2p}(\lambda_k)}, \qquad k = 0, 1, \ldots, n. \tag{6}$$

Substituting the known eigenvalues into (4), we obtain a system of algebraic equations for unknown coefficients a_{2ps}:

$$a_{2p0} + a_{2p1}\lambda_k^1 + \ldots + a_{2pm}\lambda_k^m = -\sum_{j=1,\, j\neq p}^{4} a_{2j}(\lambda_k)\, \frac{A_{2j}(\lambda_k)}{A_{2p}(\lambda_k)}, \tag{7}$$

where $k = 0, 1, \ldots, n$.

System of linear algebraic equations (7) has $m+1$ unknowns and $n+1=(m-r_0+2)$ equations. It is impossible to identify uniquely the coefficients of the polynomial since the amount of unknowns in this system is greater than the number of equations. However, by the assumption, $\lambda_0=0$ has multiplicity r_0.

The multiplicity of the root follows that

$$\begin{cases} \Delta(\lambda_0)=0, \\ \Delta'(\lambda_0)=0, \\ \quad\dots \\ \Delta^{(r_0-1)}(\lambda_0)=0, \\ \Delta^{(r_0)}(\lambda_0)\neq 0. \end{cases} \tag{8}$$

By (5) and (8) we obtain

$$\begin{cases} \Delta(\lambda_0)=\sum\limits_{j=1}^{4} a_{2j}(\lambda_0)\,A_{2j}(\lambda_0), \\ \Delta'(\lambda_0)=\sum\limits_{j=1}^{4}\left(a'_{2j}(\lambda_0)\,A_{2j}(\lambda_0)+a_{2j}(\lambda_0)A'_{2j}(\lambda_0)\right), \\ \quad\dots \\ \Delta^{(r_0-1)}(\lambda_0)=\sum\limits_{j=1}^{4}\left(C^0_{r_0-1}a^{(r_0-1)}_{2j}(\lambda_0)\,A_{2j}(\lambda_0)+\ldots+C^{r_0-1}_{r_0-1}a_{2j}(\lambda_0)A^{(r_0-1)}_{2j}\right). \end{cases} \tag{9}$$

Employing (9) and eigenvalue $\lambda_0=0$, we find the first r_0 coefficients of polynomial $a_{2ps}(\lambda)$ by means of the recurrent relations

$$\begin{aligned} a_{2pi}=&-\frac{\sum\limits_{j=1,\ j\neq p}^{4}\left(C^0_i a^{(i)}_{2j}(0)A_{2j}(0)+C^1_i a^{(i-1)}_{2j}(0)A'_{2j}(0)+\cdots+C^i_i a_{2j}(0)A^{(i)}_{2j}(0)\right)}{A_{2p}(0)} \\ &-\frac{\left(C^1_i a_{2,p,i-1}A'_{2p}(0)+C^2_i a_{2,p,i-2}A''_{2p}(0)+\cdots+C^i_i a_{2p0}A^{(i)}_{2j}(0)\right)}{A_{2p}(0)}, \end{aligned} \tag{10}$$

where $i=0,1,\ldots,r_0-1$.

Therefore, the desired polynomial reads as

$$a_{2p}(\lambda)=a_{2p0}+a_{2p1}\lambda+\ldots+a_{2\,p\,r_0-1}\lambda^{r_0-1}+\ldots+a_{2pm}\lambda^m,$$

where a_{2p0},..., $a_{2\,p\,r_0-1}$ are determined by means of recurrent relations (10), while other coefficients a_{2pr_0}, ,..., a_{2pm} are unknown. Let us find them by other $n=(m-r_0+1)$ known mutually different eigenvalues $\lambda_1,\ldots,\lambda_n$ of problem (1), (2).

We denote the known part of polynomial $a_{2p}(\lambda)$ as

$$V(\lambda):=a_{2p0}+a_{2p1}\lambda+\ldots+a_{2\,p\,r_0-1}\lambda^{(r_0-1)}.$$

Then system of equations (7) becomes

$$a_{2pr_0}\lambda_k^{r_0}+\ldots+a_{2pm}\lambda_k^m=-\sum_{j=1,\ j\neq p}^{4} a_{2j}(\lambda_k)\,\frac{A_{2j}(\lambda_k)}{A_{2p}(\lambda_k)}-V(\lambda_k), \tag{11}$$

where $A_{2p}(\lambda_k) \neq 0$, and $k = 1, 2, \ldots, m - r_0 + 1$. By the assumption, eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{m-r_0+1}$ are mutually different and are non-zero. Then we divide all the equations in system (11) by $\lambda_k^{r_0}$, $k = \overline{1, m - r_0 + 1}$, to obtain

$$a_{2pr_0} + \ldots + a_{2pm}\lambda_k^{m-r_0} = - \sum_{j=1,\, j\neq p}^{4} a_{2j}(\lambda_k)\, \frac{A_{2j}(\lambda_k)}{A_{2p}(\lambda_k)\lambda_k^{r_0}} - \frac{V(\lambda_k)}{\lambda_k^{r_0}}. \tag{12}$$

The determinant of system (12) w.r.t. unknowns a_{2ps}, $s = r_0, \ldots, m$, is the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & \lambda_1 & \ldots & \lambda_1^m \\ 1 & \lambda_2 & \ldots & \lambda_2^m \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^m \end{vmatrix} = (\lambda_n - \lambda_{n-1}) \ldots (\lambda_n - \lambda_1) \ldots (\lambda_2 - \lambda_1) \neq 0.$$

Hence, system of equations (12) has the unique solution determined, for example, by Cramer's formulae:

$$a_{2pr_0} = \frac{\Delta_1}{\Delta}, \ldots, a_{2pm} = \frac{\Delta_n}{\Delta}, \tag{13}$$

where the determinants Δ_i ($i = 1, \ \ldots, \ n$) are obtained from determinant Δ by replacing ith column by the column of the right hand sides in system of equations (12). The proof for the case $A_{2p}(\lambda_k) \neq 0$ is complete. The solution to the inverse problem is given by means of formulae (10) and (13).

If polynomial $A_{2p}(\lambda)$ vanishes at points $\lambda = \lambda_k$, it follows from representation (5) that the identity $\Delta(\lambda_k) = 0$ is possible for each $a_{2p}(\lambda_k)$. This is in the case $A_{2p}(\lambda_k) = 0$ polynomial $a_{2p}(\lambda_k)$ is recovered non-uniquely. The proof is complete. □

4. Examples

Example 1. We consider the following problem

$$\begin{aligned} &-y'' = \lambda^2 y, \\ &y'(0) + y(1) = 0, \\ &y'(1) - a_{24}(\lambda)\, y(1) = 0, \end{aligned}$$

where $a_{24}(\lambda) = a_{240} + a_{241}\lambda + a_{242}\lambda^2 + a_{243}\lambda^3 + a_{244}\lambda^4$. We need to recover the coefficients of polynomial $a_{24}(\lambda)$ by three eigenvalues. Eigenvalue $\lambda_0 = 0$ has multiplicity three and we know two other eigenvalues $\lambda_1 = \pi$, $\lambda_1 = 2\pi$. The characteristic determinant of the problem reads as:

$$\Delta(\lambda) = 1 + \lambda \sin(\lambda) - a_{24}(\lambda) \cos \lambda.$$

Since $p = 4$, we have $A_{24} = -\cos\lambda$ and $A_{24}(0) = 1 \neq 0$. Employing equation (10) and eigenvalue $\lambda_0 = 0$, we find first three coefficients of polynomial $a_{24}(\lambda)$:

$$a_{240} = 1, \qquad a_{241} = 0, \qquad a_{242} = \frac{3}{2}.$$

Then our polynomial becomes

$$a_{24}(\lambda) = 1 + \frac{3}{2}\lambda^2 + a_{243}\lambda^3 + a_{244}\lambda^4.$$

Coefficients a_{243} and a_{244} can be recovered by eigenvalues $\lambda_1 = \pi$ and $\lambda = 2\pi$ by means of formula (13):

$$a_{243} = \frac{\Delta_2}{\Delta} = -\frac{4}{\pi^3} - \frac{9}{4\,\pi}, \qquad a_{244} = \frac{\Delta_1}{\Delta} = \frac{2}{\pi^4} + \frac{3}{4\pi^2}.$$

It follows that

$$a_{24}(\lambda) = 1 + \frac{3}{2}\lambda^2 - \left(\frac{4}{\pi^3} + \frac{9}{4\pi}\right)\lambda^3 + \left(\frac{2}{\pi^4} + \frac{3}{4\pi^2}\right)\lambda^4.$$

Example 2. The characteristic determinant for the spectral problem

$$\begin{aligned} &-y'' = \lambda^2\, y,\\ &y'(0) - y'(1) = 0,\\ &y(1) - a_{22}\, y(0) = 0, \end{aligned}$$

reads as:

$$\Delta(\lambda) = (1 + a_{22})\ (\cos\lambda - 1)\,.$$

The unique recovering of coefficient a_{22} by eigenvalue $\lambda = 0$ of this problem is impossible, since, condition $A_{22}(0) \neq 0$ fails. Indeed, $A_{22}(0) = -a_{11} - a_{13}\, y_2'(0) - a_{14}\, y_2(0) = -1 + 1 \cdot \cos(0) - 0 \cdot \sin(0) = 0$.

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