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HELLY'S THEOREM AND SHIFTS OF SETS. I

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Abstract. The motivation for the considered geometric problems is the study of conditions under which an exponential system is incomplete in spaces of the functions holomorphic in a compact set C and continuous on this compact set. The exponents of this exponential system are zeroes for a sum (finite or infinite) of families of entire functions of exponential type. As C is a convex compact set, this problem happens to be closely connected to Helly's theorem on the intersection of convex sets in the following treatment. Let C and S be two sets in a finite-dimensional Euclidean space being respectively intersections and unions of some subsets. We give criteria for some parallel translation (shift) of set C to cover (respectively, to contain or to intersect) set S. These and similar criteria are formulated in terms of geometric, algebraic, and set-theoretic differences of subsets generating C and S.

Keywords: Helly's theorem, incompleteness of exponential systems, convexity, shift, geometric, algebraic, and set-theoretic differences

Mathematics Subject Classification: 52A35, 52A20

1. Introduction

1.1. The origination of our study is the following problem, which for the sake of brevity we discuss here only in a simplified one-dimensional complex version. The detailed exposition incluging multi-dimensional situation is provided in the concluding section of the second part of the work.

We consider an at most countable sequence of points $\Lambda = \{\lambda_k\}_{k\geqslant 1}$ on the complex plane $\mathbb{C}$ with no accumulation points; in this sequence there can be repetitive points. To sequence Λ we associate a system of (multiple) exponents

$$\operatorname{Exp}^{\Lambda} := \{z^p e^{\lambda_k z} \colon z \in \mathbb{C}, 0 \leqslant p \leqslant n_{\Lambda}(\lambda_k) - 1\},$$

where $n_{\Lambda}(\lambda)$ is the number of repetition of point $\lambda \in \mathbb{C}$ in sequence Λ. By Zero_L we denote a *sequence of zeroes* taken counting multiplicity for a non-zero entire function L of exponential type. At that, $\Lambda \leqslant \operatorname{Zero}_L$ means $n_{\Lambda}(\lambda) \leqslant n_{\operatorname{Zero}_L}(\lambda)$ for each $\lambda \in \mathbb{C}$. A growth indicator

$$h(\theta, L) := \limsup_{t>0, t\to+\infty} \frac{\log |L(te^{i\theta})|}{t}, \quad \theta \in \mathbb{R}, \tag{1}$$

is a continuous 2π-periodic trigonometrically convex function [1]–[3] being the support function for a convex compact set (indicator diagram) or being the support function $h_S(\theta) \equiv h(-\theta, L)$ for the adjoint diagram S of function L. Let C be a compact set in $\mathbb{C}$ and we are given a sequence of non-zero entire functions $\{L_k\}$ of exponential type with adjoint diagrams S_k, $k = 1, 2, \dots$; the sum $\sum_k L_k$ is assumed to be an entire function L of exponential type. If there exists a shift of compact set C covering all the sets in the family $\mathcal{S} = \{S_1, S_2, \dots\}$ and L is non-zero, then for

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each sequence $\Lambda \leqslant \mathrm{Zero}_L$, system Exp^Λ is incomplete in space $\mathrm{Hol}(\Omega)$ of functions holomorphic in domain Ω for each domain $\Omega \supset C$ with the natural topology of uniform convergence on compact sets. The question appears: *under which conditions does the shift of compact set C cover the union of adjoint diagrams $\bigcup_k S_k$?* This problem happened to be solvable in various ways by means of Helly's theorem provided compact set C is convex. Having a perspective of further applications, we consider the cases when compact set C is given as the intersection of convex compact sets. The answers are given in terms of geometric differences of sets or in terms of support functions. For the purpose of forthcoming applications, in particular, for the theory of entire functions of completely regular growth, which we do not touch in the present work, as well as for the completeness of the exposition, we consider also the situations, when instead of geometric differences one employs algebraic or set-theoretical differences of sets.

The work is split into two parts that is natural since the first part has a pure geometric nature, while the second part is more algebraic and first of all, is of theoretic-functional character. The results of the work were partially announced on conferences [4]–[7].

1.2. Hereinafter $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers and for $n \in \mathbb{N}$, $\mathbb{R}^n$ stands for n-dimensional (vector or affine) Euclidean space with the usual scalar product $\langle \cdot, \cdot \rangle$. The elements of this space are either vectors or points. The symbol 0 indicates both zero and zero vector, as well as the origin.

In what follows, not indicating often the precise source, we employ the conventional terminology, notations and well-known facts [8]–[13]. For instance, $\mathrm{A} \times \mathrm{B} := \{(\alpha, \beta) \colon \alpha \in \mathrm{A}, \beta \in \mathrm{B}\}$ is the Cartesian product of spaces A, B. For a set S of arbitrary nature, $\operatorname{card} S$ stands for its *cardinality* S, i.e., for a finite set S it is its *number of elements*. For $S \subset \mathbb{R}^n$, by $\operatorname{int} S$ and $\operatorname{co} S$ we denote its *interior* and *convex hull*. For $x \in \mathbb{R}^n$ we let $|x| := \sqrt{\langle x, x \rangle}$, $x \in \mathbb{R}^n$, $B(x, r) := \bigl\{ y \in \mathbb{R}^n \colon |y - x| < r \bigr\}$ is the *open ball of radius $r \geqslant 0$ centered at x.*

Convention. *Once the matter is to choose a family of elements in a set or a subfamily of sets, it is convenient to assume that there can be repetitive elements or sets in this (sub)family.*

In the first part of the work we perform studies motivated by the classical Helly's theorem on convex sets established in the beginning of twentieth century. In one of the simplest forms, this theorem can be recalled as follows [8, Introduction]: *Let a family $\mathcal{S}$ of convex sets in $\mathbb{R}^n$ is finite or each set in this family is closed and bounded. If the intersection of each $n+1$ sets in $\mathcal{S}$ is non-empty, then the total intersection of all sets in $\mathcal{S}$ is non-empty as well* (in more details Helly's theorem on convex sets is formulated later in the beginning of Section 2). Some impression (very far from complete) on immense amount of publication on Helly's theorem can be got from the bibliography of the paper. If a discussion of a result or a work is given in surveys or monographs, we cite the latter. Here we develop one of key corollaries of Helly's theorem in various directions. Let us formulate it as a theorem.

We shall call the result of a parallel translation of a set $S \subset \mathbb{R}^n$ a *shift* of set S. The following important corollary of Helly's theorem was proven independently in various generality by P. Vincensini (1939) and V. Klee (1953), while relative issues were considered by M. Edelstein (1958).

VKE theorem (Vincensini–Klee–Edelstein [8, **2.1**]). *Assume that a family $\mathcal{S}$ of convex sets is finite or contains only compact sets, and $C \subset \mathbb{R}^n$ is convex, and, in addition, is bounded and closed if $\mathcal{S}$ is infinite. Then the existence of a shift of set C covering (similarly, intersecting or containing in) each set in $\mathcal{S}$ is ensured by the existence of such shift for each $n+1$ sets in family $\mathcal{S}$.*

2. Helly's theorems on convex sets

We shall need a series of modifications of classical Helly's theorem [8]–[13].

Definition 1. *A non-zero vector $y \in \mathbb{R}^n$ is called star-shapedness direction*[1] *for a set $C \subset \mathbb{R}^n$ (w.r.t. the infinity), or is called a recession direction if for each point $c \in C$ the ray*

$$r_y(c) := \{c + ty\colon t \geqslant 0\} \tag{2}$$

is contained in C. Vector $y \in \mathbb{R}^n$ is called linearity direction, if both y and the opposite vector $-y$ are star-shapedness directions for set C, i.e., for each point $c \in C$ the straight line

$$l_y(c) := \{c + ty\colon t \in \mathbb{R}\} = r_y(c) \cup \bigl(r_{-y}(c)\bigr) = l_{-y}(c)$$

is contained in C. Set C is polyhedral, if C is the intersection of a finite number of closed half-spaces defined by a finite system of linear inequalities

$$\langle a, x\rangle - b \leqslant 0 \quad \textit{for some} \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}. \tag{3}$$

It is clear that for the empty subset in $\mathbb{R}^n$ and for whole $\mathbb{R}^n$ each non-zero vector is the star-shapedness and linearity direction. Moreover, $\varnothing \subset \mathbb{R}^n$ and $\mathbb{R}^n$ are also polyhedral, since $\varnothing$ can be considered as a set of solutions to any finite inconsistent system of linear inequalities (3), and $\mathbb{R}^n = \{x \in \mathbb{R}^n\colon \langle 0, x\rangle \leqslant 0\}$.

Helly's theorem (on convex sets). *Suppose that a family*

$$\mathcal{C} := \{C_\alpha\colon \alpha \in \mathrm{A}\}, \quad \mathrm{A} \textit{ is a set of indices}, \tag{4}$$

of convex sets C_α in $\mathbb{R}^n$ satisfies one of the conditions

(f) *set of indices* A *is finite* (finiteness condition [8, Introduction])*;*

(d) *all the sets C_α, $\alpha \in \mathrm{A}$, are closed* (closedness condition) *and for some finite subset $\mathrm{A}_0 \subset \mathrm{A}$ all the sets C_α are polyhedral as $\alpha \in \mathrm{A}_0$, and each star-shapedness direction common for all sets C_α, $\alpha \in \mathrm{A}$, is a linearity direction of sets C_α for each $\alpha \in \mathrm{A} \setminus \mathrm{A}_0$* (condition for star-shapedness direction [9, Thm. 21.5], [14]).

If the intersection of each subfamily of $n+1$ sets (cf. Convention) *in family $\mathcal{C}$ is non-empty, then the intersection of all its sets is non-empty.*

Remark 1. *Helly's theorem on convex sets holds true also under a simpler assumption which is more restrictive in comparison with condition (d):*

(b) *all sets C_α, $\alpha \in \mathrm{A}$, are closed* (closedness condition) *and for some subset $\mathrm{A}' \subset \mathrm{A}$ the intersection $\bigcap_{\alpha \in \mathrm{A}'} C_\alpha$ is non-empty and bounded* (boundedness condition [15, Thm. 5], [16, 1.1]).

It is a particular case of condition (d) since under (b) we can choose $\mathrm{A}_0 = \varnothing$, and there is no common star-shapedness directions for all sets C_α, $\alpha \in \mathrm{A}$, just by boundedness condition. Because of this, Helly's theorem on convex sets with condition (b) is not involved in the main formulation of Helly's theorem.

3. Differences of sets and auxiliary results on geometric difference

Definition 2. *Let S, C be arbitrary sets in $\mathbb{R}^n$.*

Theoretic-set difference or difference of these sets is denoted in the most usual form $C \setminus S := \{x \in \mathbb{R}^n\colon x \in C, x \notin S\}$.

For $\lambda \in \mathbb{R}$, by $\lambda S := \{\lambda s\colon s \in S\}$ we denote the multiplication of S by number λ. At that we suppose $-S := (-1)S$.

[1] One can also say that each point in C is "seen from infinity in the direction of $-y$" (cf. with the notion of a set star-shaped w.r.t. a point in this set [10, Def. 7.1]). In other words, one says that C goes to ∞ in the direction of y [9, Ch. II, Sect. 8].

Geometric sum or Minkowski sum of sets S and C coincides with its algebraic *or* vector sum *and is defined as*[1]

$$S + C := \{s + c \colon s \in S, c \in C\} \subset \mathbb{R}^n. \tag{5}$$

In particular, for $x \in \mathbb{R}^n$ by $S+x := S+\{x\} =: x+S$ we define the shift, or parallel translation, of set S by vector x.

Geometric difference, or quite often Minkowski difference [10, Def. 8.5]*, of these sets is defined as*

$$C \overset{*}{-} S := \{x \in \mathbb{R}^n \colon S + x \subset C\} \subset \mathbb{R}^n, \tag{6}$$

where we employ a widely used notation[2] *of L.S. Pontryagin* [17]*,* [18, Sect 2.C, Geometric difference]*,* [13]*,* [19]*. In particular, for $x \in \mathbb{R}^n$, by $C - x := C \overset{*}{-} \{x\} = C + (-x)$ we introduced the shift of C by $-x$. Then $C \overset{*}{-} S = \bigcap_{s \in S}(C - s)$.*

Algebraic or vector difference of these sets called often in contradiction with (6) *Minkowski difference*[3] *is defined as*

$$C - S := C + (-1)S = \{c - s \colon c \in C, s \in S\} = C + (-S). \tag{7}$$

We note that as $C \neq \varnothing$ and $\operatorname{card} S > 1$, generally speaking, $C \overset{*}{-} S \subset C - S \neq C \overset{*}{-} S$ (see [10, Ch. I, Sect. 8, Def. 8.5, Warning], [13, Prop. 1.1.1, Rem. 1.1.1]).

In the proof of main theorem 1 we shall make use of the following elementary lemma being of independent interest.

Lemma 3.1 (on inheritance of "minuend" properties by geometric difference)**.** *For a pair of arbitrary sets $S, C \subset \mathbb{R}^n$, if*

[**cl**] *set C is closed, then the geometric difference $C \overset{*}{-} S$ is closed;*
[**bd**] *C is bounded and $S \neq \varnothing$, then the geometric difference $C \overset{*}{-} S$ is bounded;*
[**co**] *set C is convex, then geometric difference $C \overset{*}{-} S$ is convex;*
[**ds**] *y is a star-shapedness direction for C, then y is a star-shapedness direction for $C \overset{*}{-} S$;*
[**dl**] *y is a linearity direction for C, then y is a linearity direction for $C \overset{*}{-} S$;*
[**ph**] *C is polyhedral, then the geometric difference $C \overset{*}{-} S$ is polyhedral.*

Part of the results (but not all of them) of this Lemma under the special assumption that set C is <u>convex</u> can be proven by the simple identity for geometric differences $C \overset{*}{-} S = C \overset{*}{-} \operatorname{co} S$ but we prefer direct proofs for all the cases.

Proof. [**cl**] It was proven in [19, Thm. 12.3], but we provide a slightly different proof in more details. If $C = \varnothing$, then $C \overset{*}{-} S = \varnothing$ as $S \neq \varnothing$ and $C \overset{*}{-} S = \mathbb{R}^n$ as also $S = \varnothing$, i.e., $C \overset{*}{-} S$ is closed anyway. If $C \neq \varnothing$, and $x_k \in C \overset{*}{-} S$, $x_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, and there exists the limit $\lim_{k\to\infty} x_k = x$, for each $s \in S$ we have $s + x_k \in C$ and $s + x = \lim_{k\to\infty}(s + x_k) \in C$ by the closedness of set C. It yields $S + x \subset C$, i.e., $x = \lim_{k\to\infty} x_k \in C \overset{*}{-} S$.

[**bd**] If $C = \varnothing$, then for $S \neq \varnothing$ set $C \overset{*}{-} S = \varnothing$ is bounded. Suppose that $C \neq \varnothing$. Then $C \subset B(0, r)$ for some $r > 0$. We fix an element $s \in S$. If $x \in C \overset{*}{-} S$, then $s + x \in B(0, r)$ and $x \in B\bigl(0, r + |s|\bigr)$. It implies that $C \overset{*}{-} S \subset B\bigl(0, r + |s|\bigr)$ is bounded since r and s are fixed.

[**co**] It was proven in [19, Thm. 12.4], but we prove slightly more (see (9) below). If $x_1, x_2 \in C \overset{*}{-} S$, then $S + x_1 \subset C$ and $S + x_2 \subset C$, and for each numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ we have $\lambda_1 S + \lambda_1 x_1 \subset \lambda_1 C$ and $\lambda_2 S + \lambda_2 x_2 \subset \lambda_2 C$. By the inclusion $(\lambda_1 + \lambda_2)S \subset \lambda_1 S + \lambda_2 S$ and the identity

[1] Some authors employ the symbol $\oplus$ for Minkowski sum.

[2] There are other widespread notations for geometric difference of sets. For instance, in [10, Def. 8.5] the usual minus symbol $-$ is used, the symbols $\div$ in [20, § 1], [21], [22] or $\overset{*}{-}$ in [23, Def. 1] are employed. etc.

[3] Sometimes the symbol $\ominus$ is employed to indicate the algebraic difference.

$\lambda_1 C + \lambda_2 C = (\lambda_1+\lambda_2)C$ as $\lambda_1\cdot\lambda_2 \geqslant 0$ for *convex* C [10, Thm. 8.2] we obtain

$$\begin{aligned}(\lambda_1+\lambda_2)S + (\lambda_1 x_1 + \lambda_2 x_2) \subset & \lambda_1 S + \lambda_2 S + \lambda_1 x_1 + \lambda_2 x_2 = (\lambda_1 S + \lambda_1 x_1) + (\lambda_2 S + \lambda_2 x_2)\\ \subset & \lambda_1 C + \lambda_2 C = (\lambda_1+\lambda_2)C \quad \text{under the condition} \quad \lambda_1\cdot\lambda_2 \geqslant 0.\end{aligned} \tag{8}$$

Thus, we have proven the statement: *for $S \subset \mathbb{R}^n$ and a convex set $C \subset \mathbb{R}^n$, for each $\lambda_1, \lambda_2 \in \mathbb{R}$ as $\lambda_1\cdot\lambda_2 \geqslant 0$, the relations*

$$\lambda_1(C \overset{*}{-} S) + \lambda_2(C \overset{*}{-} S) \subset (\lambda_1+\lambda_2)C \overset{*}{-} (\lambda_1+\lambda_2)S = (\lambda_1+\lambda_2)(C \overset{*}{-} S), \tag{9}$$

hold true, where the latter identity was provided in [13, Prop. 1.1.1, second identity after (1.1.3)]. At that, [**co**] is a particular case of this statement for $\lambda_1, \lambda_2 \geqslant 0$ and $\lambda_1 + \lambda_2 = 1$.

There is one more possible proof for [**co**]. If $S + x \subset C$, then $\operatorname{co}(S+x) \subset \operatorname{co} C$ that implies $\operatorname{co} S + x \subset \operatorname{co} C$ and $x \in \operatorname{co} C \overset{*}{-} \operatorname{co} S$. It yields $C \overset{*}{-} S \subset \operatorname{co} C \overset{*}{-} \operatorname{co} S$. It is obvious that $C \overset{*}{-} S \supset C \overset{*}{-} \operatorname{co} S$, since $S \subset \operatorname{co} S$. Thus,

$$C \overset{*}{-} \operatorname{co} S \subset C \overset{*}{-} S \subset \operatorname{co} C \overset{*}{-} \operatorname{co} S \quad \textit{for each} \quad C, S \subset \mathbb{R}^n$$

and

$$C \overset{*}{-} S = C \overset{*}{-} \operatorname{co} S \quad \textit{if } C \textit{ is a convex set.} \tag{10}$$

The geometric difference of two convex sets is a convex set [10, 8.8] that by the latter identity implies [**co**].

[**ds**] The comments after Definition 1 and 2 imply the statement for $C = \varnothing$ or $S = \varnothing$. Suppose now that y is the star-shapedness direction for C. In view of Definitions 1 and 2 it means that $C + ty \subset C$ for each number $t \geqslant 0$. Then for $x \in C \overset{*}{-} S$, i.e., $S + x \subset C$, we obtain $S + (x + ty) \subset C + ty \subset C$ for each $t \geqslant 0$. Therefore, $x + ty \in C \overset{*}{-} S$ for each $t \geqslant 0$. By Definition 1, vector y is the star-shapedness direction for $C \overset{*}{-} S$.

[**dl**] By Definition 1, this statement is an obvious implication of the previous one.

[**ph**] By the comments after Definition 2, if $C = \varnothing$, then $C \overset{*}{-} S = \varnothing$ as $S \neq \varnothing$, and $C \overset{*}{-} S = \mathbb{R}^n$ as $S = \varnothing$. Hence, it follows from the comments after Definition 1 that as $C = \varnothing$, the difference $\varnothing \overset{*}{-} S$ is polyhedral. In the same way, as $S = \varnothing$, the difference $C \overset{*}{-} S = C \overset{*}{-} \varnothing = \mathbb{R}^n$ is a polyhedral set.

Suppose that $C \neq \varnothing$ and $S \neq \varnothing$. By Definition 1, polyhedral set C is defined by a finite system of linear inequalities (3), i.e., for some finite set of vectors $a_k \in \mathbb{R}^n$ and numbers $b_k \in \mathbb{R}$, $k = 1, \dots, m \in \mathbb{N}$,

$$C = \big\{x \in \mathbb{R}^n \colon \langle a_k, x\rangle - b_k \leqslant 0, k = 1, \dots m\big\}. \tag{11}$$

Point x belongs to $C \overset{*}{-} S$, i.e., $S + x \subset C$ if and only if $x + s \in C$ for some $s \in S$. By (11), we have $x \in C \overset{*}{-} S$ if and only if

$$\langle a_k, x\rangle + \langle a_k, s\rangle - b_k \leqslant 0 \quad \textit{for each} \quad s \in S, \quad k = 1, \dots, m. \tag{12}$$

If for at least one index k we have $\sup_{s\in S}\langle a_k, s\rangle = +\infty$, then $C \overset{*}{-} S = \varnothing$ is polyhedral. Otherwise (12) is equivalent to the finite system of linear inequalities

$$\langle a_k, x\rangle - b'_k \leqslant 0, \quad k = 1, \dots, m, \quad \textit{where} \quad b'_k := b_k - \sup_{s\in S}\langle a_k, s\rangle \in \mathbb{R}, \tag{13}$$

since $S \neq \varnothing$. The latter system is determined completely by $C \overset{*}{-} S$. □

In what follows we shall also need the inverse for statements [**ds**] and [**dl**] of Lemma 3.1.

Lemma 3.2. *Let C be a closed convex set in $\mathbb{R}^n$, $S \subset \mathbb{R}^n$, $S \neq \varnothing$, and $y \in \mathbb{R}^n$ be a star-shapedness direction (respectively, linearity direction) for $C \overset{*}{-} S \neq \varnothing$. Then y is a star-shapedness direction (respectively, linearity direction) for C.*

Proof. Due to Definition 1, it is sufficient to prove the lemma for star-shapedness directions y. The hypothesis of the lemma mean that for each point $x \in C \overset{*}{-} S \neq \varnothing$, ray $r_y(x)$ in (2) is contained in $C \overset{*}{-} S$. In other words, $S + x \subset C$ implies $S + r_y(x) \subset C$. We consider an arbitrary element $s \in S \neq \varnothing$. Then $s + x \in C$ and $s + x + ty \in C$ for each $t \geqslant 0$. Thus, for some point $s + x \in C$, ray $l_y(s + x)$ is contained in C.

Proposition 1 ([9, Thm. 8.3])**.** *If $C \subset \mathbb{R}^n$ is closed and convex and for some point $c \in C$ ray $r_y(c)$ (respectively, straight line $l_y(c)$) is contained in C, then C is star-shaped (respectively, linear) in the direction of y.*

In accordance with this proposition, y is a star-shapedness direction for C. □

4. Covering by shifts and geometric difference

As it will be clarified in the end of this section, as a development of an essential part of VKE theorem [8, 1, 2.1] and as one of generalization of Helly's theorem we can regard

Theorem 1 (on covering by shifts)**.** *Let $\mathcal{C}$ be a faimly of convex sets in $\mathbb{R}^n$ in* (4) *and suppose that we are given a family of arbitrary sets*

$$\mathcal{S} := \{S_\beta \subset \mathbb{R}^n \colon \beta \in \mathrm{B}\}, \quad \mathrm{B} \textit{ is a set of indices.} \tag{14}$$

Assume that the finiteness condition

(F) $\operatorname{card} \mathrm{A} < \infty$ *and* $\operatorname{card}\{\beta \in \mathrm{B} \colon S_\beta \neq \varnothing\} < \infty$

is satisfied or for $\mathcal{C}$ condition (d) in Helly's theorem holds true but with the additional restrictions $\mathrm{A}_0 = \varnothing$ *or* $\operatorname{card} \mathrm{B} < \infty$. *We let*

$$C := \bigcap_{\alpha \in \mathrm{A}} C_\alpha, \quad S := \bigcup_{\beta \in \mathrm{B}} S_\beta. \tag{15}$$

The following four statements are pairwise equivalent (taking Convention into account):

(T) *a shift of set S is contained in set C;*
(C) *for each $n+1$ sets in $\mathcal{C}$, a shift of set S is contained in the intersection of these $n+1$ sets;*
(S) *for each $n+1$ sets $\mathcal{S}$ a shift of set C covers (includes) all these $n+1$ sets;*
(CS) *for each $n+1$ indices*

$$\alpha_1, \dots, \alpha_{n+1} \in \mathrm{A} \quad \textit{and} \quad \beta_1, \dots, \beta_{n+1} \in \mathrm{B} \tag{16}$$

the intersection

$$\bigcap_{k=1}^{n+1} \bigl(C_{\alpha_k} \overset{*}{-} S_{\beta_k}\bigr) \tag{17}$$

of geometric differences $C_{\alpha_k} \overset{*}{-} S_{\beta_k}$ is non-empty.

Due to Definition 2 of geometric difference, Condition (17) in (CS) can be replaced by each of the following equivalent conditions:

(CSC) *for each set of indices* (16) *there exists a vector $x \in \mathbb{R}^n$, for which the shifts $S_{\beta_k} + x$ are contained in C_{α_k} for each $k = 1, \dots, n+1$;*
(CSS) *for each set of indices* (16) *there exists a vector $x \in \mathbb{R}^n$, for which the shift $C_{\alpha_k} - x$ covers S_{β_k} for each $k = 1, \dots, n+1$.*

Now we are in position to prove Theorem 1.

Proof of Theorem 1. If $S = \varnothing$, then it holds $S_\beta = \varnothing$ in (15) and each statement of the theorem is true.

If at least one of sets C_α is empty, then $C = \varnothing$ and by Convention, each of four statements of the theorem implies the emptiness for S and each S_β. Hence, in these case all four statements are true. This is why in what follows in the proof, we can suppose that $S \neq \varnothing$ and $C_\alpha \neq \varnothing$. The truth of double "vertical and horizontal" implications of the sides for the "rectangle"

$$\begin{array}{ccc} (\mathrm{T}) & \Rightarrow & (\mathrm{C}) \\ \Downarrow & \nwarrow & \Downarrow \\ (\mathrm{S}) & \Rightarrow & (\mathrm{CS}) \end{array}$$

is rather obvious here even with no condition for families $\mathcal{C}$ and $\mathcal{S}$. We just observe that the implications (C)⇒(CS) and (S)⇒(CS) are even more transparent, if we consider (CS) as (CSC) and (CSS), respectively. Thus, we just need to prove the "diagonal" implication (CS)→(T). At that, empty sets S_β in family $\mathcal{S}$ makes no influence on (17) since $C_\alpha \overset{*}{-} \varnothing = \mathbb{R}^n$ for each $C_\alpha \subset \mathbb{R}^n$. Thus, passing from the set of indices B to the subset of indices $\{\beta \in \mathrm{B}\colon S_\beta \neq \varnothing\}$ and keeping the same notation B, in what follows we can assume that all sets S_β are non-empty.

Let us consider the family of sets

$$\mathcal{C} \overset{*}{-} \mathcal{S} := \{C_{\alpha,\beta} := C_\alpha \overset{*}{-} S_\beta \colon (\alpha,\beta) \in \mathrm{A} \times \mathrm{B}\}. \tag{18}$$

Lemma 4.1. *If in notations* (18) *and* (15) *the intersection*

$$\bigcap_{(\alpha,\beta)\in \mathrm{A}\times\mathrm{B}} C_{\alpha,\beta} \tag{19}$$

is non-empty, then some shift of set S is contained in C, i.e., statement (T) *is true.*

Indeed, let $x \in \mathbb{R}^n$ belongs to intersection (19). It means that

$$S_\beta + x \subset C_\alpha \quad \textit{for each } \alpha \in \mathrm{A} \textit{ and } \beta \in \mathrm{B}.$$

Hence,

$$S_\beta + x \subset \bigcap_{\alpha\in\mathrm{A}} C_\alpha \quad \textit{for each } \beta \in \mathrm{B}.$$

Therefore,

$$\bigcap_{\beta\in\mathrm{B}} S_\beta + x = \bigcap_{\beta\in\mathrm{B}} (S_\beta + x) \subset \bigcap_{\alpha\in\mathrm{A}} C_\alpha,$$

i.e., $S + x \subset C$, quod erat demonstrandum. In view of Lemma 4.1, to prove the implication (CS)⇒(T), it is sufficient to justify the applicability of Helly's theorem on convex set to family (18).

Under (CS) and by earlier conventions in family $\mathcal{C} \overset{*}{-} \mathcal{S}$

(!) all sets $C_{\alpha,\beta}$ are non-empty, and moreover, each intersection $\bigcap_{k=1}^{n+1} C_{\alpha_k,\beta_k}$ in (17) is non-empty for arbitrary set of indices (16);
(!) all the sets $C_{\alpha,\beta}$ are convex, see Statement [**co**] of Lemma 3.1;
(!) all the sets $C_{\alpha,\beta}$ are closed provided all C_α are closed, see Statement [**cl**] of Lemma 3.1;
(!) there exists at most finite number of sets $C_{\alpha,\beta}$ under finiteness condition (F), i.e., $\mathrm{card}(\mathcal{C} \overset{*}{-} \mathcal{S}) < \infty$.

Condition (F). By the above arguments, if finiteness condition (F) holds true, under our assumptions and condition (CS) finiteness condition (f) is satisfied by family $\mathcal{C} \overset{*}{-} \mathcal{S}$. Thus, Helly's theorem on convex sets is applicable to family $\mathcal{C} \overset{*}{-} \mathcal{S}$ and the implication (CS)⇒(T) is proven in this case.

Condition (d) Let $\mathrm{A}_0 = \varnothing$ and y be a common star-shapedness direction for $C_{\alpha,\beta} = C_\alpha \overset{*}{-} S_\beta$ for all $(\alpha,\beta) \in \mathrm{A}\times\mathrm{B}$. Then by Lemma 3.2, vector y is a common star-shapedness direction for all C_α, $\alpha \in \mathrm{A}$. Then by condition (d), vector y is a linearity direction for C_α for all $\alpha \in \mathrm{A}$, while Statement [**dl**] of Lemma 3.1 follows that vector y is a linearity direction for all $C_{\alpha,\beta}$ as $(\alpha,\beta) \in \mathrm{A}\times\mathrm{B}$. Hence, for family the $\mathrm{C} \overset{*}{-} \mathrm{S}$ of convex closed sets in (18) condition (d) holds true (without a finite set of indices A_0). Therefore, Helly's theorem is applicable to family $\mathcal{C} \overset{*}{-} \mathcal{S}$ and by Lemma 4.1 we obtain desired statement (T).

Suppose now that under condition (d) set of indices B is finite. We consider a partition of the set of indices $\mathrm{A}\times\mathrm{B}$ into two disjoint subsets of indices

$$\mathrm{A}\times\mathrm{B}=(\mathrm{A}_0\times\mathrm{B})\bigcup\bigl((\mathrm{A}\setminus\mathrm{A}_0)\times\mathrm{B}\bigr).$$

Then by Statement [**ph**] of Lemma 3.1 the *finite* subfamily

$$\bigl\{C_{\alpha,\beta}\colon(\alpha,\beta)\in\mathrm{A}_0\times\mathrm{B}\bigr\}$$

of family $\mathcal{C}\overset{*}{-}\mathcal{S}$ is formed by *polyhedral sets.* Let y be a star-shapedness direction for all sets of family $\mathcal{C}\overset{*}{-}\mathcal{S}$. Again by Lemma 3.2, vector y is a common star-shapedness direction for all C_α, $\alpha\in\mathrm{A}$. By condition (d) vector y is a linearity direction for C_α for all $\alpha\in\mathrm{A}\setminus\mathrm{A}_0$, and by Statement [**dl**] of Lemma 3.1 vector y is a common linearity direction for all $C_{\alpha,\beta}$ as $(\alpha,\beta)\in(\mathrm{A}\setminus\mathrm{A}_0)\times\mathrm{B}$. Thus, for family $\mathrm{C}\overset{*}{-}\mathrm{S}$ of convex closed sets in (18) condition (d) holds true, where the role of A_0 is played by $\mathrm{A}_0\times\mathrm{B}$. Therefore, we can apply Helly's theorem to family $\mathcal{C}\overset{*}{-}\mathcal{S}$ and by Lemma 4.1 we obtain desired statement (T). □

Remark 2. *Theorem 1 on shifts holds true also under condition (b) in Remark 1 since it is a particular case of condition (d) as* $\mathrm{A}_0=\varnothing$*.*

Comment 1 (on Theorem 1)**.** *Let us make sure that Helly's theorem and the most part of VKE theorem (concerning the statements* "covering", "containing"*) are particular cases of Theorem 1.*

1. *For an arbitrary one-point non-empty set S, for instance, $S=\{0\}$, and one-element family $\mathcal{S}=\{S\}$, the implication ()⇒(T) of Theorem 1 on covering by shifts gives exactly Helly's theorem on convex sets given above in the beginning of Section 2.*
2. *If family $\mathcal{C}$ has one element, i.e., it contains just one convex set C, then the implication (S)⇒(T) of Theorem 1 on covering by shifts gives VKE theorem in Introduction as particular case of "shift of set C covers everything".*
3. *If family $\mathcal{S}$ has one elements and contains just one* <u>convex</u> *set S, then the implication ()⇒(T) of Theorem 1 on covering by shifts becomes a particular case of VKE theorem concerning "contains", one just should replace the notations: S should be placed instead of C, $\mathcal{C}$ instead of $\mathcal{S}$ and assume either finiteness of $\mathcal{C}$ or compactness of S.*
4. *The proof of the part "intersects everything" of VKE theorem and its generalizations is moved to Section 4 after Theorem 2 on intersections of shifts, since surprisingly (at least, for us), this proof happened to be related not with the geometric difference, but with the algebraic difference of sets.*

Comment 2 (previous special versions of Theorem 1)**.** *The implication ()⇒(T) of Theorem 1 was established or mentioned earlier in* [8] *in the following three rather particular cases:*

% *family $\mathcal{C}$ of convex sets is finite, and S is a* convex *set* [8, **2.1**] *(VKE theorem);*
% *family $\mathcal{C}$ consists of* convex compact *sets and S is also a convex compact set* [8, **2.1**] *(VKE theorem);*
% *family $\mathcal{C}$ consists of* closed half-spaces *having common* bounded *intersection, and S is a* convex *body, i.e., a convex compact set with a non-empty interior* $\operatorname{int}S\neq\varnothing$ [8, **6.18**].

In the special case when $\mathcal{S}$ in (14) *is a family of all one-point sets $\{s\}$, $s\in S$, i.e., $\mathrm{B}:=S$ is also a set of indices and $S_s:=\{s\}$, $s\in S$, the validity of the implication (S)⇒(T) follows easily from* [8, **2.1**] *(one should let $\mathcal{K}=\mathcal{S}$) in the following two very special situations:*

% *S is finite and family $\mathcal{C}$ is formed by* one convex *set C* [8, **2.1**] *(VKE theorem);*
% *family $\mathcal{C}$ consists of* one convex compact *set C (for convex body C see* [8, **6.2**]*, and as* $\operatorname{int}C=\varnothing$*, the proof follows easily from* [8, **2.1**] *(VKE theorem)).*

5. Intersection of shifts and algebraic difference

Let us give an "algebraic" development of VKE theorem in Introduction.

Theorem 2 (on intersection of shifts)**.** *Suppose that the family*

$$\mathcal{C} := \{C_\alpha \subset \mathbb{R}^n\colon \alpha \in \mathrm{A}\}, \quad \mathrm{A} \textit{ is a set of indices,} \tag{20}$$

consists of arbitrary (cf. (4)*) of non-empty sets C_α, and the family of sets (cf.* (14)*)*

$$\mathcal{S} := \{S_\beta \subset \mathbb{R}^n\colon \beta \in \mathrm{B}\}, \quad \mathrm{B} \textit{ is a set of indices,} \tag{21}$$

also consists of arbitrary non-empty sets S_β. Assume that each algebraic, i.e., vector difference (Definition 2, (7)*)*

$$C_\alpha - S_\beta :=: C_\alpha + (-S_\beta) \quad \textit{is convex for each } \alpha \in \mathrm{A}, \beta \in \mathrm{B}. \tag{22}$$

Suppose that one of the following two conditions holds true

(F) *finiteness condition in Theorem 1, i.e.,* $\operatorname{card}\mathrm{A} + \operatorname{card}\mathrm{B} < \infty$*;*

(id) *each algebraic difference* (22) *is closed, for some finite subsets* $\mathrm{A}_0 \subset \mathrm{A}$*,* $\mathrm{B}_0 \subset \mathrm{B}$ *algebraic differences in* (22) *are polyhedral for each* $(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$*, and each star-shapedness direction common for all algebraic differences* (22) *for all* $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$ *happens to be a linearity direction for algebraic differences* (22) *for all* $(\alpha, \beta) \in (\mathrm{A} \times \mathrm{B}) \setminus (\mathrm{A}_0 \times \mathrm{B}_0)$.

The following statements are equivalent (cf. Convention in Introduction):

(I) *the exists a vector $x \in \mathbb{R}^n$ such that for each index $\beta \in \mathrm{B}$ the shift $S_\beta + x$ intersects with all C_α in $\mathcal{C}$;*

(CSI) *for each $n+1$ indices*

$$\alpha_1, \dots, \alpha_{n+1} \in \mathrm{A} \quad \textit{and} \quad \beta_1, \dots, \beta_{n+1} \in \mathrm{B}$$

the intersection

$$\bigcap_{k=1}^{n+1} (C_{\alpha_k} - S_{\beta_k}) \tag{23}$$

of algebraic differences in (22) *is non-empty.*

Proof. The implication (I)⇒(CSI) is obvious thanks to the definition and it is true for each system of non-empty sets. We prove the inverse implication (CSI)⇒(I) under both conditions (F) and (id). Since sets (22) are convex, we can apply Helly's theorem on convex sets. Here finiteness condition (f) corresponds to condition (F), while condition (d) corresponds to condition (id). If in Helly's theorem we consider sets of indices $\mathrm{A} \times \mathrm{B}$, $\mathrm{A}_0 \times \mathrm{B}_0$ instead of sets of indices A, A_0, respectively, and if we replace system of sets C_α by the system of all possible algebraic differences (22), then by Helly's theorem

$$\bigcap_{(\alpha,\beta)\in \mathrm{A}\times\mathrm{B}} (C_\alpha - S_\beta) \neq \varnothing.$$

Let x be a point in the above intersection. Id est, there always exist $c_\alpha \in C_\alpha$ and $s_\beta \in S_\beta$ satisfying $x = c_\alpha - s_\beta$ or

$$S_\beta + x \ni s_\beta + x = c_\alpha \in C_\alpha \quad \underline{\text{for each pair}} \quad (\alpha, \beta) \in \mathrm{A} \times \mathrm{B}.$$

And it is the sought vector x in (I). □

We recall that in the beginning of Section 4 after Theorem 1 and Conditions (CSC)–(CSS) the latter of four Statements (1)–(4) remained unexplained (the proof of part *"intersects everything"*). Now we can return back to the latter of Statements (1)–(4) in order to cover the gap in the proof of part *"intersects everything"* of VKE theorem and to generalize it.

Corollary 1. *Let $C \subset \mathbb{R}^n$ be non-empty and we are given family of sets* (21)*. Assume that the algebraic differences $C - S_\beta$ are convex for each $\beta \in \mathrm{B}$, where* B *is finite or, otherwise, all these algebraic differences are closed and at least one of them is bounded. If for each $n+1$ sets*

$$\{S_{\beta_1}, \dots S_{\beta_{n+1}}\} \tag{24}$$

in family (21) *some shift of set C intersects simultaneously all $n+1$ sets in* (24)*, there exists a shift of set C intersecting all sets in family $\mathcal{S}$.*

Proof. Since family $\mathcal{C} = \{C\}$ is one-point, the intersection casts into a rather simple form and the non-emptiness of intersection (23) means the latter sentence in the formulation of the corollary. If set B is infinite, the closedness of algebraic difference and, first of all, the boundedness for at least one of these algebraic difference (cf. Section 2, Remark 1, Statement (b), right after Helly's theorem) implies condition (id). Thus, the hypothesis of Theorem 2 on intersection of shifts is satisfied and the proof of the corollary is complete. □

Comment 3. *The following comments and remarks supplement Theorem 2 on intersection of shifts:*

* *If* $\operatorname{card} \mathrm{B} = 1$ *and* $S_\beta = \{0\}$*, Theorem 2 is precisely Helly's theorem on convex sets formulated above;*
* *Under conditions* $\operatorname{card} \mathrm{A} = 1$*, i.e., $\mathcal{C}$ consists of one convex set C, all S_β, $\beta \in \mathrm{B}$, are convex and set of indices* B *is finite or C and all the sets $S_\beta \in \mathcal{S}$ are compact, Theorem 2 was proven in* [8, **2.1**]*;*
* *The convexity of all algebraic differences* (22) *can be replaced to a more restrictive condition of convexity for each $C_\alpha \in \mathcal{C}$ and each $S_\beta \in \mathcal{S}$ since algebraic difference of two convex sets is convex;*
* *the part of condition (id), "*each algebraic difference in (22) is closed ...*", can be replaced by a more restrictive condition "for each pair C_α, S_β one of the sets is closed, the other is compact" since algebraic difference of a closed set and a compact set is closed;*
* *the part of condition (id), "*for some finite subsets $\mathrm{A}_0 \subset \mathrm{A}$, $\mathrm{B}_0 \subset \mathrm{B}$ algebraic differences in (22) are polyhedral for each $(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$, ...*", can be replaced by a more restrictive condition "for some finite subsets $\mathrm{A}_0 \subset \mathrm{A}$, $\mathrm{B}_0 \subset \mathrm{B}$ each of sets C_α as $\alpha \in \mathrm{A}_0$ and S_β as $\beta \in \mathrm{B}_0$ is convex and polyhedral, ...", since algebraic sum of polyhedral convex sets is polyhedral* [9, Cor. 19.3.2]*;*
* *If for some subsets $\mathrm{A}' \subset \mathrm{A}$ and $\mathrm{B}' \subset \mathrm{B}$ the intersection*

  $$\bigcap_{(\alpha,\beta)\in \mathrm{A}'\times \mathrm{B}'} (C_\alpha - S_\beta)$$

  is bounded, then there exists no star-shapedness directions common for all algebraic differences (22) *and the concluding part of condition (id) on common star-shapedness condition holds true immediately;*
* *If in pair C_α, S_β on the sets is closed, convex, and unbounded, and the other is bounded, then a star-shapedness direction for algebraic difference in* (22) *is also a star-shapedness direction (respectively, linearity direction) for unbounded set C_α or S_β (cf. Proposition 1). This can simplify the seek of common star-shapedness directions while checking condition (id) of Theorem 2.*

6. Set-theoretical differences

For the completeness of exposition we provide an analogue in a sense of Theorem 1 and 2 for set-theoretical difference of sets. For the sake of brevity, the latter in this section is called difference.

Theorem 3 (on differences of sets)**.** *Suppose that families of sets* $\mathcal{C}$ *and* $\mathcal{S}$ *are defined respectively as in* (4) *and* (14)*, and as in* (15)*,*

$$C = \bigcap_{\alpha \in \mathrm{A}} C_\alpha, \quad S = \bigcup_{\beta \in \mathrm{B}} S_\beta. \tag{25}$$

Suppose that all the differences[1]

$$C_\alpha \setminus S_\beta \quad \text{are convex for each } \alpha \in \mathrm{A} \text{ and } \beta \in \mathrm{B}. \tag{26}$$

Assume that one of the conditions holds true:

(F) *the finiteness condition in Theorem 1 and 2, i.e.,* $\operatorname{card} \mathrm{A} + \operatorname{card} \mathrm{B} < \infty$*;*

(dd) *each difference in* (26) *is closed, for some finite subsets* $\mathrm{A}_0 \subset \mathrm{A}$*,* $\mathrm{B}_0 \subset \mathrm{B}$ *the differences in* (26) *are polyhedral for all pairs* $(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$*, and each star-shapedness direction common for all differences* (26) *for each* $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$ *is a linearity direction for differences* (26) *for all* $(\alpha, \beta) \in (\mathrm{A} \times \mathrm{B}) \setminus (\mathrm{A}_0 \times \mathrm{B}_0)$*.*

The following statements are equivalent (in view of Convention):

(D) *difference* $C \setminus S$ *is non-empty;*

(CSD) *for each* $n+1$ *indices*

$$\alpha_1, \ldots, \alpha_{n+1} \in \mathrm{A} \quad \text{and} \quad \beta_1, \ldots, \beta_{n+1} \in \mathrm{B} \tag{27}$$

the intersection

$$\bigcap_{k=1}^{n+1} \bigl(C_{\alpha_k} \setminus S_{\beta_k}\bigr) \tag{28}$$

of differences in (26) *is non-empty.*

Proof. For each sets of indices A', B' and sets C_α, $\alpha \in \mathrm{A}'$, and S_β, $\beta \in \mathrm{B}'$, of arbitrary nature, the elementary general identity for differences

$$\bigcap_{(\alpha,\beta) \in \mathrm{A}' \times \mathrm{B}'} \Bigl(C_\alpha \setminus S_\beta\Bigr) = \Bigl(\bigcap_{\alpha \in \mathrm{A}'} C_\alpha\Bigr) \setminus \Bigl(\bigcup_{\beta \in \mathrm{B}'} S_\beta\Bigr)$$

holds true. For instance, as $\mathrm{A} = \mathrm{A}'$ and $\mathrm{B} = \mathrm{B}'$, in terms of notations in (25) we have

$$C \setminus S = \Bigl(\bigcap_{\alpha \in \mathrm{A}} C_\alpha\Bigr) \setminus \Bigl(\bigcup_{\beta \in \mathrm{B}} S_\beta\Bigr). \tag{29}$$

Therefore, is set $C \setminus S$ is non-empty, the same is true for the right hand side in (29). Thus, Statement (D) of the theorem implies the non-emptiness of sets (28) for any indices (27), i.e., (CSD) is proven.

We prove the implication (CSD)$\Rightarrow$(D) under both conditions (F) and (dd). Since sets (26) are convex, we can apply Helly's theorem on convex sets. Here condition (f) corresponds to condition (F), while condition (d) does to condition (CSD) provided in Helly's theorem instead of sets of indices A, A_0 we consider respectively sets of indices $\mathrm{A} \times \mathrm{B}$, $\mathrm{A}_0 \times \mathrm{B}_0$, and system of sets C_α is replaced by the system of all possible differences (26). $\square$

Comment 4 (on Theorem 3)**.** # *If* $\operatorname{card} \mathrm{B} = 1$ *and* $S_\beta = \varnothing$ *is the empty set, Theorem 3 is precisely Helly's theorem on convex sets in Section 2.*

The part of condition (dd), "each difference in (26) *is closed, . . . ", can be replaced by a more restrictive one that "each* $C_\alpha \in \mathcal{C}$ *is closed and each* $S_\beta \in \mathcal{S}$ *is open, . . . " since in this case each difference* (26) *is closed.*

[1] This often happens even if the sets are not convex.

If for some subsets $\mathrm{A}' \subset \mathrm{A}$ *and* $\mathrm{B}' \subset \mathrm{B}$ *the intersection*

$$\bigcap_{(\alpha,\beta)\in \mathrm{A}'\times\mathrm{B}'} (C_\alpha \setminus S_\beta)$$

is bounded, there exist no star-shapedness directions common for all differences (26) *and the concluding part of Condition (dd) on common star-shapedness directions holds true immediately since such directions are just absent.*

If all the sets $C_\alpha \in \mathcal{C}$ *are closed, convex, and unbounded, and all sets* $S_\beta \in \mathcal{S}$ *are bounded, a star-shapedness direction (linearity direction) for the difference in* (26) *is a star-shapedness direction (linearity direction) for unbounded set* C_α *that follows easily from Proposition 1. This can simplify the seeking of common star-shapedness directions while checking Condition (dd) of Theorem 2.*

Remark 3. *Further reach in results variations of Helly's theorem on convex sets intersecting partially with our results (especially those in Section 4 on analogues or generalizations of transversals for families of sets), in addition to the above cited sources, can be found in works by V.L. Dol'nikov, S.A. Bogatyi, N.A. Bobylev, R.N. Karasev* [24]—[27] *and many others.*

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