ISSN 2304-0122 Ufa Mathematical Journal. Vol. 6. No 3 (2014). P. 85-94.

doi:10.13108/2014-6-3-85 UDC 517.956

BEHAVIOR OF SOLUTIONS TO GAUSS-BIEBERBACH-RADEMACHER EQUATION ON PLANE

A.V. NEKLYUDOV

Abstract. We study the asymptotic behavior at infinity of solutions to Gauss-Bierbach-Rademacher equation $\Delta u = e^u$ in the domain exterior to a circle on the plane. We establish that the leading term of the asymptotics is a logarithmic function tending to $-\infty$. We also find the next-to-leading term for various values of the coefficient in the leading term.

Keywords: Semilinear elliptic equations, Gauss-Bieberbach-Rademacher equation, asymptotic behavior of solutions.

Mathematics Subject Classification: 35J15, 35J61, 35J91

1. Introduction

The equation

$$\Delta u = e^u, \tag{1}$$

appears as a model one in problems of differential geometry in relation with existence of surfaces of negative Gaussian curvature [1], the theory of automorphic functions [2], in studying the equilibrium of a charged gas [3]. Existence of solutions to equations like (1) in unbounded domains, in particular, existence of global solutions, was considered in works [1], [4]–[8]. In particular, it is well-known [1] that equation (1) has no global solutions for any number of independent variables n, while for $n \geqslant 3$ there exist no solutions defined in the exterior of a bounded domain [8]. The behavior at infinity of solutions to semi-linear elliptic equations with an exponential nonlinearity was studied mostly for cylindrical domains [9]–[13]. In the present paper we study the asymptotic behavior of solutions to two-dimensional equation (1) defined in the exterior of a circle. We employ the method of energy estimates of Saint-Venant principle kind [14]–[17] as well as the averaging principle.

We consider equation (1) in the two-dimensional domain $Q = \{x : |x| > R_0\} \subset \mathbb{R}^2_x$, where $x = (x_1, x_2)$, Δ is the two-dimensional Laplace operator. We assume that $u \in C^2(\overline{Q})$.

We introduce notations. The mean value of function $u(x)$ on the circumference $S_R = \{x : |x| = R\}$ is denoted by

$$\overline{u}(R) = \frac{1}{2\pi R}\int_{S_R} u\, ds,$$

the "heat flow" of function $u(x)$ through S_R is indicated as

$$P(R,u) = \int_{S_R} \frac{\partial u}{\partial \nu}\, ds = 2\pi R \overline{u}'(R), \tag{2}$$

A.V. Neklyudov, Behavior of solutions to Gauss-Bieberbach-Rademacher equation on plane.

Submitted March 28, 2014.

where ν is the unit outward normal to S_R. Let $Q(a,b) = \{x : a < |x| < b\}$, $0 < R_0 \leqslant a < b$. It is obvious that solution $u(x)$ to equation (1) in Q satisfies the identity

$$P(b,u) = P(a,u) + \int_{Q(a,b)} e^u\, dx. \tag{3}$$

We shall also make use of the notation $\nabla u \equiv \operatorname{grad} u$. The condition $f/g \to 1$ as the arguments of functions f and g tend to some value will be indicated by a standard notation: $f \sim g$.

2. Main results

Theorem 1. *Let $u(x)$ be a solution to equation* (1) *in Q. The relations*

$$\int_Q e^u\, dx < \infty; \qquad P(R,u) \to 2\pi C, \qquad \overline{u}(R) \sim C \ln R, \quad R \to \infty, \quad C = \text{const} \leqslant -2.$$

hold true.

Proof. It follows from (2) and (3) that

$$P(R,u) = 2\pi R\overline{u}'(R) = P(R_0,u) + \int_{Q(R_0,R)} e^u\, dx. \tag{4}$$

Let us show that the right hand side of identity (4) is negative for each $R > R_0$. Suppose the opposite, then for $R > R_1 = \text{const} > R_0$ we obtain

$$R\overline{u}'(R) > c_1 > 0.$$

Hereinafter by c_j we indicate positive constants depending only on a considered solution to (1) and independent of R, a, b, t, etc. As $R > R_2 = \text{const} > R_1$, it implies

$$\overline{u}(R) > c_2 \ln R.$$

By integral Jensen's inequality it yields

$$\int_{S_R} e^u\, ds \geqslant 2\pi R e^{\overline{u}(R)} > 2\pi R^{c_2+1}, \quad \int_{Q(R_0,R)} e^u\, dx > c_3 R^{c_2+2}, \quad R > R_3 = \text{const} > R_2.$$

Using (4) once again and integrating, we get

$$R\overline{u}'(R) > c_4 R^{c_2+2}, \quad \overline{u}(R) > c_5 R^{c_2+2}, \quad R > R_4 = \text{const} > R_3.$$

Finally, employing once again (4) and Jensen's inequality, for $R > R_5 = \text{const}$ we have

$$\overline{u}'(R) \geqslant c_6 + \frac{1}{2\pi R}\int_{Q(R_0,R)} e^u\, dx \geqslant c_6 + \frac{1}{R}\int_{R_0}^R r e^{\overline{u}(r)}\, dr > \left(\int_{R_0}^R e^{\overline{u}(r)}\, dr\right)^{1/2}.$$

Let $\int_{R_0}^R e^{\overline{u}(r)}\, dr = z(R)$, then $\overline{u}(R) = \ln z'(R)$ and the latter inequality can be written as

$$\frac{z''}{z'} > z^{1/2}$$

that for $R > R_6$ follows

$$z' > c_7 z^{3/2}.$$

It implies easily that $z(R) \to \infty$, $R \to R_7 - 0$ for some $R_7 > R_6$. This is impossible for a solution defined for $|x| > R_0$. Thus, the obtained contradiction means that the right hand side in (4) is negative for each $R > R_0$ that implies immediately the first statement of the theorem.

It follows from (4) that

$$P(R,u) \to 2\pi C, \quad \overline{u}(R) \sim C \ln R, \quad R \to \infty, \quad C = \text{const} \leqslant 0.$$

Jensen's inequality yields

$$\int_{R_0}^\infty r e^{\overline{u}(r)}\, dr \leqslant \frac{1}{2\pi}\int_Q e^u\, dx < \infty.$$

It follows that $C \leqslant -2$. The proof is complete. $\square$

Lemma 1. *Let $f \in C^1(\overline{Q}) \cap L_1(Q)$,*

$$\int_{R_0}^{\infty} r\Big(\int_{S_r} |f|\,dx\Big)^2\,dr < \infty.$$

Then there exists a solution $V(x)$ to equation

$$\Delta V = f$$

in Q satisfying the estimates

$$\int_{Q(R_0,R)} |\nabla V|^2\,dx \leqslant c_0 \ln R, \quad |\overline{V}(R)| \leqslant c_1 \ln R \tag{5}$$

as $R > R_1 = \text{const} > R_0$. At that, if $f > 0$ in Q, then $V \leqslant 0$ in Q.

If in addition the conditions

$$\int_{R_0}^{\infty} \frac{dr}{r} \int_{Q(r,\infty)} |f|\,dx < \infty, \quad \int_Q |x|^2 f^2\,dx < \infty \tag{6}$$

hold true, then

$$\int_Q |\nabla V|^2\,dx < \infty, \quad V(x) \to C = \text{const}, \quad |x| \to \infty.$$

Proof. For each natural $N > R_0$, in domain $Q(R_0, N)$ we consider solution V_N to the boundary value problem

$$\Delta V_N = f, \quad V_N\big|_{S_{R_0}} = 0, \quad \frac{\partial V_N}{\partial \nu}\bigg|_{S_N} = C_N,$$

where

$$C_N = -\frac{1}{2\pi N}\int_{Q(N,\infty)} f\,dx.$$

It is clear that as $N \geqslant R > R_0$,

$$P(R, V_N) = P(N, V_N) - \int_{Q(R,N)} f\,dx = -\int_{Q(R,\infty)} f\,dx. \tag{7}$$

In view of the identity $2\pi R\overline{V}'_N(R) = P(R, V_N)$ we obtain that as $N \geqslant R >> R_0$

$$\big|\overline{V_N}(R)\big| \leqslant c_2 \ln R. \tag{8}$$

Let us estimate the Dirichlet integral for solution V_N. It is obvious that

$$\int_{Q(R_0,N)} |\nabla V_N|^2\,dx = C_N \int_{S_N} V_N\,ds - \int_{Q(R_0,N)} f V_N\,dx. \tag{9}$$

Let us estimate the integrals in the right hand side of (9). Due to (8) we get

$$\left|C_N \int_{S_N} V_N\,ds\right| = 2\pi N\big|C_N \overline{V_N}(N)\big| \leqslant c_3 \ln N. \tag{10}$$

Since by the embedding theorem for functions of one variable and Poincaré inequality

$$\sup_{S_r} |V_N - \overline{V_N}(r)| \leqslant c_4 r^{1/2}\big(\int_{S_r} |\nabla V_N|^2\,ds\big)^{1/2},$$

we have

$$\begin{aligned}\left|\int_{S_r} f(V_N - \overline{V_N}(r))\,ds\right| &\leqslant c_4 r^{1/2}\Big(\int_{S_r} |\nabla V_N|^2\,ds\Big)^{1/2}\int_{S_r} |f|\,ds\\ &\leqslant \frac{1}{2}\int_{S_r} |\nabla V_N|^2\,ds + c_5 r\Big(\int_{S_r} |f|\,ds\Big)^2.\end{aligned} \tag{11}$$

By (8) we get

$$\left|\overline{V_N}(r)\int_{S_r} f\,ds\right| \leqslant c_6\ln r\int_{S_r}|f|\,ds. \tag{12}$$

It follows from (9)–(12) that

$$\begin{aligned}\int_{Q(R_0,N)}|\nabla V_N|^2\,dx \leqslant & c_3\ln N+\frac{1}{2}\int_{Q(R_0,N)}|\nabla V_N|^2\,dx\\ &+c_5\int_{R_1}^N r\bigg(\int_{S_r}|f|\,ds\bigg)^2 dr+c_6\ln N\int_{Q(R_1,N)}|f|\,dx.\end{aligned}$$

Thus, we obtain

$$\int_{Q(R_0,N)}|\nabla V_N|^2\,dx \leqslant c_7\ln N. \tag{13}$$

Let us estimate the Dirichlet integral for function V_N over domain $Q(R_0,R)$ for arbitrary $R\in(R_0,N)$:

$$I(R)\equiv\int_{Q(R_0,R)}|\nabla V_N|^2\,dx=\int_{S_R}\frac{\partial V_N}{\partial\nu}V_N\,ds-\int_{Q(R_0,R)}fV_N\,dx.$$

Estimating the second term in accordance with (11)-(12), for $R\geqslant R_1>R_0$ we get

$$\int_{Q(R_0,R)}|\nabla V_N|^2\,dx\leqslant c_8\ln R+\frac{1}{2}\int_{Q(R_0,R)}|\nabla V_N|^2\,dx+\int_{S_R}\frac{\partial V_N}{\partial\nu}V_N\,ds.$$

Employing Poincaré inequality and (7), (8), we arrive at

$$\begin{aligned}I(R)\equiv\int_{Q(R_0,R)}|\nabla V_N|^2\,dx &\leqslant 2\int_{S_R}\frac{\partial V_N}{\partial\nu}V_N\,ds+2c_8\ln R\\ &=2P(R,V_N)\overline{V_N}(R)+2\int_{S_R}\frac{\partial V_N}{\partial\nu}\big(V_N-\overline{V_N}(R)\big)\,ds+2c_8\ln R\\ &\leqslant c_9\bigg(R\int_{S_R}|\nabla V_N|^2\,ds+\ln R\bigg)=c_9(RI'(R)+\ln R).\end{aligned}$$

We integrate this inequality from R to $N\geqslant R^2$ to obtain by (13) that

$$I(R)\leqslant I(N)\Big(\frac{R}{N}\Big)^{\delta}+c_{10}R^{\delta}\int_R^N\frac{\ln r}{r^{\delta+1}}\,dr\leqslant c_{11}\ln R,\quad \delta>0.$$

Thus, for each fixed $R>R_0$, sequence V_N is uniformly bounded in Sobolev space $W_2^1(Q(R_0,R))$. Applying standard diagonal process, we obtain a sequence V_{N_k} converging to some function V weakly in $W_2^1(Q(R_0,R))$ and strongly in $L_2(Q(R_0,R))$ for each $R>R_0$. Since $V_{N_k}-V_{N_l}$ are harmonic functions, the convergence of these functions and of their derivatives is uniform in $Q(R_0,R)$. Thus, function V satisfies equation (1) and in view of (8) it satisfies also estimates (5).

If $f>0$ in Q, by the maximum principle one can see easily that $V_N<0$ in $Q(R_0,N)$ and $V\leqslant 0$ in Q.

Let function f satisfies also conditions (6). Then it follows from (6) and (7) that

$$\int_{R_0}^{\infty}|\overline{V}'(r)|\,dr=\frac{1}{2\pi}\int_{R_0}^{\infty}\frac{|P(r,V)|}{r}\,dr<\infty,\quad \overline{V}(R)\to C_0=\text{const},\quad R\to\infty.$$

In the same way (7) also follows the uniform boundedness of $|\overline{V}_N(R)|$. Hence, by estimates similar to (9)–(12), we obtain

$$\int_{Q(R_0,N)}|\nabla V_N|^2\,dx\leqslant c_{12}$$

that implies the finiteness of the Dirichlet integral over Q for V.

Let us show that in this case $V(x) \to C_0$, $|x| \to \infty$. As $R > 2R_0$, in accordance with De Giorgi type estimates [18] and Poincaré inequality for $x \in S_R$ we have

$$\begin{aligned}|V(x) - \overline{V}(R)|^2 \leqslant & c_{13}\left(R^{-2}\int_{Q(R/2,3R/2)}\left(V(z) - \overline{V}(R)\right)^2 dz + R^2\int_{Q(R/2,3R/2)} f^2(z)\,dz\right)\\ \leqslant & c_{14}\left(\int_{Q(R/2,3R/2)}|\nabla V(z)|^2\,dz + \int_{Q(R/2,3R/2)}|z|^2 f^2(z)\,dz\right) \to 0, \quad R \to \infty.\end{aligned}$$

Since $\overline{V}(R) \to C_0$, $R \to \infty$, we obtain that $V(x) \to C_0$, $|x| \to \infty$. The proof is complete. □

Lemma 2. *Let $g(r) > 0$ be a non-increasing on $[r_0, \infty)$ measurable function and*

$$\int_{r_0}^{\infty} r g(r)\,dr < \infty.$$

Then

$$\int_{r_0}^{\infty} r^3 g^2(r) dr < \infty.$$

Proof. It follows easily from monotonicity of $g(r)$ that $g(r) \leqslant r^{-2}$ as $r > r_1 = \text{const} > 0$. Then $r^3 g^2(r) \leqslant r g(r)$, $r > r_1$, that implies the statement of the lemma. □

Lemma 3. *Let $u(x)$ be a solution to equation (1) in Q. Then*

$$r^{-1}\int_{S_r} e^u\,ds \leqslant c_0, \qquad \int_{R_0}^{\infty} r\left(\int_{S_r} e^u\,ds\right)^2 dr < \infty.$$

Proof. Due to Theorem 1 and Lemma 2 it is sufficient to prove that $g'(r) < 0$ for each $r > R_0$, where

$$g(r) = r^{-1}\int_{S_r} e^u\,ds.$$

We have

$$g'(r) = r^{-1}\int_{S_r} e^u \frac{\partial u}{\partial \nu}\,ds.$$

We assume that $g'(r_1) \geqslant 0$ for some $r_1 > R_0$ and we choose an arbitrary $r > r_1$. Let $\theta = \theta(|x|) \geqslant 0$ be a cut-off function belonging to C^2 such that $\theta(|x|) = 1$ as $|x| \leqslant r$, $\theta(|x|) = 0$ as $|x| \geqslant r+1$, $(\theta'(|x|))^2 \leqslant c_1\theta(|x|)$ as $r \leqslant |x| \leqslant r+1$, $c_1 = \text{const} > 0$. Multiplying both sides of equation (1) by $e^u\theta$ and integrating over domain $Q(r_1, r+1)$, we obtain

$$\begin{aligned}\int_{Q(r_1,r+1)}\left(e^{2u} + |\nabla u|^2 e^u\right)\theta\,dx = & - r_1 g'(r_1) - \int_{Q(r,r+1)} e^u \frac{\partial u}{\partial |x|}\theta'\,dx\\ \leqslant & \int_{Q(r,r+1)} e^u\left(|\nabla u|^2\theta + c_2\right)dx.\end{aligned}$$

Hence,

$$\int_{Q(r_1,r)} e^{2u}\,dx \leqslant c_2\int_{Q(r,r+1)} e^u\,dx \to 0, \quad r \to \infty,$$

that is impossible. This contradiction shows that $g'(r) < 0$ for each $r > R_0$ and it completes the proof. □

Theorem 2. *Let $u(x)$ be a solution to equation (1) in Q obeying*

$$\overline{u}(R) \sim C\ln R, \quad C = \text{const} < -2, \quad R \to \infty.$$

Then

$$u(x) = C\ln|x| + C_1 + o(1), \quad |x| \to \infty, \quad C_1 = \text{const}.$$

Proof. Let us prove first that for each $\varepsilon > 0$ as $|x| > R_1 = R_1(\varepsilon)$, the estimate

$$u(x) \leqslant (C+\varepsilon)\ln|x|$$

holds true. We observe that by Theorem 1 and Lemma 3, function $f(x) = e^{u(x)}$ satisfies the hypothesis of Lemma 1, except, probably, conditions (6). We consider the harmonic function $U = u - V$, where V is the solution to equation $\Delta V = e^u$, the existence of which was established in Lemma 1. Let us estimate the Fourier coefficients w.r.t. φ for function U on circumference S_r. Since by Lemma 3 and Theorem 1

$$\int_0^{2\pi} |u(r,\varphi)|\,d\varphi = 2\int_0^{2\pi} u^+(r,\varphi)\,d\varphi - 2\pi\overline{u}(r,\varphi)) \leqslant 2\int_0^{2\pi} e^{u(r,\varphi)}\,d\varphi + c_1\ln r \leqslant c_2\ln r,$$

where $u^+ = \max\{u, 0\}$, employing estimates $|\overline{V}(r)| \leqslant c_3\ln r$, $V \leqslant 0$ and Lemma 3, we obtain

$$\int_0^{2\pi} |U(r,\varphi)|\,d\varphi \leqslant \int_0^{2\pi} \big(|u(r,\varphi)| + |V(r,\varphi)|\big)\,d\varphi \leqslant c_4\ln r, \quad r \geqslant r_1 = \text{const} > R_0.$$

Hence, the expansion of U into the Fourier series w.r.t. φ reads as

$$U = a_0\ln r + b_0 + \sum_{k=1}^{\infty} r^{-k}(a_k\cos k\varphi + b_k\sin k\varphi).$$

Then in view of the estimate for the Dirichlet integral of V in Lemma 1 we obtain

$$\int_{Q(R_0,R)} |\nabla u|^2\,dx \leqslant c_4\ln R. \tag{14}$$

We fix $\varepsilon > 0$ such that $C + \varepsilon < -2$. As $R > R_2 = R_2(\varepsilon)$,

$$\overline{u}(R) \leqslant (C+\varepsilon/2)\ln R.$$

By (14) for each $R > 2R_2$ there exists $r_1 \in (R/2, R)$ satisfying

$$\int_{S_{r_1}} |\nabla u|^2\,ds \leqslant 2c_4\frac{\ln R}{R}.$$

Then, by the embedding theorem and Poincaré inequality, for $x \in S_{r_1}$ we get the estimate

$$\begin{aligned} u(x) - (C+\varepsilon/2)\ln R <& (u(x) - \overline{u}(r_1)) + (\overline{u}(r_1) - (C+\varepsilon/2)\ln r_1) \\ <& c_5 r_1^{1/2}\bigg(\int_{S_{r_1}} |\nabla u|^2\,ds\bigg)^{1/2} \leqslant c_6\ln^{1/2} R, \\ u(x) \leqslant& (C+\varepsilon)\ln R, \quad R > R_3(\varepsilon). \end{aligned}$$

By analogy, the same inequality holds true as $x \in S_{r_2}$ for some $r_2 \in (R, 3R/2)$ provided R is great enough. In accordance with the maximum principle, this inequality holds true in $Q(r_1, r_2)$ and, in particular, as $|x| = R$. Hence, for $|x| > R_4(\varepsilon)$ we have

$$u(x) \leqslant (C+\varepsilon)\ln|x|.$$

It follows that $e^{u(x)} \leqslant c_7|x|^{-2-\delta}$ in Q, $\delta > 0$. Thus, function $f(x) = e^{u(x)}$ satisfies conditions (6). It yields that function $V \to C_0$, $|x| \to \infty$. Hence,

$$u(x) = U + V = C\ln|x| + C_1 + o(1), \quad C < -2 \ , |x| \to \infty.$$

The proof is complete. □

We proceed to the case $C = -2$, i.e., $\overline{u}(R) \sim -2\ln R$. It is clear that a direct analogue of theorem 2 does not hold, since by Theorem 1 $\int_Q e^u\,dx < \infty$ and therefore, solution can not be represented as $u(x) = -2\ln|x| + C_1 + o(1)$.

Lemma 4. *Let $u(x)$ be a solution to equation* (1) *in Q. Then the Dirichlet integral for function $w(x) = u(x) - \overline{u}(|x|)$ over Q is finite:*

$$\int_Q |\nabla w|^2\, dx < \infty.$$

Proof. Since $\Delta(\overline{u}(|x|)) = \overline{\Delta u}(|x|)$ [19], function w solves the equation

$$\Delta w = h(x) \equiv e^u - \overline{e^u}.$$

We have

$$\int_{Q(R_0,R)} |\nabla w|^2\, dx = -\int_{Q(R_0,R)} hw\, dx + \int_{S_R} w\frac{\partial w}{\partial \nu}\, ds - \int_{S_{R_0}} w\frac{\partial w}{\partial \nu}\, ds. \tag{15}$$

Function $h(x)$ satisfies the estimate like (11):

$$\Big|\int_{Q(R_0,R)} hw\, dx\Big| \leqslant \frac{1}{2}\int_{Q(R_0,R)} |\nabla w|^2\, dx + c_1\int_{R_0}^{R} r\Big(\int_{S_r} |h|\, ds\Big)^2 dr. \tag{16}$$

By Lemma 3,

$$\int_{R_0}^{\infty} r\Big(\int_{S_r} |h|\, ds\Big)^2 dr < \infty. \tag{17}$$

It follows from (15)–(17) that

$$\int_{Q(R_0,R)} |\nabla w|^2\, dx \leqslant 2\int_{S_R} w\frac{\partial w}{\partial \nu}\, ds + c_2.$$

Applying Cauchy-Schwarz inequality and taking into consideration that $\overline{w}(R) = 0$ and Poincaré inequality, we obtain

$$\begin{aligned}J(R) \equiv \int_{Q(R_0,R)} |\nabla w|^2\, dx &\leqslant 2\Big(\int_{S_R} w^2\, ds\Big)^{1/2}\Big(\int_{S_R} |\nabla w|^2\, ds\Big)^{1/2} + c_2\\ &\leqslant c_3 R\int_{S_R} |\nabla w|^2\, ds + c_2 \equiv c_3 R J'(R) + c_2.\end{aligned}$$

It implies that either function $J(R)$ is bounded or it grows faster than $\ln R$. The latter is impossible by (14). The proof is complete. □

Lemma 5. *Let $u(x)$ be a solution to* (1) *in Q and $\overline{u}(R) \sim -2\ln R$, $R \to \infty$. Then for each $\varepsilon > 0$ and each $R \geqslant R_1(\varepsilon)$ the estimate*

$$\overline{u}(R) \leqslant -2\ln R - 2\ln\ln R + \ln 2 + \varepsilon$$

holds true.

Proof. Let us prove first that the inequality

$$\overline{u}(R) > -2\ln R - 2\ln\ln R + \ln 2 + \varepsilon \tag{18}$$

can not be true for each $R \geqslant R_1 = \text{const} \geqslant \mathrm{R}_0$. Suppose the opposite and let (18) is valid for some $\varepsilon > 0$ and for each sufficiently great R. Then

$$\int_{Q(R,\infty)} e^u\, dx \geqslant 2\pi\int_R^{\infty} re^{\overline{u}(r)}\, dr \geqslant 2\pi M_0\int_R^{\infty}\frac{dr}{r\ln^2 r} = \frac{2\pi M_0}{\ln R}, \quad M_0 = \text{const} > 2.$$

Since $P(R,u) \to -4\pi$, $R \to \infty$, we obtain by taking into consideration (3)

$$\overline{u}'(R) = \frac{1}{2\pi R}P(R,u) = \frac{1}{2\pi R}\Big(-4\pi - \int_{Q(R,\infty)} e^u\, dx\Big) \leqslant -\frac{2}{R} - \frac{M_0}{R\ln R}$$

for each $R > R_1$ that contradicts inequality (18). Hence, (18) can not hold true simultaneously for each R starting from some R_1. It means that the limit inferior of the function $z(R) =$

$\overline{u}(R) + 2\ln R + 2\ln\ln R - \ln 2$ as $R \to \infty$ is non-positive. In order to prove the statement of the lemma, it is sufficient that $z(R)$ has no positive local maxima. If there exists a maximum point R, then

$$0 = z'(R) = \overline{u}'(R) + \frac{2}{R} + \frac{2}{R\ln R} = \frac{P(R,u)}{2\pi R} + \frac{2}{R} + \frac{2}{R\ln R}.$$

Then at this point

$$\begin{aligned} z''(R) =& \overline{u}''(R) - \frac{2}{R^2} - \frac{2(1+\ln R)}{R^2\ln^2 R} \\ =& \frac{1}{2\pi}\Big(- \frac{P(R,u)}{R^2} + \frac{P'(R,u)}{R}\Big) - \frac{2}{R^2} - \frac{2(1+\ln R)}{R^2\ln^2 R} \\ =& \frac{2}{R^2} + \frac{2}{R^2\ln R} + \frac{1}{2\pi R}\int_{S_R} e^u\,ds - \frac{2}{R^2} - \frac{2(1+\ln R)}{R^2\ln^2 R} > e^{\overline{u}(R)} - \frac{2}{R^2\ln^2 R} > 0. \end{aligned}$$

Hence, $z''(R) > 0$ that is impossible at a maximum. The proof is complete. □

Lemma 6. *Let $u(x)$ be a solution to* (1) *in Q, $\overline{u}(R) \sim -2\ln R$, $R \to \infty$. Then*

$$u(x) = \overline{u}(|x|) + o(1), \quad |x| \to \infty.$$

Proof. We fix arbitrary $R > 2R_0$. By Lemma 4, for some $r_1 \in (R/2, R)$ the estimate

$$\int_{S_{r_1}} |\nabla w|^2\,ds \leqslant \frac{c_1}{R}$$

holds true, where $w(x) = u(x) - \overline{u}(|x|)$. Hence,

$$\sup_{S_{r_1}} |w| \leqslant c_2 r_1^{1/2}\Big(\int_{S_{r_1}} |\nabla w|^2\,ds\Big)^{1/2} \leqslant c_3.$$

Thus, employing Lemma 5, we obtain that for each $x \in S_{r_1}$

$$u(x) \leqslant \overline{u}(|x|) + c_3 \leqslant -2\ln R - 2\ln\ln R + c_4.$$

In the same way, for some $r_2 \in (R, 3R/2)$ we have

$$u(x) \leqslant \overline{u}(|x|) + c_5 \leqslant -2\ln R - 2\ln\ln R + c_6$$

as $x \in S_{r_2}$. In accordance with the maximum principle, for each $x \in S_R$ we obtain

$$u(x) \leqslant -2\ln|x| - 2\ln\ln|x| + c_7,$$

hence,

$$e^{u(x)} \leqslant \frac{c_8}{|x|^2\ln^2|x|}, \quad |\Delta w| \leqslant \frac{c_9}{|x|^2\ln^2|x|}.$$

In accordance with De Giorgi estimate and Poincaré inequality we obtain

$$\begin{aligned} \sup_{S_R} |w|^2 \leqslant& c_{10}\Big(R^{-2}\int_{Q(R/2,3R/2)} w^2\,dx + R^2\int_{Q(R/2,3R/2)} (\Delta w)^2\,dx\Big) \\ \leqslant& c_{11}\Big(\int_{Q(R/2,3R/2)} |\nabla w|^2\,dx + \ln^{-4} R\Big) \to 0, \quad R \to \infty. \end{aligned}$$

The proof is complete. □

Lemma 7. *Let $u(x)$ be a solution to equation* (1) *in Q obeying $\overline{u}(R) \sim -2\ln R$. Then for each $\varepsilon > 0$ and each $R \geqslant R_1(\varepsilon)$ the estimate*

$$\overline{u}(R) \geqslant -2\ln R - 2\ln\ln R + \ln 2 - \varepsilon$$

holds true.

Proof. We arguing in the same was as in the proof of Lemma 5. At that, integral $\int_{S_R} e^u\, ds$ should be estimated from above instead of from below and instead of integral Jensen's inequality one needs to employ small deviation $u(x)$ from its mean over circumference S_R established in Lemma 6.

We assume that for each $R \geqslant R_1$ the inequality

$$\overline{u}(R) < -2\ln R - 2\ln\ln R + \ln 2 - \varepsilon \tag{19}$$

holds true. Then for each $R \geqslant R_2$ we have $u(x) < -2\ln R - 2\ln\ln R + \ln 2 - \varepsilon/2$,

$$\int_{Q(R,\infty)} e^u\, dx \leqslant 2\pi M_1 \int_R^\infty \frac{dr}{r\ln^2 r} = \frac{2\pi M_1}{\ln R}, \quad M_1 = \mathrm{const} < 2.$$

It yields

$$\overline{u}'(R) = \frac{1}{2\pi R} P(R,u) = \frac{1}{2\pi R}\left(-4\pi - \int_{Q(R,\infty)} e^u\, dx \right) \geqslant -\frac{2}{R} - \frac{M_1}{R\ln R}$$

that contradicts to (19). Hence, (19) can not hold true for each $R \geqslant R_1$. By analogy with the proof of Lemma 5 let us show that the function $z(R) = \overline{u}(R) + 2\ln R + 2\ln\ln R - \ln 2$ can not have negative minima separated uniformly from zero. Indeed, at such minimum we obtain

$$z''(R) = \frac{1}{2\pi R}\int_{S_R} e^u\, ds - \frac{2}{R^2\ln^2 R} < 0$$

for sufficiently great R that is impossible at a minimum. The proof is complete. □

Thus, Theorem 2 and Lemmata 5–7 imply immediately the main result of the work.

Theorem 3. *As* $|x| \to \infty$, *each solution to equation* (1) *in* Q *behaves either as*
1) $u(x) = C\ln|x| + C_1 + o(1)$, $C = \mathrm{const} < -2$; $C_1 = \mathrm{const}$;
or as
2) $u(x) = -2\ln|x| - 2\ln\ln|x| + \ln 2 + o(1)$.

Examples of solutions to equation (1) behaving at infinity in accordance with the first or second options are the solutions $u = -\ln|x| - 2\ln\big(|x|-1\big) + \ln 2$ and $u = -2\ln|x| - 2\ln\ln|x| + \ln 2$, respectively.

In conclusion, we mention that since in the multidimensional case ($n \geqslant 3$) equation (1) has no solutions in exteriors of a ball [8], the problem on finding the asymptotics for solutions to (1) in the exterior domains is restricted by the two-dimensional case.

BIBLIOGRAPHY

1. I.N. Vekua. *Some properties of solutions of Gauss equation* // Trudy Mat. Inst. Steklov. **64**, 5-8 (1961). (in Russian).
2. L. Bieberbach. $\Delta u = e^u$ *und die automorphen Funktionen* // Math. Ann. **77**:2, 173-212 (1916).
3. H. Rademacher. *Die Differential- und Integralgleichungen der Mechanik und Physik.* Braunschweig, Vieweg (1935).
4. O.A. Oleinik. *On the equation* $\Delta u + k(x)e^u = 0$ // Uspekhi Matem. Nauk. **33**:2, 204-205 (1978). [Russ. Math. Surv. **33**:2, 243-244 (1978).]
5. J.N. Flavin, R.J. Knops, L.E Payne. *Asymptotic behavior of solutions to semi-linear elliptic equations on the half-cylinder* // Z. Angew. Math. Phys. **43**:3, 405-421 (1992).
6. H. Usami. *Note on the inequality* $\Delta u \geqslant k(x)e^u$ *in* $\mathbb{R}^n$ // Hiroshima Math. J. **18**:3, 661-668 (1988).
7. Kuo-Shung Cheng, Chang-Shou Lin. *On the conformal gaussian curvature equation in* $\mathbb{R}^2$ // J. Diff. Equat. **146**:1, 226-250 (1998).
8. A.V. Neklyudov. *On the absence of global solutions to the gauss equation and solutions in external areas* // Izv. VUZov. Matem. 1, 55-60 (2014). [Russ. Math. **58**:1, 47-51 (2014).]

9. V.A. Kondrat'ev, O.A. Oleinik. *On asymptotics for solutions to nonlinear elliptic equations* // Uspekhi Matem. Nauk. **48**:4, 184-185 (1993). [Russ. Math. Surv. **48**:4, (1993).]
10. O.A. Oleinik. *Some asymptotic problems in the theory of partial differential equations.* Cambridge Univ. Press, Cambridge, 1996.
11. A.I. Nasrullaev. *Asymptotics of solutions of Neumann's problem for the equation* $\Delta u - e^u = 0$ *in a semi-infinite cylinder* // Uspekhi Matem. Nauk. **50**:3, 161-162 (1995). [Russ. Math. Surv. **50**:3, 630-632 (1995).]
12. A.V. Neklyudov. *The behavior of solutions of semilinear elliptic equations of second order of the form* $Lu = e^u$ *in the infinite cylinder* // Matem. Zametki. **85**:3, 408-420 (2009). [Math. Notes. **85**:3-4, 397-408 (2009).]
13. A.V. Neklyudov. *The behavior of solutions of the nonlinear biharmonic equation in an unbounded domain* // Matem. Zametki. **95**:2, 248-256 (2014). [Math. Notes. **95**:1-2, 224-231 (2014).]
14. O.A. Oleinik, G.A. Iosif'yan. *On the behavior at infinity of solutions of second order elliptic equations in domains with noncompact boundary* // Matem. Sborn. **112**:4, 588-610 (1980). [Math. USSR. Sbornik. **40**:4, 527-548 (1980).]
15. O.A. Oleinik, G.A. Yosifian. *On the asymptotic behavior at infinity of solutions in linear elasticity* // Arch. Ration. Mech. Anal. **78**:1, 29-53 (1982).
16. A.V. Neklyudov. *On Neumann problem for high order divergent elliptic equations in unbounded domains close to a cylinder* // Trudy Sem. I.G. Petrovskogo. **16**, 191-217 (1991). (in Russian).
17. A.V. Neklyudov. *On solutions to Robin problem for Laplace equation in semi-infinite cylinder* // Bull. Udmurt Univ. 2, 48-58 (2013). (in Russian).
18. D. Gilbarg, N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer, Berlin (1983).
19. I. Kametaka, O.A. Oleĭnik. *On the asymptotic properties and necessary conditions for existence of solutions of nonlinear second order elliptic equations* // Matem. Sborn. **107**:4, 572-600 (1978). [Math. USSR. Sbornik. **35**:6, 823-849 (1979).]

Alexei Vladimirovich Neklyudov,
Bauman Moscow State Technical University,
Rubtsovskaya quay, 2/18,
105005, Moscow, Russia
E-mail: `nekl5@yandex.ru`