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PROPERLY DISTRIBUTED SUBSEQUENCE ON THE LINE

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Abstract. In the article we consider first order sequences of complex numbers. We prove that a sequence of nonzero minimal density contains a subsequence of the same density. We also prove that a real sequence of nonzero minimal density contains a properly distributed subsequence. Basing on this fact, we prove a result on representation of an entire function of exponential type with real zeros as a product of two entire functions with the same properties. Moreover, one of these functions has a regular growth. As a corollary, we obtain a result on completeness of exponential systems with real exponents in the space of analytic functions in a bounded convex domain of the complex plane.

Keywords: entire function, regular growth, zero set

Mathematics Subject Classification: 30D10

1. Introduction

In the paper we mostly study real sequences of first order. We find the conditions under which such sequences contain a properly distributed set of a prescribed density. On the basis of these conditions we prove the result on representing an entire function of exponential type with real zeroes by a product of two functions of the same type, one of those has the a regular growth. As a corollary, we obtain the result on the completeness of exponential systems with real exponents in the space of functions analytic in a bounded convex planar domain.

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers taken in the ascending order of its absolute values. At that we assume that it can be multiple, i.e. some of λ_k can coincide. We denote by $n(r,\Lambda)$ the number of the terms in sequence Λ located in the circle $|\lambda| < r$, $r > 0$. The lower and upper densities of Λ are respectively the quantities

$$\underline{n}(\Lambda) = \varliminf_{r\to\infty} \frac{n(r,\Lambda)}{r}, \qquad \overline{n}(\Lambda) = \varlimsup_{r\to\infty} \frac{n(r,\Lambda)}{r}.$$

Sequence Λ is said to have density $n(\Lambda)$ if $\underline{n}(\Lambda) = \overline{n}(\Lambda) = n(\Lambda)$. It is easy to see that in this case the identity

$$n(\Lambda) = \lim_{k\to\infty} \frac{k}{|\lambda_k|}$$

holds true.

Maximal and minimal densities of sequence Λ are the quantities

$$\overline{n}_0(\Lambda) = \varlimsup_{\delta\to 0} \varlimsup_{r\to\infty} \frac{n(r,\Lambda) - n((1-\delta)r,\Lambda)}{\delta r}, \quad \underline{n}_0(\Lambda) = \varliminf_{\delta\to 0} \varliminf_{r\to\infty} \frac{n(r,\Lambda) - n((1-\delta)r,\Lambda)}{\delta r}.$$

Lemma 1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be such $\overline{n}(\Lambda) < \infty$. The inequalities*

$$\underline{n}_0(\Lambda) \leqslant \underline{n}(\Lambda) \leqslant \overline{n}(\Lambda) \leqslant \overline{n}_0(\Lambda) \tag{1}$$

hold true.

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Proof. We have

$$\begin{aligned}\overline{n}_0(\Lambda) =&\varlimsup_{\delta\to 0}\varlimsup_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r} \geqslant \varlimsup_{\delta\to 0}\left(\varlimsup_{r\to\infty}\frac{n(r,\Lambda)}{\delta r}-\varlimsup_{r\to\infty}\frac{n((1-\delta)r,\Lambda)}{\delta r}\right)\\ =&\varlimsup_{\delta\to 0}\left(\frac{\overline{n}(\Lambda)}{\delta}-(1-\delta)\varlimsup_{r\to\infty}\frac{n((1-\delta)r,\Lambda)}{(1-\delta)\delta r}\right)=\varlimsup_{\delta\to 0}\left(\frac{\overline{n}(\Lambda)}{\delta}-(1-\delta)\frac{\overline{n}(\Lambda)}{\delta}\right)=\overline{n}(\Lambda).\end{aligned}$$

Thus, $\overline{n}(\Lambda)\leqslant\overline{n}_0(\Lambda)$.

Inequalities $\underline{n}(\Lambda)\leqslant\overline{n}(\Lambda)$ follow directly from the definitions of these quantuties.

To prove (1), it remains to show that $\underline{n}_0(\Lambda)\leqslant\underline{n}(\Lambda)$. Let $\delta\in(0,1)$. We have

$$\begin{aligned}\varliminf_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r}\leqslant&\varliminf_{r\to\infty}\frac{n(r,\Lambda)}{\delta r}-(1-\delta)\varliminf_{r\to\infty}\frac{n((1-\delta)r,\Lambda)}{\delta r}\\ =&\frac{\underline{n}(\Lambda)}{\delta}-(1-\delta)\frac{\underline{n}(\Lambda)}{\delta}=\underline{n}(\Lambda).\end{aligned}$$

It implies the required inequality. The proof is complete. □

Let $\Lambda=\{\lambda_k\}_{k=1}^{\infty}$ and $\widetilde{\Lambda}=\{\tilde{\lambda}_n\}_{n=1}^{\infty}$. If Λ is a subsequence of $\widetilde{\Lambda}$, we shall write $\Lambda\subset\widetilde{\Lambda}$. The proof of the following statement is based on the approach employed in the proof of Lemma 5 in work [1].

Lemma 2. *Let $\tau\geqslant 0$ and $\widetilde{\Lambda}=\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ be such that $\underline{n}_0(\widetilde{\Lambda})\geqslant\tau$. Then there exists a sequence $\Lambda\subset\widetilde{\Lambda}$ having density τ.*

Proof. Since the arguments of the terms in $\widetilde{\Lambda}$ makes no influence of its various densities, we can assume that sequence $\widetilde{\Lambda}$ lies on a non-negative real semi-axis. We can also assume that $\tau>0$, since the case $\tau=0$ is trivial.

Let $\alpha=1/\tau$, and $\widetilde{\Lambda}_m$ be the set of all the terms in $\widetilde{\Lambda}$ belonging the semi-interval $[(m-1)\alpha,m\alpha)$, $m\geqslant 1$. We seek sequence Λ as the union $\Lambda=\bigcup_{m\geqslant 1}\Lambda_m$, where Λ_m is a subset of $\widetilde{\Lambda}_m$. We construct sets Λ_m, $m\geqslant 1$, by induction following the restriction: for each $m\geqslant 1$ the total amount of points in sets $\Lambda_1,\dots,\Lambda_m$ should be less or equal m.

Let $m=1$. If $\widetilde{\Lambda}_1$ is non-empty, we choose an arbitrary point $\tilde{\lambda}_{n(1)}$ in $\widetilde{\Lambda}_1$ and let $\lambda_1=\tilde{\lambda}_{n(1)}$, $\Lambda_1=\{\lambda_1\}$. Otherwise we let $\Lambda_1=\emptyset$. By construction, the number of points in set Λ_1 does not exceed one.

Assume that we have constructed sets Λ_m for each $m<p$ and the number of points in these sets satisfy the aforementioned restriction. Let us define Λ_p. If total amount of points in sets $\Lambda_1,\dots,\Lambda_{p-1},\widetilde{\Lambda}_p$ is less or equal p, as Λ_p we take the set $\widetilde{\Lambda}_p$. Otherwise, as Λ_p we choose an arbitrary subset $\widetilde{\Lambda}_p$ such that the total amount of potins in sets $\Lambda_1,\dots,\Lambda_p$ is p. Hence, the aforementioned restriction is satisfied.

Let us show that our construction is well-defined, i.e., at the last step in set $\widetilde{\Lambda}_p$ there are always a necessary amount of points. Suppose this is not true. In other words, for each subset Λ_p (including an empty set) of set $\widetilde{\Lambda}_p$, the total amount of points in $\Lambda_1,\dots,\Lambda_p$ is not p. Then for each $\Lambda_p\subset\widetilde{\Lambda}_p$ the total amount of points in $\Lambda_1,\dots,\Lambda_p$ is either strictly less than p or strictly greater than p. The first case is impossible since on the last step of construction it is assumed for $\Lambda_p=\widetilde{\Lambda}_p$ that the total amounts of points in $\Lambda_1,\dots,\Lambda_p$ is strictly greater than p. The second case is also impossible since by the assumption of the induction the total amount of points in sets $\Lambda_1,\dots,\Lambda_{p-1}$ is less or equal $p-1$, and therefore, for $\Lambda_p=\emptyset$ the total amount of points in $\Lambda_1,\dots,\Lambda_p$ is also less than p. Thus, our construction is well-defined.

Let us show that Λ is the sought set, i.e., $\Lambda\subset\widetilde{\Lambda}$ and $n(r,\Lambda)/r\to\tau$ as $r\to\infty$. The former is valid by the construction. Let us show the latter. Let $r>0$ and $q(r)$ stands for the maximal natural number satisfying the inequality $\alpha q(r)\leqslant r$. By construction, the quantity

$n(\alpha(q(r)+1),\Lambda)$ coincides with the total amount of point in sets $\Lambda_1,\dots,\Lambda_{q(r)+1}$; in accordance with the above restriction this amount does not exceed $q(r)+1$. Therefore,

$$\overline{n}(\Lambda)=\varlimsup_{r\to\infty}\frac{n(r,\Lambda)}{r}\leqslant\varlimsup_{r\to\infty}\frac{n(\alpha(q(r)+1),\Lambda)}{\alpha q(r)}\leqslant\varlimsup_{r\to\infty}\frac{q(r)+1}{\alpha q(r)}=\frac{1}{\alpha}=\tau. \tag{2}$$

Let us prove the inequality $\underline{n}(\Lambda)\geqslant\tau$. In view of (1), it is sufficient to show that $\underline{n}_0(\Lambda)\geqslant\tau$. We fix $\varepsilon>0$. In accordance with the hypothesis of the lemma and the definition of $\underline{n}_0(\widetilde\Lambda)$ we find $\delta_0>0$ such that

$$\varliminf_{r\to\infty}\frac{n(r,\widetilde\Lambda)-n((1-\delta')r,\widetilde\Lambda)}{\delta' r}\geqslant\underline{n}_0(\widetilde\Lambda)-\varepsilon\geqslant\tau-\varepsilon,\qquad\delta'\in(0,\delta_0). \tag{3}$$

Let $r>0$. By construction, $\Lambda_p\subset\widetilde\Lambda_p$. We denote by $p(r)$ the maximal index such that $\alpha p(r)\leqslant r$ and $\Lambda_{p(r)}$ is a proper subset of $\widetilde\Lambda_{p(r)}$. We can assume that for large r such index exists since otherwise sequences Λ and $\widetilde\Lambda$ coincide. Then the required inequality holds by the condition: $\underline{n}_0(\Lambda)=\underline{n}_0(\widetilde\Lambda)\geqslant\tau$.

We fix $\delta\in(0,\delta_0)$. We choose a sequence $r_j\to\infty$ such that

$$\varliminf_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r}=\lim_{j\to\infty}\frac{n(r_j,\Lambda)-n((1-\delta)r_j,\Lambda)}{\delta r_j}. \tag{4}$$

In accordance with the construction and the choice of numbers $p(r)$ and $q(r)$, the intersections of the semi-interval $[\alpha p(r),\alpha q(r))$ with sets Λ and $\widetilde\Lambda$ coincide. This is why the identity

$$n(\alpha q(r),\Lambda)-n(\tilde r,\Lambda)=n(\alpha q(r),\widetilde\Lambda)-n(\tilde r,\widetilde\Lambda),\qquad\alpha p(r)\leqslant\tilde r<\alpha q(r) \tag{5}$$

holds true. Let $\delta'\in(0,\delta)$. Then by the definition of $q(r)$ there exists $r(\delta')>0$ such that $(1-\delta')r'\geqslant(1-\delta)r$ as $r\geqslant r(\delta')$, where $r'=\alpha q(r)$. If $\alpha p(r_{j(k)})\leqslant(1-\delta)r_{j(k)}$ for some subsequence $\{r_{j(k)}\}$, by (3)–(5) we obtain

$$\begin{aligned}
\varliminf_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r}&=\lim_{k\to\infty}\frac{n(r_{j(k)},\Lambda)-n((1-\delta)r_{j(k)},\Lambda)}{\delta r_{j(k)}}\\
&\geqslant\varliminf_{k\to\infty}\frac{n(\alpha q(r_{j(k)}),\Lambda)-n((1-\delta)r_{j(k)},\Lambda)}{\delta r_{j(k)}}=\varliminf_{r\to\infty}\frac{\alpha q(r_{j(k)}),\widetilde\Lambda)-n((1-\delta)r_{j(k)},\widetilde\Lambda)}{\delta r_{j(k)}}\\
&\geqslant\varliminf_{r\to\infty}\frac{n(\alpha q(r),\widetilde\Lambda)-n((1-\delta)r,\widetilde\Lambda)}{\delta r}\geqslant\frac{\delta'}{\delta}\varliminf_{r\to\infty}\frac{r'\left(n(r',\widetilde\Lambda)-n((1-\delta')r',\widetilde\Lambda)\right)}{r\delta' r'}\\
&=\frac{\delta'}{\delta}\varliminf_{r'\to\infty}\frac{n(\alpha q(r),\widetilde\Lambda)-n((1-\delta')r',\widetilde\Lambda)}{\delta' r'}\geqslant\frac{\delta'}{\delta}(\tau-\varepsilon).
\end{aligned}$$

Since the latter inequality holds for each $\delta'\in(0,\delta)$, then

$$\varliminf_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r}\geqslant\tau-\varepsilon. \tag{6}$$

Thus, we can assume that $\alpha p(r_j)>(1-\delta)r_j$ for each $j\geqslant1$. Passing to a subsequence, we can also suppose that

$$\varliminf_{j\to\infty}\frac{n(r_j,\Lambda)-n(\alpha p(r_j),\Lambda)}{\delta r_j}=\lim_{j\to\infty}\frac{n(r_j,\Lambda)-n(\alpha p(r_j),\Lambda)}{\delta r_j}.$$

Hence, we have

$$\begin{aligned}
&\lim_{j\to\infty}\frac{n(r_j,\Lambda)-n(\alpha p(r_j),\Lambda)+n(\alpha p(r_j),\Lambda)-n((1-\delta)r_j,\Lambda)}{\delta r_j}\\
&\qquad=\lim_{j\to\infty}\frac{n(r_j,\Lambda)-n(\alpha p(r_j),\Lambda)}{\delta r_j}+\lim_{j\to\infty}\frac{n(\alpha p(r_j),\Lambda)-n((1-\delta)r_j,\Lambda)}{\delta r_j}.
\end{aligned} \tag{7}$$

Let us estimate separately the terms in (7). Since $\alpha p(r_j) \in ((1-\delta)r_j, r_j)$, passing to a subsequence we can assume that $p(r_j)/r_j$ converges to some γ, at that, $\alpha\gamma \in [1-\delta, 1]$. We proceed to the second term in (7). By the choice of number $p(r)$, set $\Lambda_{p(r_j)}$ is a proper subset of $\widetilde{\Lambda}_{p(r_j)}$. Then by construction the identity $n(\alpha p(r_j), \Lambda) = p(r_j)$ holds true. Moreover, in accordance with the above restriction, we have the inequality $n(\alpha m(r_j), \Lambda) \leqslant m(r_j)$, where $m(r_j)$ is a minimal natural number such that $\alpha m(r_j) \geqslant (1-\delta)r_j$. Therefore,

$$\lim_{j\to\infty} \frac{n(\alpha p(r_j), \Lambda) - n((1-\delta)r_j, \Lambda)}{\delta r_j} \geqslant \lim_{j\to\infty} \frac{p(r_j) - m(r_j)}{\delta r_j} = \frac{\gamma}{\delta} - \frac{1-\delta}{\alpha\delta}. \tag{8}$$

If $\alpha\gamma = 1$, by (8), (7) and (4) we obtain

$$\lim_{r\to\infty} \frac{n(r, \Lambda) - n((1-\delta)r, \Lambda)}{\delta r} \geqslant \tau. \tag{9}$$

Let $\alpha\gamma < 1$ and $\tilde{\delta} \in (0, 1-\alpha\gamma) \subset (0, \delta)$. Then there exists an index j_0 such that $\alpha p(r_j) \leqslant (1-\tilde{\delta})r'_j$, $j \geqslant j_0$, where $r'_j = \alpha q(r_j)$. This is why in view of (5) and (3) we obtain

$$\begin{aligned}\lim_{j\to\infty} \frac{n(r_j, \Lambda) - n(\alpha p(r_j, \delta), \Lambda)}{\delta r_j} &\geqslant \lim_{j\to\infty} \frac{n(\alpha q(r_j), \Lambda) - n((1-\tilde{\delta})r'_j, \Lambda)}{\delta r_j} \\ &= \frac{\tilde{\delta}}{\delta} \lim_{j\to\infty} \frac{r'_j\left(n(r'_j, \widetilde{\Lambda}) - n((1-\tilde{\delta})r'_j, \widetilde{\Lambda})\right)}{r_j \tilde{\delta} r'_j} \geqslant \frac{\tilde{\delta}}{\delta}(\tau - \varepsilon).\end{aligned}$$

By (4), (7), (8) it yields

$$\lim_{r\to\infty} \frac{n(r, \Lambda) - n((1-\delta)r, \Lambda)}{\delta r} \geqslant \frac{\gamma}{\delta} - \frac{1-\delta}{\alpha\delta} + \frac{\tilde{\delta}}{\delta}(\tau - \varepsilon).$$

Since $\alpha = 1/\tau$ and $\tilde{\delta}$ can be arbitrarily close to $1 - \alpha\gamma < \delta$, then

$$\lim_{r\to\infty} \frac{n(r, \Lambda) - n((1-\delta)r, \Lambda)}{\delta r} \geqslant \frac{\alpha\gamma}{\delta}\tau - \frac{1-\delta}{\delta}\tau + \frac{1-\alpha\gamma}{\delta}(\tau - \varepsilon) = \tau - \frac{1-\alpha\gamma}{\delta}\varepsilon \geqslant \tau - \varepsilon.$$

Thus, in view of (9), (6) and the arbitrariness of number $\varepsilon > 0$ we obtain the inequality $\underline{n}_0(\Lambda) \geqslant \tau$. By (1) and (2) it completes the proof. □

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ and $r > 0$. We define

$$V(r, \Lambda) = \sum_{0<|\lambda_k|<r} \frac{1}{\lambda_k}.$$

In what follows we consider only real sequences Λ and we represent them as $\Lambda = \Omega \cup \Xi$, where $\Omega = \{\omega_k\}_{k=1}^{\infty}$ and $\Xi = \{\xi_k\}_{k=1}^{\infty}$ are taken in the ascending order of their absolute values. These sequences consists of non-negative and negative terms of Λ.

Lemma 3. *Let $\Lambda = \Omega \cup \Xi$, where Ω and Ξ have the same density τ. Then for each $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for each $r_2 > r_1 > r(\varepsilon)$ the inequality*

$$|V(r_2, \Lambda) - V(r_1, \Lambda)| \leqslant \ln(r_2/r_1) + \varepsilon$$

holds true.

Proof. Without loss of generality we can assume that $\omega_k \neq 0$, $k \geqslant 1$. In accordance with Euler's representation we have

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \beta + \varepsilon(n), \tag{10}$$

where β is Euler constant and $\varepsilon(n) \to 0$ as $n \to \infty$. By the assumption, Ω and Ξ have density τ, i.e., the identities

$$\omega_k = |\omega_k| = k/(\tau + \delta'(k)), \quad \xi_k = -|\xi_k| = -k/(\tau + \delta''(k)) \tag{11}$$

$$n(r,\Omega) = \tau r + \varepsilon'(r) r, \qquad n(r,\Xi) = \tau r + \varepsilon''(r) r, \tag{12}$$

hold true, where $\delta'(k), \delta''(k) \to 0$ as $k \to \infty$ and $\varepsilon'(r), \varepsilon''(r) \to 0$ as $r \to \infty$. We fix $\tilde\varepsilon > 0$. We choose index m such that

$$|\delta'(k)| \leqslant \tilde\varepsilon, \quad |\delta''(k)| \leqslant \tilde\varepsilon, \quad |\varepsilon(n)| \leqslant \tilde\varepsilon, \quad k, n \geqslant m. \tag{13}$$

We choose $r(\tilde\varepsilon) > 0$ so that

$$n(r_1,\Omega) \geqslant m, \;\; n(r_1,\Xi) \geqslant m, \;\; \left|\ln \frac{\tau + \varepsilon'(r_2)}{\tau + \varepsilon'(r_1)}\right| \leqslant \tilde\varepsilon, \;\; \left|\ln \frac{\tau + \varepsilon''(r_2)}{\tau + \varepsilon''(r_1)}\right| \leqslant \tilde\varepsilon, \;\; r_2 > r_1 > r(\tilde\varepsilon). \tag{14}$$

Then by (10)–(14) we obtain

$$\begin{aligned}
|V(r_2,\Lambda) - V(r_1,\Lambda)| &= \left| \sum_{k=n(r_1,\Omega)+1}^{n(r_2,\Omega)} \frac{\tau + \delta'(k)}{k} - \sum_{k=n(r_1,\Xi)+1}^{n(r_2,\Xi)} \frac{\tau + \delta''(k)}{k} \right| \\
&\leqslant \left| \sum_{k=n(r_1,\Omega)+1}^{n(r_2,\Omega)} \frac{\tau}{k} - \sum_{k=n(r_1,\Xi)+1}^{n(r_2,\Xi)} \frac{\tau}{k} \right| + \sum_{k=n(r_1,\Omega)+1}^{n(r_2,\Omega)} \frac{|\delta'(k)|}{k} + \sum_{k=n(r_1,\Xi)+1}^{n(r_2,\Xi)} \frac{|\delta''(k)|}{k} \\
&\leqslant \tau \left| \ln \frac{n(r_2,\Omega)}{n(r_1,\Omega)} - \ln \frac{n(r_2,\Xi)}{n(r_1,\Xi)} \right| + 4\tau\tilde\varepsilon + \tilde\varepsilon \ln \frac{n(r_2,\Omega)}{n(r_1,\Omega)} + \tilde\varepsilon \ln \frac{n(r_2,\Xi)}{n(r_1,\Xi)} + 4\tilde\varepsilon^2 \\
&\leqslant 6\tau\tilde\varepsilon + 2\tilde\varepsilon \ln \frac{r_2}{r_1} + 6\tilde\varepsilon^2, \qquad r_2 > r_1 > r(\tilde\varepsilon).
\end{aligned}$$

It follows easily the required inequality. The proof is complete. □

Lemma 4. *Let $\Lambda = \Omega \cup \Xi$, where Ω and Ξ have the same density τ. Then there exists a set of zero density $T \subset \Lambda$ such that*

$$\lim_{r\to\infty} (V(r,\Lambda) - V(r,T)) = 0.$$

Proof. If $\tau = 0$, then Λ has the zero density. In this case the statement of the lemma is trivial since as T we can take Λ.

Let $\tau > 0$. We seek sequence $T \subset \Lambda$, $T = \{t_p\}$ as the union $T = \cup_{m\geqslant 1} T_m$, where $T_m = \{t_p\}_{p=p(m)}^{p(m+1)-1}$. We construct sets T_m by induction. Let $m = 1$. As $T_1 = \{t_p\}_{p=p(1)=1}^{p(2)-1}$ we take a set formed by all the elements in Λ belonging to the interval $(-2, 2)$. We suppose that we have constructed sets T_m for each $m < l$. Let us define T_l. We consider two cases.

1) $V(2^l,\Lambda) - \sum_{p=1}^{p(l)-1} 1/t_p \geqslant 0$.

a) If

$$V(2^l,\Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{2^{l-1} \leqslant \omega_k < 2^l} \frac{1}{\omega_k} \geqslant 0,$$

then as T_l we take the set (probably empty) of all the elements in Ω belonging to the semi-interval $[2^{l-1}, 2^l)$.

b) Let

$$V(2^l,\Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{2^{l-1} \leqslant \omega_k < 2^l} \frac{1}{\omega_k} < 0.$$

Then semi-interval $[2^{l-1}, 2^l)$ contains points ω_k of sequence Ω. By $k(l)$ we denote the minimal of indices k for which $\omega_k \geqslant 2^{l-1}$. As T_l we take the set of the elements $\omega_{k(l)}, \dots, \omega_{k'(l)}$, where $k'(l)$ is the minimal index satisfying the inequality

$$V(2^l, \Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{k=k(l)}^{k'(l)} \frac{1}{\omega_k} < 0.$$

By the choice of $k'(l)$, the point $\omega_{k'(l)}$ together with all the elements in set T_l belong the semi-interval $[2^{l-1}, 2^l)$.

2) $V(2^l, \Lambda) - \sum\limits_{p=1}^{p(l)-1} 1/t_p < 0$.

a) If

$$V(2^l, \Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{-2^l < \xi_n \leqslant -2^{l-1}} \frac{1}{\xi_n} < 0,$$

as T_l we take the set of all the elements Ξ lying in the semi-interval $(-2^l, -2^{l-1}]$.

b) Let

$$V(2^l, \Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{-2^l < \xi_n \leqslant -2^{l-1}} \frac{1}{\xi_n} \geqslant 0.$$

Then semi-interval $(-2^l, -2^{l-1}]$ contains points ξ_n of sequence Ξ. By $n(l)$ we denote the minimal of indices n satisfying $\xi_n \leqslant -2^{l-1}$. As T_l, we take the set of the elements $\xi_{n(l)}, \dots, \xi_{n'(l)}$, where $n'(l)$ is the minimal index satisfying the estimate

$$V(2^l, \Lambda) - \sum_{p=1}^{p(l)-1} \frac{1}{t_p} - \sum_{n=n(l)}^{n'(l)} \frac{1}{\xi_n} \geqslant 0.$$

By the choice of $n'(l)$, point $\omega_{k'(l)}$ together with all the elements in T_l belong to the semi-interval $[2^{l-1}, 2^l)$.

Thus, sequence T is defined completely. We note that by construction, one of the following possibilities occurs:

a) the quantity $V(2^m, \Lambda) - V(2^m, T)$ preserves the sign (number 0 is assigned with sign "+") while passing from $m = l-1$ to $m = l$ ($l > 1$) and thus, the inequality

$$|V(2^l, \Lambda) - V(2^l, T)| \leqslant |V(2^{l-1}, \Lambda) - V(2^{l-1}, T)| \tag{15}$$

holds true, while set T_l consists of all the elements in Λ lying in the annulus $B(0, 2^l) \setminus B(0, 2^{l-1})$ and having the same sign as $V(2^l, \Lambda) - V(2^l, T)$.

b) quantity $V(2^m, \Lambda) - V(2^m, T)$ changes the sign while passing from $m = l-1$ to $m = l$ and thus, the estimate

$$|V(2^l, \Lambda) - V(2^l, T)| \leqslant 1/\omega_{k'(l)}(1/|\xi_{n'(l)}|) \leqslant 1/2^{l-1} \tag{16}$$

holds true. It is implied by the choice of indices $k'(l)$ and $n'(l)$.

Suppose that for some index $m(0)$ the quantity $V(2^m, \Lambda) - V(2^m, T)$ preserves the sign for all $m \geqslant m(0)$, for instance, "+". Then the part of sequence T formed by the elements lying outside the circle $B(0, 2^{m(0)})$ coincides with the corresponding part of sequence Ω. Then by (15), the series $\sum 1/\xi_n$ converges. It means that $n/\xi_n \to 0$ as $n \to \infty$, i.e., sequence Ξ has the zero density. It contradicts to the inequality $\tau > 0$.

Thus, there exists a sequence of indices $m(j)$, $j \geqslant 1$, such that the quantity $V(2^m, \Lambda) - V(2^m, T)$ changes the sign while passing from $m = m(j)$ to $m = m(j)+1$. Then it follows from (15) and (16) that

$$|V(2^m, \Lambda) - V(2^m, T)| \to 0, \quad m \to \infty. \tag{17}$$

Let us prove that T has the zero density. Let s_m be then number of the elements in T_m. By construction, all of them have the same sign and lie in the annulus $B(0,2^m)\setminus B(0,2^{m-1})$. Therefore, by (17) and Lemma 3 we obtain

$$\frac{s_m}{2^m}\leqslant\left|\sum_{p=p(m)}^{p(m+1)-1}\frac{1}{t_p}\right|=|V(2^m,T)-V(2^{m-1},T)|\leqslant|V(2^m,\Lambda)-V(2^{m-1},\Lambda)|$$
$$+|V(2^m,\Lambda)-V(2^m,T)|+|V(2^{m-1},\Lambda)-V(2^{m-1},T)|\to 0,\quad m\to\infty.$$

We fix $\varepsilon>0$. Then there exists an index $m(\varepsilon)$ such that

$$s_m/2^m\leqslant\varepsilon,\quad m\geqslant m(\varepsilon). \tag{18}$$

Let $r>2^{m(\varepsilon)}$ and n be the minimal of the indices for which $r\leqslant 2^n$. Then by (18)

$$\frac{n(r,T)}{r}\leqslant\frac{n(2^{m(\varepsilon)},T)}{r}+\frac{n(2^n,T)-n(2^{m(\varepsilon)},T)}{2^{n-1}}=\frac{n(2^{m(\varepsilon)},T)}{r}$$
$$+\frac{s_{m(\varepsilon)+1}+s_{m(\varepsilon)+2}+\dots+s_n}{2^{n-1}}\leqslant\frac{n(2^{m(\varepsilon)},T)}{r}+2\varepsilon\left(1+\frac{1}{2}+\dots+\frac{1}{2^{n-m(\varepsilon)-1}}\right).$$

It follows that $\overline{n}(T)\leqslant 2\varepsilon$. Since $\varepsilon>0$ is arbitrary and $\underline{n}(T)\geqslant 0$, it leads to the required identity $n(T)=0$.

It remains to show that $V(r,\Lambda)-V(r,T)\to 0$ as $r\to\infty$. Let $r>2$. We choose index m such that $2^m\leqslant r<2^{m+1}$. As it has been proved above, T has the zero density. This is why for sequence T, as well as for Λ, Lemma 3 holds true. Then by (17) we have

$$|V(r,\Lambda)-V(r,T)|\leqslant|V(2^m,\Lambda)-V(2^m,T)|+|V(r,\Lambda)-V(2^m,\Lambda)|+|V(r,T)-V(2^m,T)|$$
$$\leqslant|V(2^m,\Lambda)-V(2^m,T)|+2\varepsilon(r)(\ln 2+1)\to 0,\quad r\to\infty.$$

The proof is complete. □

We recall that set $\Lambda=\Omega\cup\Xi$ is called properly distributed (see [2, Ch. II, Sect. 1]), if sequences Ω and Ξ have densities and there exists $\lim_{r\to\infty}V(r,\Lambda)$.

Theorem 1. *Let $\tau\geqslant 0$ and $\widetilde{\Lambda}=\widetilde{\Omega}\cup\widetilde{\Xi}$ be such that $\underline{n}(\widetilde{\Omega})\geqslant\tau$ and $\underline{n}(\widetilde{\Xi})\geqslant\tau$. Then there exists a properly distributed set $\Lambda\subset\widetilde{\Lambda}$, $\Lambda=\Omega\cup\Xi$, where Ω and Ξ have the same density τ and $\lim_{r\to\infty}V(r,\Lambda)=0$.*

Proof. In accordance with Lemma 2, there exists a sequence $\Lambda'\subset\widetilde{\Lambda}$, $\Lambda'=\Omega'\cup\Xi'$ such that $n(\Omega')=n(\Xi')=\tau$. Then by Lemma 4 there exists a sequence $T\subset\Lambda'$ satisfying condition $\lim_{r\to\infty}(V(r,\Lambda')-V(r,T))=0$. Thus, $\Lambda=\Lambda'\setminus T$ possesses all the required properties. The proof is complete. □

Let us consider some applications of Theorem 1.

Let f be an entire function of exponential type (i.e., there exist $A,B>0$ such that the estimate $|f(z)|\leqslant A+B|z|$, $z\in\mathbb{C}$ holds true). The upper indicator of f is the function

$$h_f(\lambda)=\varlimsup_{t\to\infty}\ln|f(t\lambda)|/t,\quad\lambda\in\mathbb{C}.$$

Indicator h_f is a convex order one positively homogeneous function which coincides with the complex support function H_K of some convex compact set K (in other words, with the usual support function of the compact set complex conjugate with K) called adjoint diagram of f (see, for instance, [3, Ch. I, Sect. 5, Thm. 5.4]):

$$h_f(\lambda)=H_K(\lambda)=\sup_{z\in K}\operatorname{Re}(\lambda z),\quad\lambda\in\mathbb{C}.$$

Function f is said to have a regular growth (see [2, Ch. III]) if

$$h_f(\lambda) = \lim_{t\to\infty, t\notin E} \ln|f(t\lambda)|/t, \qquad \lambda \in \mathbb{C},$$

where E is a set of relative zero measure on the ray $(0, +\infty)$, i.e., the Lebesgue measure of its intersection with the interval $(0, r)$ is infinitesimally small w.r.t. r as $r \to +\infty$.

Let K be a convex compact set. The points $z(\varphi)$ being the intersection of support line $l(\varphi) = \{z : \mathrm{Re}\,(ze^{i\varphi} = H_K(e^{i\varphi})\}$ and the boundary ∂K are called support points of line $l(\varphi)$.

In accordance of [2, Ch. III, Thm. 4], function f has a regular growth if and only if its multiple zero set Λ (i.e., each zero of f appears in Λ counting its multiplicity) is properly distributed. At that, if Λ is real ($\Lambda = \Omega \cup \Xi$), the identities

$$n(\Omega) = S_K(\varphi_1, \varphi_2)/2\pi, \qquad -\pi/2 < \varphi_1 < \varphi_2 < \pi/2,$$
$$n(\Xi) = S_K(\varphi_1, \varphi_2)/2\pi, \qquad \pi/2 < \varphi_1 < \varphi_2 < 3\pi/2,$$

hold true, see [2, Ch. II, Sect. 1, Formula (2.07)]. Here K is the adjoint diagrma, and $S_K(\varphi_1, \varphi_2)$ is the length of boundary ∂K (measured in the clockwise direction) between support points $z(\varphi_1)$ and $z(\varphi_2)$. We observe that for all mentioned φ_1, φ_2 quantity $S_K(\varphi_1, \varphi_2)$ is constant. It is possible if and only if K is the vertical segment of length $2\pi\tau$, where $\tau = n(\Omega) = n(\Xi)$.

Theorem 2. *Let f be an entire function of exponential type with a real zero set $\widetilde{\Lambda} = \widetilde{\Omega} \cup \widetilde{\Xi}$. Then the following statements are equivalent:*

1) The inequalities $\underline{n}(\widetilde{\Omega}) \geqslant \tau$, $\underline{n}(\widetilde{\Xi}) \geqslant \tau$ hold true.

2) The representation $f = f_1 f_2$ is valid, where f_1, f_2 are entire function of exponential type, f_1 is a function of regular growth, its adjoint diagram is a vertical segment of length $2\pi\tau$ and $h_{f_2} \equiv h_f - h_{f_1}$.

Proof. Assume that Statement 1) holds true. Then by Theorem 1 there exists a properly distributed set $\Lambda \subset \widetilde{\Lambda}$, $\Lambda = \Omega \cup \Xi$, where Ω and Ξ have the same density τ. By f_1 we denote the canonical function of set Λ. It has exponential type and regular growth (see [2, Ch. II, Thm. 4]). At that, as it was mentioned above, the adjoint diagram f_1 coincides with the vertical segment of length $2\pi\tau$. We let $f_2 = f/f_1$. Since the zero set of f_1 is a part of zero set of f, function f_2 is entire. Then by Corollary 2 of Theorem 5 in [2, Ch. III] the identity $h_{f_2} \equiv h_f - h_{f_1}$ holds true. In particular, it implies that f_2 is of exponential type.

Assume that Statement 2 holds true. Then zero set $\Lambda = \Omega \cup \Xi$ of function f_1 satisfies the identities $n(\Omega) = n(\Xi) = \tau$. Since Λ is a part of $\widetilde{\Lambda}$, it yields $\underline{n}(\widetilde{\Omega}) \geqslant \tau$, $\underline{n}(\widetilde{\Xi}) \geqslant \tau$. The proof is complete. □

Theorem 3. *Let f be an entire function of exponential type with real zero set $\widetilde{\Lambda} = \widetilde{\Omega} \cup \widetilde{\Xi}$, where $\widetilde{\Omega}$ and $\widetilde{\Xi}$ possess densities. Then the representation $f = f_1 f_2$ holds true, where f_1, f_2 are entire function of exponential type, f_1 is a function of regular growth, its adjoint diagram is a vertical segment, $h_{f_2} \equiv h_f - h_{f_1}$ and the zero set of f_2 has the zero density.*

Proof. Let us show first that the densities of $\widetilde{\Omega}$ and $\widetilde{\Xi}$ are same. Since f has an exponential type, by the Lindelöf's theorem (see [2, Ch. I, Thm. 15]) there exists $c > 0$ such that $|V(r, \widetilde{\Lambda})| \leqslant c$, $r > 0$. Let $n(\widetilde{\Omega}) = \tau$ and $\underline{n}(\widetilde{\Xi}) = \gamma$. Assume that $\tau \neq \gamma$, for instance, $\tau > \gamma$ (the case $\tau < \gamma$ is studied in the same way). As in Lemma 3, we have (without loss of generality we can assume that $0 \notin \widetilde{\Omega}$):

$$\begin{aligned} V(r,\widetilde{\Lambda}) = &\sum_{k=1}^{n(r,\widetilde{\Omega})} \frac{\tau+\delta'(k)}{k} - \sum_{k=1}^{n(r,\widetilde{\Xi})} \frac{\gamma+\delta''(k)}{k} = \tau \ln n(r,\widetilde{\Omega}) - \gamma \ln n(r,\widetilde{\Xi}) \\ &+ \sum_{k=1}^{n(r,\widetilde{\Omega})} \frac{\delta'(k)}{k} - \sum_{k=1}^{n(r,\widetilde{\Xi})} \frac{\delta''(k)}{k} + \varepsilon(n(r,\widetilde{\Omega})) - \varepsilon(n(r,\widetilde{\Xi})), \\ n(r,\widetilde{\Omega}) =& \tau r + \varepsilon'(r) r, \quad n(r,\widetilde{\Xi}) = \tau r + \varepsilon''(r) r, \end{aligned}$$

where $\varepsilon(k)$, $\delta'(k)$, $\delta''(k) \to 0$, $k \to \infty$ and $\varepsilon'(r), \varepsilon''(r) \to 0$, $r \to \infty$. We fix $\varepsilon > 0$. We choose index m such that $|\delta'(k)| \leqslant \varepsilon$, $|\delta''(k)| \leqslant \varepsilon$,$|\varepsilon(n)| \leqslant \varepsilon$, $k, n \geqslant m$. Then we choose $r(\varepsilon) > 0$ so that $n(r,\widetilde{\Omega}) \geqslant m$, $n(r,\widetilde{\Xi}) \geqslant m$, $r > r(\varepsilon)$. By (10) it implies

$$\begin{aligned} |V(r,\widetilde{\Lambda})| \geqslant & (\tau-\gamma)\ln r - \tau|\ln(\tau+\varepsilon\varepsilon'(r))| - \gamma|\ln(\gamma+\varepsilon''(r))| - \sum_{k=1}^{m} \frac{|\delta'(k)| + |\delta''(k)|}{k} \\ & - 2\varepsilon \ln r - \varepsilon|\ln(\tau+\varepsilon'(r))| - \varepsilon\varepsilon|\ln(\gamma+\varepsilon''(r))| - 2\varepsilon - 2\beta, \quad r > r(\varepsilon). \end{aligned}$$

It yields that $|V(r,\widetilde{\Lambda})| \to \infty$ as $r \to \infty$. It contradicts the boundedness of $|V(r,\widetilde{\Lambda})|$. Thus, sequences $\widetilde{\Omega}$ and $\widetilde{\Xi}$ have the same density τ.

As in Theorem 2, the representation $f = f_1 f_2$ holds true, where f_1, f_2 are entire functions of exponential type, f_1 has a regular growth, its adjoint diagram is a vertical segment of length $2\pi\tau$, and $h_{f_2} \equiv h_f - h_{f_1}$. At that, $\widetilde{\Lambda} = \Lambda \cup \Lambda'$, where $\Lambda = \Omega \cup \Xi$, Λ' are zero sets of functions f_1, f_2, respectively, and $n(\Omega) = n(\Xi) = \tau$. Since the densities of sequences $\widetilde{\Omega}$ and $\widetilde{\Xi}$ are also equal to τ, we get

$$\lim_{r\to\infty} \frac{n(r,\Lambda')}{r} = \lim_{r\to\infty} \frac{n(r,\widetilde{\Lambda}) - n(r,\Lambda)}{r} = \lim_{r\to\infty} \frac{n(r,\widetilde{\Omega}) + n(r,\widetilde{\Xi}) - n(r,\Omega) - n(r,\Xi)}{r} = 0.$$

The proof is complete. □

Let D be a convex set in $\mathbb{C}$, $H(D)$ is the space of functions analytic in D with the topology of uniform convergence on compact subsets in D, and $H^*(D)$ is a strongly dual space for $H(D)$. By P_D we denote the space of entire functions of exponential type whose adjoint diagrams lie in domain D. The Laplace transform $f(\lambda) = \nu(\exp(\lambda z))$ makes an isomorphism (see, for instance, [4, Ch. III, Sect. 12, Thm. 12.3]) between $H^*(D)$ and P_D.

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ and $\mathcal{E}(\Lambda) = \{z^{n-1}\exp(\lambda_k z)\}_{n=1,k}^{n(k)}$, where we go over various points λ_k, and $n(k)$ is the multiplicity of λ_k (i.e., the number of elements in sequence Λ coinciding with λ_k). By the Hahn-Banach theorem, system $\mathcal{E}(\Lambda)$ is incomplete in space $H(D)$ if and only if there exists a non-zero functional $\nu \in H^*(D)$ vanishing on the elements of the system. Hence, the incompleteness of $\mathcal{E}(\Lambda)$ is equivalent to the existence of function $f \in P_D$ vanishing on the points λ_k with multiplicity at least $n(k)$, $k = 1, 2, \ldots$ If domain D is non-empty, for the sake of convenience we assume that each system $\mathcal{E}(\Lambda)$ is complete in $H(D)$.

Let $\tau > 0$, $I(\tau) = [-i\pi\tau, i\pi\tau]$, and $D(\tau)$ denotes a convex domain covered while segment $I(\tau)$ moves inside D. By the definition, the inclusion $D(\tau) \subset D$ holds true. Domain $D(\tau)$ is empty if none of the shifts of segment $I(\tau)$ lies in D. We obviously have the representation $D(\tau) = D'(\tau) + I(\tau)$, where

$$D'(\tau) = \{z \in \mathbb{C} : \mathrm{Re}\,(z\lambda) < H_D(\lambda) - \pi\tau|\mathrm{Im}\,\lambda|, \quad \forall \lambda : |\lambda| = 1\},$$

$\pi\tau|\mathrm{Im}\,\lambda|$ is the support function of segment $I(\tau)$.

Theorem 4. *Let $\tau > 0$, D be a convex domain and $\widetilde{\Lambda} = \widetilde{\Omega} \cup \widetilde{\Xi}$ is a real sequence such that $\underline{n}(\widetilde{\Omega}) \geqslant \tau$, $\underline{n}(\widetilde{\Xi}) \geqslant \tau$. System $\mathcal{E}(\widetilde{\Lambda})$ is complete in $H(D)$ if and only if it is complete in $H(D(\tau))$.*

Proof. Suppose that $\mathcal{E}(\widetilde{\Lambda})$ is incomplete in space $H(D(\tau))$. Then there exists an entire function f of exponential type vanishing at the points λ_k of the multiplicity at least $n(k)$, $k = 1, 2, \ldots$, whose adjoint diagram lies in $D(\tau)$. Since $D(\tau) \subset D$, we have $f \in P_D$. It follows that $\mathcal{E}(\widetilde{\Lambda})$ is incomplete in space $H(D)$.

Suppose that $\mathcal{E}(\widetilde{\Lambda})$ is incomplete in $H(D)$. Then there exists $f \in P_D$ vanishing at points λ_k of multiplicity at least $n(k)$, $k = 1, 2, \ldots$ Let us show that $f \in P_{D(\tau)}$.

In accordance with Theorem 1, there exists a properly distributed set $\Lambda \subset \widetilde{\Lambda}$, $\Lambda = \Omega \cup \Xi$, where Ω and Ξ have the same density τ. By f_1 we denote the canonical function of set Λ. It is of exponential type and regular growth, while its adjoint diagram K_1 is represented as $I(\tau) + z_0$ (z_0 is some point in the plane). We let $f_2 = f/f_1$. Since the zero set of f_1 is a part of zero set for f, function f_2 is entire. Then in accordance with Corollary 2 of Theorem 5 in [2, Ch. III] the identity $h_{f_2} \equiv h_f - h_{f_1}$ holds true. In particular, it follows that f_2 is of exponential type. Let K and K_2 be adjoint diagrams of functions f and f_2, respectively. Then

$$H_{K_2} \equiv h_{f_2} \equiv h_f - h_{f_1} \equiv H_K - H_{K_1}.$$

Thus, $H_K \equiv H_{K_1} + H_{K_2}$, i.e., $K = K_1 + K_2 = I(\tau) + z_0 + K_2 \subset D$. Hence, if $z \in K$, then z belongs to a shift of segment $I(\tau)$ lying in domain D. It means that the inclusion $K \subset D(\tau)$ is true. It follows that $\mathcal{E}(\widetilde{\Lambda})$ is incomplete in $H(D(\tau))$. The proof is complete. □

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