BEHAVIOR OF SOLUTIONS TO
GAUSS-BIEBERBACH-RADEMACHER
EQUATION ON PLANE

A.V. NEKLYUDOV

Abstract. We study the asymptotic behavior at infinity of solutions to Gauss-Bieberbach-
Rademacher equation $\Delta u = e^u$ in the domain exterior to a circle on the plane. We establish
that the leading term of the asymptotics is a logarithmic function tending to $-\infty$. We also
find the next-to-leading term for various values of the coefficient in the leading term.

Keywords: Semilinear elliptic equations, Gauss-Bieberbach-Rademacher equation,
asymptotic behavior of solutions.

Mathematics Subject Classification: 35J15, 35J61, 35J91

1. Introduction

The equation

$$\Delta u = e^u, \quad (1)$$

appears as a model one in problems of differential geometry in relation with existence of surfaces
of negative Gaussian curvature [1], the theory of automorphic functions [2], in studying the
equilibrium of a charged gas [3]. Existence of solutions to equations like (1) in unbounded
domains, in particular, existence of global solutions, was considered in works [1], [4]–[8]. In
particular, it is well-known [1] that equation (1) has no global solutions for any number of
independent variables $n$, while for $n \geq 3$ there exist no solutions defined in the exterior of a
bounded domain [8]. The behavior at infinity of solutions to semi-linear elliptic equations with
an exponential nonlinearity was studied mostly for cylindrical domains [9]–[13]. In the present
paper we study the asymptotic behavior of solutions to two-dimensional equation (1) defined
in the exterior of a circle. We employ the method of energy estimates of Saint-Venant principle
kind [14]–[17] as well as the averaging principle.

We consider equation (1) in the two-dimensional domain $Q = \{x : |x| > R_0\} \subset \mathbb{R}^2$, where
$x = (x_1, x_2)$, $\Delta$ is the two-dimensional Laplace operator. We assume that $u \in C^2(Q)$.

We introduce notations. The mean value of function $u(x)$ on the circumference
$S_R = \{x : |x| = R\}$ is denoted by

$$\bar{u}(R) = \frac{1}{2\pi R} \int_{S_R} u \, ds,$$

the “heat flow” of function $u(x)$ through $S_R$ is indicated as

$$P(R, u) = \int_{S_R} \frac{\partial u}{\partial \nu} \, ds = 2\pi R \bar{u}'(R), \quad (2)$$

A.V. NEKLYUDOV, Behavior of solutions to Gauss-Bieberbach-Rademacher equation on plane.
© NEKLYUDOV A.V. 2014.
where $\nu$ is the unit outward normal to $S_R$. Let $Q(a,b) = \{ x : a < |x| < b \}$, $0 < R_0 \leq a < b$. It is obvious that solution $u(x)$ to equation (1) in $Q$ satisfies the identity

$$P(b, u) = P(a, u) + \int_{Q(a,b)} e^u \, dx. \tag{3}$$

We shall also make use of the notation $\nabla u \equiv \text{grad } u$. The condition $f/g \to 1$ as the arguments of functions $f$ and $g$ tend to some value will be indicated by a standard notation: $f \sim g$.

2. MAIN RESULTS

Theorem 1. Let $u(x)$ be a solution to equation (1) in $Q$. The relations

$$\int_Q e^u \, dx < \infty; \quad P(R, u) \to 2\pi C, \quad \pi(R) \sim C \ln R, \quad R \to \infty, \quad C = \text{const} \leq -2.$$  

hold true.

Proof. It follows from (2) and (3) that

$$P(R, u) = 2\pi R\pi'(R) = P(R_0, u) + \int_{Q(R_0,R)} e^u \, dx. \tag{4}$$

Let us show that the right hand side of identity (4) is negative for each $R > R_0$. Suppose the opposite, then for $R > R_1 = \text{const} > R_0$ we obtain

$$R\pi'(R) > c_1 > 0.$$  

Hereinafter by $c_j$ we indicate positive constants depending only on a considered solution to (1) and independent of $R, a, b, t,$ etc. As $R > R_2 = \text{const} > R_1$, it implies

$$\pi(R) > c_2 \ln R.$$  

By integral Jensen’s inequality it yields

$$\int_{S_R} e^u \, ds \geq 2\pi R e^{\pi(R)} > 2\pi R^{c_2+1}, \quad \int_{Q(R_0,R)} e^u \, dx > c_3 R^{c_2+2}, \quad R > R_3 = \text{const} > R_2.$$  

Using (4) once again and integrating, we get

$$R\pi'(R) > c_4 R^{c_2+2}, \quad \pi(R) > c_5 R^{c_2+2}, \quad R > R_4 = \text{const} > R_3.$$  

Finally, employing once again (4) and Jensen’s inequality, for $R > R_5 = \text{const}$ we have

$$\pi'(R) \geq c_6 + \frac{1}{2\pi R} \int_{Q(R_0,R)} e^u \, dx \geq c_6 + \frac{1}{R} \int_{R_0}^R r e^{\pi(r)} \, dr > \left( \int_{R_0}^R r e^{\pi(r)} \, dr \right)^{1/2}.$$  

Let $\int_{R_0}^R r e^{\pi(r)} \, dr = z(R)$, then $\pi(R) = \ln z'(R)$ and the latter inequality can be written as

$$\frac{z''}{z'} > z^{1/2}$$  

that for $R > R_6$ follows

$$z' > c_7 z^{3/2}.$$  

It implies easily that $z(R) \to \infty, \ R \to R_7 - 0$ for some $R_7 > R_6$. This is impossible for a solution defined for $|x| > R_0$. Thus, the obtained contradiction means that the right hand side in (4) is negative for each $R > R_0$ that implies immediately the first statement of the theorem.

It follows from (4) that

$$P(R, u) \to 2\pi C, \quad \pi(R) \sim C \ln R, \quad R \to \infty, \quad C = \text{const} \leq 0.$$  

Jensen’s inequality yields

$$\int_{R_0}^\infty r e^{\pi(r)} \, dr \leq \frac{1}{2\pi} \int_Q e^u \, dx < \infty.$$
It follows that $C \leq -2$. The proof is complete.

**Lemma 1.** Let $f \in C^1(\overline{Q}) \cap L_1(Q)$,

$$
\int_{R_0}^{\infty} r \left( \int_{S_r} |f| \, dx \right)^2 \, dr < \infty.
$$

Then there exists a solution $V(x)$ to equation

$$
\Delta V = f
$$
in $Q$ satisfying the estimates

$$
\int_{Q(R_0,R)} |\nabla V|^2 \, dx \leq c_0 \ln R, \quad |\overline{V}(R)| \leq c_1 \ln R \tag{5}
$$
as $R > R_1 = \text{const} > R_0$. At that, if $f > 0$ in $Q$, then $V \leq 0$ in $Q$.

If in addition the conditions

$$
\int_{R_0}^{\infty} \frac{dr}{r} \int_{Q(r,\infty)} |f| \, dx < \infty, \quad \int_{Q} |x|^2 f^2 \, dx < \infty \tag{6}
$$
hold true, then

$$
\int_{Q} |\nabla V|^2 \, dx < \infty, \quad V(x) \to C = \text{const}, \quad |x| \to \infty.
$$

**Proof.** For each natural $N > R_0$, in domain $Q(R_0,N)$ we consider solution $V_N$ to the boundary value problem

$$
\Delta V_N = f, \quad V_N|_{S_{R_0}} = 0, \quad \frac{\partial V_N}{\partial n}|_{S_N} = C_N,
$$
where

$$
C_N = \frac{-1}{2\pi N} \int_{Q(N,\infty)} f \, dx.
$$

It is clear that as $N \geq R > R_0$,

$$
P(R,V_N) = P(N,V_N) - \int_{Q(R,N)} f \, dx = - \int_{Q(R,\infty)} f \, dx. \tag{7}
$$

In view of the identity $2\pi R \overline{V}_N'(R) = P(R,V_N)$ we obtain that as $N \geq R >> R_0$

$$
|\overline{V}_N'(R)| \leq c_2 \ln R. \tag{8}
$$

Let us estimate the Dirichlet integral for solution $V_N$. It is obvious that

$$
\int_{Q(R_0,N)} |\nabla V_N|^2 \, dx = C_N \int_{S_{R_0}} V_N \, ds - \int_{Q(R_0,N)} f V_N \, dx. \tag{9}
$$

Let us estimate the integrals in the right hand side of (9). Due to (8) we get

$$
\left| C_N \int_{S_N} V_N \, ds \right| = 2\pi N |C_N V_N(N)| \leq c_3 \ln N. \tag{10}
$$

Since by the embedding theorem for functions of one variable and Poincaré inequality

$$
\sup_{S_r} |V_N - \overline{V}_N(r)| \leq c_4 r^{1/2} \left( \int_{S_r} |\nabla V_N|^2 \, ds \right)^{1/2},
$$
we have

$$
\left| \int_{S_r} f(V_N - \overline{V}_N(r)) \, ds \right| \leq c_4 r^{1/2} \left( \int_{S_r} |\nabla V_N|^2 \, ds \right)^{1/2} \int_{S_r} |f| \, ds
$$

$$
\leq \frac{1}{2} \int_{S_r} |\nabla V_N|^2 \, ds + c_3 r \left( \int_{S_r} |f| \, ds \right)^2. \tag{11}
$$
By (8) we get
\[ |V_N(r) \int_{S_r} f \, ds| \leq c_6 \ln r \int_{S_r} |f| \, ds. \tag{12} \]

It follows from (9)-(12) that
\[ \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx \leq c_3 \ln N + \frac{1}{2} \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx \]
\[ + c_5 \int_{R_1}^N r \left( \int_{S_r} |f| \, ds \right)^2 \, dr + c_6 \ln N \int_{Q(R_1, N)} |f| \, dx. \]

Thus, we obtain
\[ \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx \leq c_7 \ln N. \tag{13} \]

Let us estimate the Dirichlet integral for function \( V_N \) over domain \( Q(R_0, R) \) for arbitrary \( R \in (R_0, N) \):
\[ I(R) \equiv \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx = \int_{S_R} \frac{\partial V_N}{\partial V} V_N \, ds - \int_{Q(R_0, R)} fV_N \, dx. \]

Estimating the second term in accordance with (11)-(12), for \( R \geq R_1 > R_0 \) we get
\[ \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx \leq c_8 \ln R + \frac{1}{2} \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx + \int_{S_R} \frac{\partial V_N}{\partial V} V_N \, ds. \]

Employing Poincaré inequality and (7), (8), we arrive at
\[ I(R) \equiv \int_{Q(R_0, R)} |\nabla V_N|^2 \, dx \leq 2 \int_{S_R} \frac{\partial V_N}{\partial V} V_N \, ds + 2c_8 \ln R \]
\[ = 2P(R, V_N)\nabla N(R) + 2 \int_{S_R} \frac{\partial V_N}{\partial V} (V_N - \nabla N(R)) \, ds + 2c_8 \ln R \]
\[ \leq c_9 \left( R \int_{S_R} |\nabla V_N|^2 \, ds + \ln R \right) = c_9 (RI'(R) + \ln R). \]

We integrate this inequality from \( R \) to \( N \geq R^2 \) to obtain by (13) that
\[ I(R) \leq I(N) \left( \frac{R}{N} \right)^\delta + c_{10} R^\delta \int_{R}^{N} \frac{\ln r}{r^{\frac{\delta}{\delta + 1}}} \, dr \leq c_{11} \ln R, \quad \delta > 0. \]

Thus, for each fixed \( R > R_0 \), sequence \( V_N \) is uniformly bounded in Sobolev space \( W^1_2(Q(R_0, R)) \).

Applying standard diagonal process, we obtain a sequence \( V_{N_k} \) converging to some function \( V \) weakly in \( W^1_2(Q(R_0, R)) \) and strongly in \( L^2(Q(R_0, R)) \) for each \( R > R_0 \). Since \( V_{N_k} - V_{N_k} \) are harmonic functions, the convergence of these functions and of their derivatives is uniform in \( Q(R_0, R) \). Thus, function \( V \) satisfies equation (1) and in view of (8) it satisfies also estimates (5).

If \( f > 0 \) in \( Q \), by the maximum principle one can see easily that \( V_N < 0 \) in \( Q(R_0, N) \) and \( V \leq 0 \) in \( Q \).

Let function \( f \) satisfies also conditions (6). Then it follows from (6) and (7) that
\[ \int_{R_0}^\infty |\nu(r)| \, dr = \frac{1}{2\pi} \int_{R_0}^\infty \frac{|P(r, V)|}{r} \, dr < \infty, \quad \nu(R) \rightarrow C_0 = \text{const}, \quad R \rightarrow \infty. \]

In the same way (7) also follows the uniform boundedness of \( |\nabla V_N(R)| \). Hence, by estimates similar to (9)-(12), we obtain
\[ \int_{Q(R_0, N)} |\nabla V_N|^2 \, dx \leq c_{12} \]

A.V. NEKLYUDOV
that implies the finiteness of the Dirichlet integral over $Q$ for $V$.

Let us show that in this case $V(x) \to C_0$, $|x| \to \infty$. As $R > 2R_0$, in accordance with De Giorgi type estimates [18] and Poincaré inequality for $Q$ that implies the finiteness of the Dirichlet integral over $V$.

Due to Theorem 1 and Lemma 2 it is sufficient to prove that $\int_{Q(R/2,3R/2)} |V(x) - \nabla(R)|^2 \leq c_{13} \left( R^{-2} \int_{Q(R/2,3R/2)} (V(z) - \nabla(R))^2 \, dz + R^2 \int_{Q(R/2,3R/2)} f^2(z) \, dz \right) \leq c_{14} \left( \int_{Q(R/2,3R/2)} \left| \nabla V(z) \right|^2 \, dz + \int_{Q(R/2,3R/2)} \left| z \right|^2 f^2(z) \, dz \right) \to 0$, $R \to \infty$.

Since $\nabla(R) \to C_0$, $R \to \infty$, we obtain that $V(x) \to C_0$, $|x| \to \infty$. The proof is complete.

**Lemma 2.** Let $g(r) > 0$ be a non-increasing on $[r_0, \infty)$ measurable function and

$$\int_{r_0}^{\infty} r g(r) \, dr < \infty.$$ 

Then

$$\int_{r_0}^{\infty} r^3 g^2(r) \, dr < \infty.$$ 

**Proof.** It follows easily from monotonicity of $g(r)$ that $g(r) \leq r^{-2}$ as $r > r_1 = \text{const} > 0$. Then $r^3 g^2(r) \leq rg(r)$, $r > r_1$, that implies the statement of the lemma.

**Lemma 3.** Let $u(x)$ be a solution to equation (1) in $Q$. Then

$$\int_{S_r} e^u \, ds \leq c_0, \quad \int_{R_0}^{\infty} r \left( \int_{S_r} e^u \, ds \right)^2 \, dr < \infty.$$ 

**Proof.** Due to Theorem 1 and Lemma 2 it is sufficient to prove that $g'(r) < 0$ for each $r > R_0$, where

$$g(r) = r^{-1} \int_{S_r} e^u \, ds.$$ 

We have

$$g'(r) = r^{-1} \int_{S_r} e^u \frac{\partial u}{\partial \nu} \, ds.$$ 

We assume that $g'(r) \geq 0$ for some $r_1 > R_0$ and we choose an arbitrary $r > r_1$. Let $\theta = \theta(|x|) \geq 0$ be a cut-off function belonging to $C^2$ such that $\theta(|x|) = 1$ as $|x| \leq r$, $\theta(|x|) = 0$ as $|x| \geq r + 1$, $(\theta'(|x|))^2 \leq c_1 \theta(|x|)$ as $r \leq |x| \leq r + 1$, $c_1 = \text{const} > 0$. Multiplying both sides of equation (1) by $e^\theta$ and integrating over domain $Q(r_1, r + 1)$, we obtain

$$\int_{Q(r_1, r+1)} \left( e^{2u} + |\nabla u|^2 e^u \right) \theta \, dx = -r_1 g'(r_1) - \int_{Q(r, r+1)} e^u \frac{\partial u}{\partial \nu} \theta' \, dx \leq \int_{Q(r, r+1)} e^u \left( |\nabla u|^2 \theta + c_2 \right) \, dx.$$ 

Hence,

$$\int_{Q(r_1, r)} e^{2u} \, dx \leq c_2 \int_{Q(r, r+1)} e^u \, dx \to 0, \quad r \to \infty,$$

that is impossible. This contradiction shows that $g'(r) < 0$ for each $r > R_0$ and it completes the proof.

**Theorem 2.** Let $u(x)$ be a solution to equation (1) in $Q$ obeying

$$u(R) \sim C \ln R, \quad C = \text{const} < -2, \quad R \to \infty.$$ 

Then

$$u(x) = C \ln |x| + C_1 + o(1), \quad |x| \to \infty, \quad C_1 = \text{const}.$$
Proof. Let us prove first that for each \( \varepsilon > 0 \) as \( |x| > R_1 = R_1(\varepsilon) \), the estimate
\[
u(x) \leq (C + \varepsilon) \ln |x|
\]
holds true. We observe that by Theorem 1 and Lemma 3, function \( f(x) = e^{u(x)} \) satisfies the hypothesis of Lemma 1, except, probably, conditions [6]. We consider the harmonic function \( U = u - V \), where \( V \) is the solution to equation \( \Delta V = e^u \), the existence of which was established in Lemma 1. Let us estimate the Fourier coefficients w.r.t. \( \varphi \) for function \( U \) on circumference \( S_r \). Since by Lemma 3 and Theorem 1
\[
\int_0^{2\pi} |u(r, \varphi)| d\varphi = 2 \int_0^{2\pi} u^+(r, \varphi) d\varphi - 2\pi \overline{u}(r, \varphi) \leq 2 \int_0^{2\pi} e^{u(r, \varphi)} d\varphi + c_1 \ln r \leq c_2 \ln r,
\]
where \( u^+ = \max\{u, 0\} \), employing estimates \( |\nabla(r)| \leq c_3 \ln r, V \leq 0 \) and Lemma 3, we obtain
\[
\int_0^{2\pi} |U(r, \varphi)| d\varphi \leq \int_0^{2\pi} \left( |u(r, \varphi)| + |V(r, \varphi)| \right) d\varphi \leq c_4 \ln r, \quad r \geq r_1 = \text{const} > R_0.
\]
Hence, the expansion of \( U \) into the Fourier series w.r.t. \( \varphi \) reads as
\[
U = a_0 \ln r + b_0 + \sum_{k=1}^{\infty} r^{-k}(a_k \cos k\varphi + b_k \sin k\varphi).
\]
Then in view of the estimate for the Dirichlet integral of \( V \) in Lemma 1 we obtain
\[
\int_{Q(R_0, R)} |\nabla u|^2 dx \leq c_4 \ln R. \tag{14}
\]
We fix \( \varepsilon > 0 \) such that \( C + \varepsilon < -2 \). As \( R > R_2 = R_2(\varepsilon) \),
\[
\overline{u}(R) \leq (C + \varepsilon/2) \ln R.
\]
By (14) for each \( R > 2R_2 \) there exists \( r_1 \in (R/2, R) \) satisfying
\[
\int_{S_{r_1}} |\nabla u|^2 ds \leq 2c_4 \frac{\ln R}{R}.
\]
Then, by the embedding theorem and Poincaré inequality, for \( x \in S_{r_1} \) we get the estimate
\[
u(x) - (C + \varepsilon/2) \ln R < (u(x) - \overline{u}(r_1)) + (\overline{u}(r_1) - (C + \varepsilon/2) \ln r_1)
\]
\[
\leq c_5 r_1^{1/2} \int_{S_{r_1}} |\nabla u|^2 ds \leq c_6 \ln^{1/2} R,
\]
\[
u(x) \leq (C + \varepsilon) \ln |x|, \quad R > R_3(\varepsilon).
\]
By analogy, the same inequality holds true as \( x \in S_{r_2} \) for some \( r_2 \in (R, 3R/2) \) provided \( R \) is great enough. In accordance with the maximum principle, this inequality holds true in \( Q(r_1, r_2) \) and, in particular, as \( |x| = R \). Hence, for \( |x| > R_4(\varepsilon) \) we have
\[
u(x) \leq (C + \varepsilon) \ln |x|.
\]
It follows that \( e^{u(x)} \leq c_7 |x|^{-2-\delta} \) in \( Q, \delta > 0 \). Thus, function \( f(x) = e^{u(x)} \) satisfies conditions [6]. It yields that function \( V \to C_0, |x| \to \infty \). Hence,
\[
u(x) = U + V = C \ln |x| + C_1 + o(1), \quad C < -2, \quad |x| \to \infty.
\]
The proof is complete. \( \square \)

We proceed to the case \( C = -2 \), i.e., \( \overline{u}(R) \sim -2 \ln R \). It is clear that a direct analogue of theorem 2 does not hold, since by Theorem 1 \( \int_Q e^{u} dx < \infty \) and therefore, solution can not be represented as \( u(x) = -2 \ln |x| + C_1 + o(1) \).
Lemma 4. Let $u(x)$ be a solution to equation (1) in $Q$. Then the Dirichlet integral for function $w(x) = u(x) - \pi(|x|)$ over $Q$ is finite:

\[ \int_Q |\nabla w|^2 \, dx < \infty. \]

Proof. Since $\Delta(\pi(|x|)) = \Delta u(|x|)$ \[19\], function $w$ solves the equation

\[ \Delta w = h(x) \equiv e^u - e^{\pi}. \]

We have

\[ \int_{Q(R_0,R)} |\nabla w|^2 \, dx = - \int_{Q(R_0,R)} hw \, dx + \int_{S_R} w \frac{\partial w}{\partial \nu} \, ds - \int_{S_{R_0}} w \frac{\partial w}{\partial \nu} \, ds. \]  \hspace{1cm} (15)

Function $h(x)$ satisfies the estimate like \[11\]:

\[ \left| \int_{Q(R_0,R)} hw \, dx \right| \leq \frac{1}{2} \int_{Q(R_0,R)} |\nabla w|^2 dx + c_1 \int_{R_0}^R \left( \int_{S_r} |h| \, ds \right)^2 dr. \]  \hspace{1cm} (16)

By Lemma 3,

\[ \int_{R_0}^\infty r \left( \int_{S_r} |h| \, ds \right)^2 \, dr < \infty. \]  \hspace{1cm} (17)

It follows from \[13\]–\[17\] that

\[ \int_{Q(R_0,R)} |\nabla w|^2 \, dx \leq 2 \int_{S_R} w \frac{\partial w}{\partial \nu} \, ds + c_2. \]

Applying Cauchy-Schwarz inequality and taking into consideration that $\pi(R) = 0$ and Poincaré inequality, we obtain

\[ J(R) = \int_{Q(R_0,R)} |\nabla w|^2 \, dx \leq 2 \left( \int_{S_R} w^2 \, ds \right)^{1/2} \left( \int_{S_R} |\nabla w|^2 \, ds \right)^{1/2} + c_2 \leq c_3 R \int_{S_R} |\nabla w|^2 \, ds + c_2 \equiv c_3 RJ'(R) + c_2. \]

It implies that either function $J(R)$ is bounded or it grows faster than $\ln R$. The latter is impossible by \[11\]. The proof is complete. \[ \square \]

Lemma 5. Let $u(x)$ be a solution to \[1\] in $Q$ and $\pi(R) \sim -2 \ln R$, $R \to \infty$. Then for each $\varepsilon > 0$ and each $R \geq R_1(\varepsilon)$ the estimate

\[ \pi(R) \leq -2 \ln R - 2 \ln \ln R + \ln 2 + \varepsilon \]

holds true.

Proof. Let us prove first that the inequality

\[ \pi(R) > -2 \ln R - 2 \ln \ln R + \ln 2 + \varepsilon \]  \hspace{1cm} (18)

can not be true for each $R \geq R_1 = \text{const} \geq R_0$. Suppose the opposite and let (18) is valid for some $\varepsilon > 0$ and for each sufficiently great $R$. Then

\[ \int_{Q(R,\infty)} e^u \, dx \geq 2\pi \int_r^\infty r e^{\pi(r)} \, dr \geq 2\pi M_0 \int_R^\infty \frac{dr}{r \ln^2 r} = \frac{2\pi M_0}{\ln R}, \quad M_0 = \text{const} > 2. \]

Since $P(R,u) \to -4\pi$, $R \to \infty$, we obtain by taking into consideration \[3\]

\[ \pi'(R) = \frac{1}{2\pi R} P(R,u) = \frac{1}{2\pi R} \left( -4\pi - \int_{Q(R,\infty)} e^u \, dx \right) \leq -\frac{2}{R} - \frac{M_0}{R \ln R} \]

for each $R > R_1$ that contradicts inequality \[18\]. Hence, \[18\] can not hold true simultaneously for each $R$ starting from some $R_1$. It means that the limit inferior of the function $z(R) =$
\( \pi(R) + 2 \ln R + 2 \ln \ln R - \ln 2 \) as \( R \to \infty \) is non-positive. In order to prove the statement of the lemma, it is sufficient that \( \zeta(R) \) has no positive local maxima. If there exists a maximum point \( R \), then

\[
0 = z'(R) = \pi'(R) + \frac{2}{R} + \frac{2}{R \ln R} = \frac{P(R, u)}{2\pi R} + \frac{2}{R} + \frac{2}{R \ln R}. 
\]

Then at this point

\[
z''(R) = \pi''(R) - \frac{2}{R^2} - \frac{2(1 + \ln R)}{R^2 \ln^2 R} 
= \frac{1}{2\pi} \left( - \frac{P(R, u)}{R} \right) - \frac{2}{R^2} - \frac{2(1 + \ln R)}{R^2 \ln^2 R} 
= \frac{2}{R^2} + \frac{2}{R^2 \ln R} + \frac{1}{2\pi R} \int_{S_R} e^u \, ds - \frac{2}{R^2} - \frac{2(1 + \ln R)}{R^2 \ln^2 R} > e^{\pi(u)} - \frac{2}{R^2 \ln^2 R} > 0. 
\]

Hence, \( z''(R) > 0 \) that is impossible at a maximum. The proof is complete. \( \square \)

**Lemma 6.** Let \( u(x) \) be a solution to \( \{1\} \) in \( Q \), \( \pi(R) \sim -2 \ln R \), \( R \to \infty \). Then

\[
u(x) = \pi(|x|) + o(1), \quad |x| \to \infty. 
\]

**Proof.** We fix arbitrary \( R > 2R_0 \). By Lemma 4, for some \( r_1 \in (R/2, R) \) the estimate

\[
\int_{S_{r_1}} |\nabla w|^2 \, ds \leq \frac{c_1}{R}
\]

holds true, where \( w(x) = u(x) - \pi(|x|) \). Hence,

\[
\sup_{S_{r_1}} |w| \leq c_2 r_1^{1/2} \left( \int_{S_{r_1}} |\nabla w|^2 \, ds \right)^{1/2} \leq c_3.
\]

Thus, employing Lemma 5, we obtain that for each \( x \in S_{r_1} \)

\[
u(x) \leq \pi(|x|) + c_3 \leq -2 \ln R - 2 \ln \ln R + c_4.
\]

In the same way, for some \( r_2 \in (R, 3R/2) \) we have

\[
u(x) \leq \pi(|x|) + c_5 \leq -2 \ln R - 2 \ln \ln R + c_6
\]

as \( x \in S_{r_2} \). In accordance with the maximum principle, for each \( x \in S_R \) we obtain

\[
u(x) \leq -2 \ln |x| - 2 \ln \ln |x| + c_7,
\]

hence,

\[
e^\nu(x) \leq \frac{c_8}{|x|^2 \ln^2 |x|}, \quad |\Delta w| \leq \frac{c_9}{|x|^2 \ln^2 |x|}.
\]

In accordance with De Giorgi estimate and Poincaré inequality we obtain

\[
\sup_{S_R} |w|^2 \leq c_{10} \left( R^{-2} \int_{Q(R/2, 3R/2)} w^2 \, dx + R^2 \int_{Q(R/2, 3R/2)} (\Delta w)^2 \, dx \right) 
\leq c_{11} \left( \int_{Q(R/2, 3R/2)} |\nabla w|^2 \, dx + \ln^{-4} R \right) \to 0, \quad R \to \infty.
\]

The proof is complete. \( \square \)

**Lemma 7.** Let \( u(x) \) be a solution to equation \( \{1\} \) in \( Q \) obeying \( \pi(R) \sim -2 \ln R \). Then for each \( \varepsilon > 0 \) and each \( R \geq R_1(\varepsilon) \) the estimate

\[
\pi(R) \geq -2 \ln R - 2 \ln \ln R + \ln 2 - \varepsilon
\]

holds true.
Proof. We arguing in the same was as in the proof of Lemma 5. At that, integral \( \int_{S_R} e^u \, ds \) should be estimated from above instead of from below and instead of integral Jensen’s inequality one needs to employ small deviation \( u(x) \) from its mean over circumference \( S_R \) established in Lemma 6.

We assume that for each \( R \geq R_1 \) the inequality
\[
\overline{u}(R) < -2 \ln R - 2 \ln \ln R + \ln 2 - \varepsilon
\]  
holds true. Then for each \( R \geq R_2 \) we have \( u(x) < -2 \ln R - 2 \ln \ln R + \ln 2 - \varepsilon/2, \)
\[
\int_{Q(R,\infty)} e^u \, dx \leq 2\pi M_1 \int_R^\infty \frac{dr}{r \ln^2 r} = \frac{2\pi M_1}{\ln R}, \quad M_1 = \text{const} < 2.
\]
It yields
\[
\overline{u}(R) = \frac{1}{2\pi R} P(R, u) = \frac{1}{2\pi R} \left(-4\pi - \int_{Q(R,\infty)} e^u \, dx\right) \geq \frac{-2}{R} - \frac{M_1}{R \ln R}
\]
that contradicts to (19). Hence, (19) can not hold true for each \( R \geq R_1 \). By analogy with the proof of Lemma 5 let us show that the function \( z(R) = \overline{u}(R) + 2 \ln R + 2 \ln \ln R - \ln 2 \) can not have negative minima separated uniformly from zero. Indeed, at such minimum we obtain
\[
z''(R) = \frac{1}{2\pi R} \int_{S_R} e^u \, ds - \frac{2}{R^2 \ln^2 R} < 0
\]
for sufficiently great \( R \) that is impossible at a minimum. The proof is complete. \( \Box \)

Thus, Theorem 2 and Lemmata 5–7 imply immediately the main result of the work.

**Theorem 3.** As \( |x| \to \infty \), each solution to equation (1) in \( Q \) behaves either as
1) \( u(x) = C \ln |x| + C_1 + o(1), \quad C = \text{const} < -2; \ C_1 = \text{const}; \)
or as
2) \( u(x) = -2 \ln |x| - 2 \ln \ln |x| + \ln 2 + o(1). \)

Examples of solutions to equation (1) behaving at infinity in accordance with the first or second options are the solutions \( u = -\ln |x| - 2 \ln (|x| - 1) + \ln 2 \) and \( u = -2 \ln |x| - 2 \ln \ln |x| + \ln 2 \), respectively.

In conclusion, we mention that since in the multidimensional case \( (n \geq 3) \) equation (1) has no solutions in exteriors of a ball \( S \), the problem on finding the asymptotics for solutions to (1) in the exterior domains is restricted by the two-dimensional case.

**BIBLIOGRAPHY**


Alexei Vladimirovich Neklyudov,
Bauman Moscow State Technical University,
Rubtsovskaya quay, 2/18,
105005, Moscow, Russia
E-mail: nekl15@yandex.ru