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# ON A.F. LEONT'EV'S INTERPOLATING FUNCTION

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**Abstract.** We introduce and study an abstract version of an interpolating functional. It is defined by means of Pommiez operator acting in an countable inductive limit of weighted Fréchet spaces of entire functions and of an entire function of two complex variables. The properties of the corresponding Pommiez operator are studied. The A.F.Leont'ev's interpolating function used widely in the theory of exponentional series and convolution operators and as well as the interpolating functional applied earlier for solving the problem on the existence of a continuous linear right inverse to the operator of representation of analytic functions on a bounded convex domain in C by quasipolynomial series are partial cases of the introduced interpolating functional.

**Keywords:** A.F.Leont'ev's interpolating function, interpolating functional, Pommiez operator

**Mathematics Subject Classification:** 30B50, 46A13, 47B38

## Introduction

Let $G$ be a bounded convex domain in $\mathbb{C}$; $A(\bar{G})$ is the space of germs of functions analytic on the closure $\bar{G}$ of the domain $G$ with the natural topology of inductive limit of Banach spaces sequence. A.F. Leont'ev introduced (see [1, Ch. IV, Sect. 2]) interpolating function $\omega_L(\mu, f)$ defined by a special entire function $L$ of exponential type and applied it for calculating the coefficients in expansions of functions in $A(\bar{G})$ into exponential series with exponents being the zeroes of $L$. The interpolating function was introduced also in other situations and was employed for calculating coefficients of exponential series or generalized exponential series for functions in various spaces. It was also applied for other issues in theory of exponential series, in the theory of exponential polynomial, convolution operators, in interpolation problems.

For solving the problem on existence of linear continuous right inverse (LCRI) for the operator of representation by the series of quasi-polynomials of functions analytic in $G$, in [2, Sect. 3] there was introduced interpolating functional $\Omega_Q(\mu, z, f)$ being an analogue of interpolating function $\omega_L(\mu, f)$ defined by some entire function $Q(\mu, z)$ of two complex variables $\mu$, $z$. Functional $\Omega_Q$ and its analogues were used for solving problem on existence of LCRI for the operator of representation by the quasi-monomials series of functions analytic in a bounded convex domain in $\mathbb{C}$ [3]; for the operator of representation by the Mittag-Leffler functions of functions analytic in $\rho$-convex domain ($\rho > 0$) [5]. Moreover, a version of functional $\Omega_Q$ was employed in solving the problem on existence of LCRI for the operator of representation by generalized exponentials series of Björling type ultra-distributions on a multi-dimensional real cube [6].

In the present work we introduce an abstract version of interpolating functional, in particular, of interpolating A.F. Leont'ev function. It is introduced by means of Pommiez operator acting in a weighted $(LF)$-space of entire in $\mathbb{C}$ functions. In this connection, in Section 1 we study the properties of Pommiez operator. The interpolating functional is introduced and studied

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in Section 2. In Section 3 we provide the realizations of interpolating functions for particular spaces. In the present paper we restrict ourselves by the examples motivated our study. The interpolating functional can be also useful in many other situations in which the dual space for the main space is realized as a weighted space of entire functions. We expect to devote a separate paper to analyzing such situations as well as to applications of interpolating functional to the theory of exponential series and to convolution operators.

## 1. Pommiez operators and their properties

In this section we study the Pommiez operator acting in some weighted $(LF)$-space (i.e., in the countable inductive limit of Fréchet space) $E$ of entire functions. For a continuous function $v : \mathbb{C} \to \mathbb{R}$ and a function $f : \mathbb{C} \to \mathbb{C}$ we denote

$$p_v(f) := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{\exp v(z)}.$$

Let continuous functions $v_{n,k} : \mathbb{C} \to \mathbb{R}$ be such that

$$v_{n,k+1} \leqslant v_{n,k} \leqslant v_{n+1,k},\ \ n,k \in \mathbb{N}.$$

As usually, $A(\mathbb{C})$ indicates the space of entire (in $\mathbb{C}$) functions. We introduce Banach spaces

$$E_{nk} := \{f \in A(\mathbb{C}) : p_{v_{n,k}}(f) < +\infty\},\ \ n,k \in \mathbb{N},$$

and weighted Fréchet spaces

$$E_n := \{f \in A(\mathbb{C}) : p_{v_{n,k}}(f) < +\infty \quad \forall k \in \mathbb{N}\}, \quad n \in \mathbb{N}.$$

We note that $E_n$ is continuously embedded into $E_{n+1}$ for each $n \in \mathbb{N}$. We define weighted $(LF)$-space $E$ as follows:

$$E := \operatorname*{ind}_{n\to} E_n.$$

We make the following assumptions for functions $v_{n,k}$:

$$\forall n\ \exists m\ \forall k\ \exists s\ \exists C \geqslant 0 : \sup_{|t-z|\leqslant 1} v_{n,s}(t) \leqslant \inf_{|t-z|\leqslant 1} v_{m,k}(t) + C,\ \ z \in \mathbb{C}, \tag{1}$$

and

$$\forall n\ \exists m\ \forall k\ \exists s : \lim_{z\to\infty} (v_{m,k}(z) - v_{n,s}(z)) = +\infty. \tag{2}$$

For $f : \mathbb{C} \to \mathbb{C}$, $h \in \mathbb{C}$ we let $\tau_h(f)(z) := f(z+h)$, $z \in \mathbb{C}$.

**Proposition 1.** *1) Suppose that condition (1) is satisfied. Then*

(a) *Space $E$ is invariant w.r.t. the differentiation, i.e., for each function $f \in E$ we have $f' \in E$.*

(b) *Space $E$ is invariant w.r.t. a shift, i.e., $\tau_h(f) \in E$ for each $f \in E$ and $h \in \mathbb{C}$.*

*2) Suppose that condition (2) is satisfied. Then for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that each bounded in $E_n$ set is relatively compact in $E_m$.*

*Proof.* Statement 1) is obvious.

2) Suppose a set $B$ is bounded in $E_n$. For the given $n$, we choose $m \in \mathbb{N}$ by condition (2). We then choose a sequence $f_j \in B$, $j \in \mathbb{N}$. Since it is bounded on each compact set in $\mathbb{C}$, by Montel theorem there exists a subsequence $(f_{j_r})_{r\in\mathbb{N}}$ converging uniformly on each compact set in $\mathbb{C}$ to a function $f \in A(\mathbb{C})$. It is obvious that $f \in E_n \subseteq E_m$. For $k \in \mathbb{N}$ we define $s \in \mathbb{N}$ by (2). Since $\sup\limits_{r\in\mathbb{N}} p_{v_{n,s}}(f_{j_r}) < +\infty$, then

$$p_{v_{m,k}}(f_{j_r} - f) \to 0 \text{ as } r \to \infty.$$

Hence, set $B$ is relatively compact in $E_m$. □

**Lemma 2.** *Suppose that conditions (1) and (2) hold. For each $f \in E$, $z \in \mathbb{C}$ there exists $m \in \mathbb{N}$ such that*

$$\lim_{\mu \to z} \frac{\tau_\mu(f) - \tau_z(f)}{\mu - z} = \tau_z(f')$$

*in $E_m$.*

*Proof.* It is obvious that

$$\lim_{\mu \to z} \frac{\tau_\mu(f)(t) - \tau_z(f)(t)}{\mu - z} = f'(t+z) = \tau_z(f')(t)$$

for each $t \in \mathbb{C}$. The maximum modulus principle and condition (1) yield that set $\{\frac{\tau_\mu(f)-\tau_z(f)}{\mu-z} : 0 < |\mu - z| \leqslant 1\}$ is bounded in some space $E_n$, and thus, it is relatively compact in some space $E_m$. Therefore, in $E_m$ there exists $\lim\limits_{\mu \to z} \frac{\tau_\mu(f)-\tau_z(f)}{\mu-z}$ equal to $\tau_z(f')$. □

We shall assume that space $E$ contains a function which is not identically zero. Then there exists a function $g_0 \in E$ such that $g_0(0) = 1$.

We fix $z \in \mathbb{C}$. An operator $D_z : E \to A(\mathbb{C})$ is introduced as follows: for $f \in E$

$$D_z(f)(t) := \begin{cases} \frac{f(t)-g_0(t-z)f(z)}{t-z}, & t \neq z, \\ f'(z) - g_0'(0)f(z), & t = z. \end{cases}$$

**Remark 3.** *Earlier operator $D_z$ was studied and applied in the case $g_0 \equiv 1$ in the spaces of analytic functions with no restrictions for their growth (cf., for instance, works [7]-[12] and the references therein). In those cases it was referred to as Pommiez operator. We shall employ the same name also for operator $D_z$ introduced above.*

Let us prove some properties of operator $D_z$.

**Lemma 4.** *For each $\mu, z \in \mathbb{C}$ the identity*

$$D_\mu(f) - D_z(f) = (\mu - z)D_\mu(D_z(f)) + f(z)D_\mu(\tau_{-z}(g_0)), \;\; f \in E \tag{3}$$

*holds true.*

*Proof.* We choose $\mu, z \in \mathbb{C}$, $\mu \neq z$. If $t \neq z$, $t \neq \mu$, then

$$\begin{aligned} D_\mu(f)(t) - D_z(f)(t) =& \frac{f(t) - g_0(t-\mu)f(\mu)}{t-\mu} - \frac{f(t) - g_0(t-z)f(z)}{t-z} \\ =& \frac{f(t)(\mu - z) + g_0(t-z)f(z)(t-\mu) - g_0(t-\mu)f(\mu)(t-z)}{(t-\mu)(t-z)} \end{aligned}$$

and

$$\begin{aligned} D_\mu(D_z(f))(t) =& \frac{\frac{f(t)-g_0(t-z)f(z)}{t-z} - g_0(t-\mu)\frac{f(\mu)-g_0(\mu-z)f(z)}{\mu-z}}{t-\mu} \\ =& \Big(f(t)(\mu - z) - g_0(t-z)f(z)(\mu-z) - g_0(t-\mu)f(\mu)(t-z) \\ & + g_0(t-\mu)g_0(\mu-z)f(z)(t-z)\Big) / \Big((\mu-z)(t-z)(t-\mu)\Big). \end{aligned}$$

Hence,

$$\begin{aligned} (\mu - z)D_\mu(D_z(f))(t) =& \Big(f(t)(\mu - z) + g_0(t-z)f(z)(t-\mu) - g_0(t-\mu)f(\mu)(t-z) \\ & - g_0(t-z)f(z)(t-\mu) - g_0(t-z)f(z)(\mu-z) \\ & + g_0(t-\mu)g_0(\mu-z)f(z)(t-z)\Big) / \Big((t-z)(t-\mu)\Big) \\ =& D_\mu(f)(t) - D_z(f)(t) \end{aligned}$$

$$+\frac{-g_0(t-z)f(z)(t-z)+g_0(t-\mu)g_0(\mu-z)f(z)(t-z)}{(t-z)(t-\mu)}$$
$$=D_\mu(f)(t)-D_z(f)(t)+f(z)\frac{g_0(t-\mu)g_0(\mu-z)-g_0(t-z)}{t-\mu}$$
$$=D_\mu(f)(t)-D_z(f)(t)-f(z)D_\mu(\tau_{-z}(g_0))(t).$$

It is clear that the identity

$$(\mu-z)D_\mu(D_z(f))(t)=D_\mu(f)(t)-D_z(f)(t)-f(z)D_\mu(\tau_{-z}(g_0))(t)$$

holds true as $t=\mu$ and $t=z$ since the functions in both sides of this identity are entire. Since $D_\mu(\tau_{-z}(g_0))(t)=0$, as $t\in\mathbb{C}$, $\mu=z$, the latter identity holds true also for $\mu=z$. □

**Remark 5.** *If $g_0\equiv 1$, identity (3) becomes*

$$D_\mu-D_z=(\mu-z)D_\mu\circ D_z.$$

**Lemma 6.** *Suppose that condition (1) holds true. Then*

(i) *For each $n\in\mathbb{N}$ and bounded in $\mathbb{C}$ set $M$ there exists $m\in\mathbb{N}$ such that for each $z\in M$ operator $D_z$ linearly and continuously maps $E_n$ into $E_m$.*

(ii) *For each $n\in\mathbb{N}$, bounded in $E_n$ set $B$, and bounded in $\mathbb{C}$ set $M$ there exists $m\in\mathbb{N}$ such that the set*

$$\{D_z(f):\ z\in M, f\in B\}$$

*is bounded in $E_m$.*

*Proof.* (i): Due to (1) there exists $n_1\in\mathbb{N}$ for which the set

$$\{\tau_{-z}(g_0): z\in M\}$$

is bounded in $E_{n_1}$. We choose $m\in\mathbb{N}$ associated with $n_2:=\max\{n,n_1\}$ by means of (1) for $k\in\mathbb{N}$ we define $s\in\mathbb{N}$ by (1) as well.

We take $f\in E_n$ and fix $z\in M$. Suppose that $|t-z|>1$. Then

$$\frac{|D_z(f)(t)|}{\exp v_{m,k}(t)}\leqslant\frac{|f(t)-g_0(t-z)f(z)|}{\exp v_{m,k}(t)}\leqslant\frac{|f(t)|}{\exp v_{m,k}(t)}+|f(z)|\frac{|g_0(t-z)|}{\exp v_{m,k}(t)}. \tag{4}$$

If $|t-z|\leqslant 1$,

$$\begin{aligned}\frac{|D_z(f)(t)|}{\exp v_{m,k}(t)}&\leqslant\frac{\sup\limits_{|w-z|=1}|f(w)-g_0(w-z)f(z)|}{\exp v_{m,k}(t)}\leqslant\frac{\sup\limits_{|w-z|=1}|f(w)|}{\exp v_{m,k}(t)}+|f(z)|\frac{\sup\limits_{|w-z|=1}|g_0(w-z)|}{\exp v_{m,k}(t)}\\&\leqslant\Big(p_{v_{n_2,s}}(f)+|f(z)|p_{v_{n_2,s}}(\tau_{-z}(g_0))\Big)\exp\Big(\sup_{|w-z|\leqslant1}v_{n_2,s}(w)-\inf_{|w-z|\leqslant1}v_{m,k}(w)\Big).\end{aligned}\tag{5}$$

Thus, for each $k\in\mathbb{N}$

$$p_{v_{m,k}}(D_z(f))<+\infty,$$

i.e., $D_z(f)\in E_m$. Hence, for each $z\in M$ operator $D_z$ maps linearly $E_n$ into $E_m$. Since the graph of operator $D_z:E_n\to E_m$ is closed, by the closed graph theorem [13, Thm. 6.7.1], operators $D_z:E_n\to E_m$, $z\in M$, are continuous.

(ii): Suppose that a set $B$ is bounded in $E_n$, i.e., $\sup\limits_{f\in B}p_{v_{n,l}}(f)<+\infty$ for each $l\in\mathbb{N}$. It follows from condition (1) that the set $\{\tau_{-z}(g_0): z\in M\}$ is bounded in some space $E_{n_1}$. We let $n_2:=\max\{n;n_1\}$ and choose the associated $m$ by means of (1). Fixing then $k$, we choose $s$ by (1). Since $B$ is bounded in $E_n$, then $\sup\limits_{z\in M,f\in B}|f(z)|<+\infty$. Thanks to inequalities (4)-(5) and taking into consideration that sets $B$ and $\{\tau_{-z}(g_0): z\in M\}$ are bounded in $E_m$, we obtain:

$$\sup_{z\in M,f\in B}p_{m,k}(D_z(f))<+\infty.$$

Thus, the set $\{D_z(f) : z \in M, f \in B\}$ is bounded in $E_m$. □

**Lemma 7.** *Suppose that conditions (1) and (2) hold true. Then*

(iii) *For each $n \in \mathbb{N}$, each bounded in $E_n$ set $B$ there exists $m \in \mathbb{N}$ such that $\lim\limits_{\mu\to z} D_\mu(f) = D_z(f)$ in $E_m$ uniformly in $f$ on $B$.*

(iv) *For each $f \in E$, $z \in \mathbb{C}$ there exists $r \in \mathbb{N}$ such that in $E_r$ the limit $\lim\limits_{\mu\to z} \frac{D_\mu(\tau_{-z}(f))}{\mu - z}$ is well-defined and is equal to $D_z(\tau_{-z}(f'))$.*

(v) *For each $f \in E$, $z \in \mathbb{C}$ there exists $r \in \mathbb{N}$ such that in $E_r$*

$$\lim_{\mu\to z} \frac{D_\mu(f) - D_z(f)}{\mu - z} = D_z^2(f) + f(z) D_z(\tau_{-z}(g_0')).$$

*Proof.* (iii): Let $n \in \mathbb{N}$ and a set $B$ be bounded in $E_n$. By Statement 2 in Proposition 1 there exists $m_1 \in \mathbb{N}$ such that $B$ is relatively compact in $E_{m_1}$.

It is clear that $D_\mu(f) \to D_z(f)$ as $\mu \to z$ pointwise for each function $f \in E$. By Statement (ii) of Lemma 6 there exists $m_2$ for which the set

$$\{D_\mu(f) : \ |\mu - z| \leqslant 1, f \in B\}$$

is bounded in $E_{m_2}$. Hence, by Statement 2 of Proposition 1 this set is relatively compact in some space $E_{m_3}$, where $m_3 \geqslant m_1$. It follows that for each $f \in B$ in $E_{m_3}$ the limit $\lim\limits_{\mu\to z} D_\mu(f)$ is well-defined and it is equal to $D_z(f)$. By Statement (i) of Lemma 6 there exists $m \geqslant m_3$ such that operators $D_\mu$, $|\mu - z| \leqslant 1$, linearly and continuously map $E_{m_1}$ into $E_m$. By Banach-Steinhaus theorem [13, Corollary 7.1.4], $\lim\limits_{\mu\to z} D_\mu(f) = D_z(f)$ in $E_m$ uniformly in $f$ on $B$, i.e.,

$$\lim_{\mu\to z} \sup_{f\in B} p_{v_{m,k}}(D_\mu(f) - D_z(f)) = 0$$

for each $k \in \mathbb{N}$.

(iv): We fix $f \in E$ and $z \in \mathbb{C}$. As $\mu \neq z$, since $D_\mu(\tau_{-\mu}(f)) = 0$,

$$\frac{D_\mu(\tau_{-z}(f))}{\mu - z} = D_\mu\Big(\frac{\tau_{-z}(f) - \tau_{-\mu}(f)}{\mu - z}\Big). \tag{6}$$

By Lemma 2, there exists $m \in \mathbb{N}$ such that in $E_m$ the limit $\lim\limits_{\mu\to z} \frac{\tau_{-z}(f)-\tau_{-\mu}(f)}{\mu - z}$ is well-defined and it is equal to $\tau_{-z}(f')$, and the set $B = \left\{\frac{\tau_{-z}(f)-\tau_{-\mu}(f)}{\mu-z} : \ 0 < |\mu - z| \leqslant 1\right\}$ is relatively compact in $E_m$ (see the proof of Lemma 2). By (iii) and Statement (i) of Lemma 6 there exists $r \in \mathbb{N}$ for which $\lim\limits_{\mu\to z} D_\mu(g) = D_z(g)$ in $E_r$ uniformly in $g$ on $B$ and operator $D_z$ linearly and continuously maps $E_m$ into $E_r$. Employing this fact and identity (6), it is easy to show that in $E_r$ the limit $\lim\limits_{\mu\to z} \frac{D_\mu(\tau_{-z}(f))}{\mu-z}$ is defined and it is equal to $D_z(\tau_{-z}(f'))$.

(v): Due to identity (3), as $\mu \neq z$

$$\frac{D_\mu(f) - D_z(f)}{\mu - z} = D_\mu(D_z(f)) + f(z)\frac{D_\mu(\tau_{-z}(g_0))}{\mu - z},$$

and Statement (v) follows from (iii) and (iv). □

Let us prove one more result on the estimate of growth $D_\mu(f)(t)$ w.r.t. $t$ and $\mu$ for $f \in E$.

**Lemma 8.** *Suppose condition (1) holds true and $g_0 \equiv 1$. Then $\forall f \in E$ $\exists m$ $\forall k, l$ $\exists A \geqslant 0$:*

$$|D_\mu(f)(t)| \leqslant A \exp(v_{m,k}(\mu) + v_{m,l}(t)), \ t, \mu \in \mathbb{C}.$$

*Proof.* We observe that function $g_0 \equiv 1$ belongs to $E$ if and only if there exists $n_0 \in \mathbb{N}$ such that for each $n \geqslant n_0$ and $s \in \mathbb{N}$

$$\inf_{z\in\mathbb{C}} v_{n,s}(z) > -\infty. \tag{7}$$

Without loss of generality we can assume that $n_0 = 1$. Let $f \in E_r$. By condition (1), there exists $m \geqslant r$ such that for each $l \in \mathbb{N}$ there exist $s \in \mathbb{N}$ and $C \geqslant 0$ for which

$$\sup_{|w-t|\leqslant 2} v_{r,s}(w) \leqslant v_{m,l}(t) + C, \ t \in \mathbb{C}.$$

For $k, l \in \mathbb{N}$ (employing the maximum principle if $|t-\mu| \leqslant 1$) we obtain: for each $t, \mu \in \mathbb{C}$

$$|D_\mu(f)(t)| \leqslant \sup_{|w-t|\leqslant 2} |f(w)| + |f(\mu)| \leqslant p_{v_{r,s}}(f)\exp(v_{m,l}(t)+C) + p_{v_{m,k}}(f)\exp v_{m,k}(\mu)$$

$$= \Big(p_{v_{r,s}}(f)\exp(C - v_{m,k}(\mu)) + p_{v_{m,k}}(f)\exp(-v_{m,l}(t))\Big)\exp(v_{m,k}(\mu) + v_{m,l}(t)).$$

It remains to note that due to (7)

$$\sup_{t,\mu\in\mathbb{C}} \left(p_{v_{r,s}}(f)\exp(C - v_{m,k}(\mu)) + p_{v_{m,k}}(f)\exp(-v_{m,l}(t))\right) < +\infty.$$

□

## 2. $Q$-interpolating functional and its properties

In what follows $E$ is the space of entire functions defined in Section 1 and the family of function $(v_{n,k})_{n,k\in\mathbb{N}}$ defining this space obeys conditions (1) and (2). Assume that $F$ is a complex locally convex space (LCS) possessing the following properties:

(F1) $(F, E)$ is a dual pair w.r.t. the bilinear form $\langle x, f\rangle$, $x \in F$, $f \in E$.

(F2) The topologies in $F$ and $E$ majorize weak topologies $\sigma(F, E)$ and $\sigma(E, F)$, respectively.

(F3) There exist elements $e_\lambda \in F$, $\lambda \in \mathbb{C}$ such that

$$\langle e_\lambda, g\rangle = g(\lambda), \quad g \in E, \quad \lambda \in \mathbb{C}.$$

**Remark 9.** *A natural example of space $F$ obeying conditions (F1)-(F3) is a topologically dual space $E'$ for $E$ with a topology majorizing the weak topology $\sigma(E', E)$. In this case $e_\lambda$ are delta-functions:*

$$\langle e_\lambda, f\rangle = e_\lambda(f) = f(\lambda), \ \lambda \in \mathbb{C}, \ f \in E.$$

On used here notions in the duality theory see, for instance, [14, Ch. 2].

**Definition 10.** *Let $Q$ be an entire in $\mathbb{C}^2$ function such that $Q(\cdot, z) \in E$ for each $z \in \mathbb{C}$. $Q$-interpolating functional is the mapping $\Omega_Q : \mathbb{C}^2 \times F \to \mathbb{C}$ defined by the identity*

$$\Omega_Q(\mu, z, x) := \langle x, D_\mu(Q(\cdot, z))\rangle, \qquad \mu, z \in \mathbb{C}, \quad x \in F.$$

Let us prove some properties of functional $\Omega_Q$. We denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For LCS $H$ the symbol $H'$ denotes the topologically dual space of $H$.

**Theorem 11.** (i) *For each $\mu, z, \lambda \in \mathbb{C}$*

$$(\lambda - \mu)\Omega_Q(\mu, z, e_\lambda) = Q(\lambda, z) - g_0(\lambda - \mu)Q(\mu, z).$$

(ii) *$\Omega_Q(\mu, z, \cdot) \in F'$ for each $\mu, z \in \mathbb{C}$.*

(iii) *Suppose that a mapping $z \mapsto Q(\cdot, z)$ possesses the following property: for each compact set $M$ in $\mathbb{C}$ there exists $n \in \mathbb{N}$ such that for each $s \in \mathbb{N}$*

$$\sup_{z\in M} p_{v_{n,s}}(Q(\cdot, z)) < +\infty.$$

*Then $\Omega_Q(\cdot, \cdot, x) \in A(\mathbb{C}^2)$ for each $x \in F$.*

(iv) *If $g_0 \equiv 1$, then $\Omega_Q(\cdot, z, x) \in E$ for each $z \in \mathbb{C}$ and $x \in F$.*

*Proof.* (i): Bearing in mind property (F3), for $\mu, z, \lambda \in \mathbb{C}$, $\mu \neq \lambda$, we obtain:

$$\begin{aligned}(\lambda-\mu)\Omega_Q(\mu,z,e_\lambda) =&(\lambda-\mu)\langle e_\lambda, D_\mu(Q(\cdot,z))\rangle = (\lambda-\mu)D_\mu(Q(\cdot,z))(\lambda)\\ =&(\lambda-\mu)\frac{Q(\lambda,z)-g_0(\lambda-\mu)Q(\mu,z)}{\lambda-\mu} = Q(\lambda,z)-g_0(\lambda-\mu)Q(\mu,z).\end{aligned}$$

If $\mu=\lambda$, identity (i) is obvious.

Statement (ii) is implied by property (F2).

(iii): We fix $x \in F$ and $z \in \mathbb{C}$ and take $\mu \in \mathbb{C}$. By Statement (v) of Lemma 7 there exists $r \in \mathbb{N}$ such that in $E_r$ the limit $\lim\limits_{\triangle\mu\to 0}\frac{D_{\mu+\triangle\mu}-D_\mu}{\triangle\mu}(Q(\cdot,z))$ is well-defined and is equal to $D^2_\mu(Q(\cdot,z))+Q(\mu,z)D_\mu(\tau_{-\mu}(g_0')) =: h$. Hence, due to (F2), the limit $\lim\limits_{\triangle\mu\to 0}\frac{\Omega_Q(\mu+\triangle\mu,z,x)-\Omega_Q(\mu,z,x)}{\triangle\mu}$ is well-defined and is equal to $\langle x,h\rangle$. Thus, function $\Omega_Q(\mu,z,x)$ is entire w.r.t. $\mu$.

We fix $x \in F$ and $\mu \in \mathbb{C}$. Expanding entire function $Q(t,z)$ into the power in $z$ series for a fixed $t \in \mathbb{C}$, we obtain:

$$Q(t,z)=\sum_{j=0}^{\infty} a_j(t)z^j,\ t,z\in\mathbb{C}, \tag{8}$$

where $a_j \in A(\mathbb{C})$. We choose $z \in \mathbb{C}$. By Cauchy inequalities

$$|a_j(t)| \leqslant \frac{\sup\limits_{|\xi|\leqslant|z|+1}|Q(t,\xi)|}{(|z|+1)^j},\ j\in\mathbb{N}_0,\ t\in\mathbb{C}.$$

Let $n \in \mathbb{N}$ be such that for each $s \in \mathbb{N}$

$$C_s := \sup_{|\xi|\leqslant|z|+1} p_{v_{n,s}}(Q(\cdot,\xi)) < +\infty.$$

We fix $t \in \mathbb{C}$. Then for each $j \in \mathbb{N}_0$

$$|a_j(t)| \leqslant \frac{C_s \exp v_{n,s}(t)}{(|z|+1)^j}.$$

Thus, series (8) converges absolutely in some space $E_n$ w.r.t. $t$ to $Q(t,z)$, where $n$ depends on $z$. By Statement (i) in Lemma 6 there exists $m \in \mathbb{N}$ such that $D_\mu$ linearly and continuously maps $E_n$ into $E_m$. By property (F2), the linear functional

$$g \mapsto \langle x,g\rangle,\ g\in E, \tag{9}$$

is continuous on $E$, and thus, its restriction on each space $E_l$, $l \in \mathbb{N}$, is continuous, too. In particular, it is valid for $E_m$ [14, Ch. 5, Prop. 5]. Therefore,

$$\begin{aligned}\Omega_Q(\mu,z,x) &= \langle x,(D_\mu)_t(Q(t,z))\rangle = \Big\langle x,(D_\mu)_t\Big(\sum_{j=0}^{\infty} a_j(t)z^j\Big)\Big\rangle\\ &= \Big\langle x,\sum_{j=0}^{\infty} D_\mu(a_j)z^j\Big\rangle = \sum_{j=0}^{\infty}\langle x,D_\mu(a_j)\rangle z^j,\end{aligned}$$

and the latter numerical series converges absolutely. Thus, function $\Omega_Q(\mu,z,x)$ is entire w.r.t. $z$. By Hartogs' theorem [15, Ch. 1, Sect. 2, Subsect. 6], $\Omega_Q(z,\mu,x)$ is entire in $\mathbb{C}^2$ function w.r.t. $(\mu,z)$ for each $x \in F$.

(iv): We fix $z \in \mathbb{C}$ and $x \in F$. By (iii), $\Omega_Q(\mu,z,x)$ is an entire in $\mu$ function. Since linear functional (9) is continuous on $E = \operatorname*{ind}\limits_{n\to}\operatorname*{proj}\limits_{\leftarrow s} E_{ns}$, then $\forall n \in \mathbb{N}\ \exists s \in \mathbb{N}\ \exists \tilde{B} \geqslant 0$:

$$|\Omega_Q(\mu,z,x)| \leqslant \tilde{B}p_{v_{n,s}}(D_\mu(Q(\cdot,z))). \tag{10}$$

By Lemma 8, $\exists m\ \forall k,l\ \exists A \geqslant 0$:

$$|D_\mu(Q(\cdot,z))(t)| \leqslant A\exp(v_{m,k}(\mu)+v_{m,l}(t)),\ \ \mu,t\in\mathbb{C}. \tag{11}$$

Inequalities (10) and (11) with $n=m$, $l=s$ imply that for each $k\in\mathbb{N}$

$$|\Omega_Q(\mu,z,x)| \leqslant A\tilde{B}\exp v_{m,k}(\mu),\ \ \mu\in\mathbb{C}.$$

Hence, $\Omega_Q(\cdot,z,x)\in E$. □

## 3. Examples

**1)** Interpolating function $\omega_L(\mu,x)$ introduced by A.F. Leont'ev (see [1, Ch. IV, Sect. 2]) is a particular case of functional $\Omega_Q$.

Let $G$ be a bounded convex domain in $\mathbb{C}$; $\bar{G}$ be the closure of $G$ in $\mathbb{C}$; $0\in G$; $A(\bar{G})$ be the space of functions analytic on $\bar{G}$ with the natural topology of inductive limit of Banach spaces sequence. Let $H_G$ be the support function of $\bar{G}$, i.e.,

$$H_G(z) := \sup_{t\in\bar{G}} \operatorname{Re}(zt),\ \ z\in\mathbb{C}.$$

We let $F := A(\bar{G})$. As $E$ we consider the weighted Fréchet space

$$E := \left\{ f\in A(\mathbb{C}) \ \Big|\ \forall n\in\mathbb{N}\ \|f\|_n := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{\exp(H_G(z)+|z|/n)} < +\infty \right\},$$

i.e., in our case

$$v_{n,k}(z) = H_G(z)+|z|/k,\ \ n,k\in\mathbb{N},\ \ z\in\mathbb{C},$$

and all LCS's $E_n$ coincide. The family of functions $(v_{n,k})_{n,k\in\mathbb{N}}$ satisfy conditions (1) and (2).

By $\mathcal{F}$ we denote the Laplace transform:

$$\mathcal{F}(\varphi)(z) := \varphi_t(\exp(tz)),\ \ z\in\mathbb{C},\ \ \varphi\in A(\bar{G})'.$$

As it is known [16, Thm. 4.5.3], $\mathcal{F}$ is a topological isomorphism of strongly dual space $A(\bar{G})'_b$ of $A(\bar{G})$ on $E$. The bilinear form

$$\langle x,f\rangle := \mathcal{F}^{-1}(f)(x),\ \ x\in F,\ \ f\in E, \tag{12}$$

defines the duality between $F$ and $E$, i.e., condition (F1) is satisfied. Due to (12), condition (F2) holds as well. If $e_\lambda(z) := \exp(\lambda z)$, $\lambda, z\in\mathbb{C}$, then

$$\langle e_\lambda, f\rangle = f(\lambda),\ \ \lambda\in\mathbb{C},\ \ f\in E,$$

and thus, condition (F3) is satisfied. Let $L$ be an entire function of exponential type with the adjoint diagram $\bar{G}$. In accordance with [1, Ch. IV, Sect. 2],

$$\omega_L(\mu,x) = \frac{1}{2\pi i}\int_C \gamma(t)\Big(\int_0^t x(t-\xi)e^{\mu\xi}d\xi\Big)dt,$$

where $\gamma$ is the function associated with $L$ in the Borel sense, $C$ is a contour enveloping $\bar{G}$ and located in the analyticity domain of $x$ and $\gamma$.

We let $Q(\mu,z) := L(\mu)$, $\mu,z\in\mathbb{C}$. Since $0\in G$, as $g_0$ we can take $g_0\equiv 1$. Let us show that

$$\Omega_Q(\mu,z,x) = \omega_L(\mu,x),\ \ \mu,z\in\mathbb{C},\ \ x\in F.$$

Since for $\mu,z\in\mathbb{C}$ the linear functionals $\Omega_Q(\mu,z,\cdot)$ and $\omega_L(\mu,\cdot)$ are continuous on $F$ (Statement (ii) of Theorem 11 and [1, Property 5, P. 243], respectively), by the completeness of the family $\{e_\lambda : \lambda\in\mathbb{C}\}$ in $F$, it is sufficient to show that

$$\Omega_Q(\mu,z,e_\lambda) = \omega_L(\mu,e_\lambda)$$

for $\lambda \in \mathbb{C}$. Since $\Omega(\mu, z) = L(\mu)$ for each $\mu, z \in \mathbb{C}$, then

$$\Omega_Q(\mu, z, e_\lambda) = \frac{Q(\lambda, z) - Q(\mu, z)}{\lambda - \mu} = \frac{L(\lambda) - L(\mu)}{\lambda - \mu}.$$

By [1, Property 3, P. 242], we also have

$$\omega_L(\mu, e_\lambda) = \frac{L(\lambda) - L(\mu)}{\lambda - \mu}.$$

**2)** Interpolating functional introduced in [2, Sect. 3] on the base of A.F. Leont'ev interpolating function is also a particular case of the functional studied in the present work.

Let $G$, $H_G$ be as in 1); $F := A(G)$ be the space of functions analytic in $G$ with the topology of uniform convergence on compact sets in $G$. As $E$ we consider the countable inductive limit of weighted Banach spaces:

$$E := \left\{ f \in A(\mathbb{C}) \;\middle|\; \exists n \in \mathbb{N}\ |f|_n := \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp(H_G(z) - |z|/n)} < +\infty \right\},$$

i.e., in the present case

$$v_{n,k}(z) = H_G(z) - |z|/n,\ n, k \in \mathbb{N},\ z \in \mathbb{C},$$

and all LCS's $E_n$ are Banach spaces. The family of functions $(v_{n,k})_{n,k \in \mathbb{N}}$ satisfies conditions (1) and (2). Let $e_\lambda(z) := \exp(\lambda z)$, $\lambda, z \in \mathbb{C}$.

The Laplace transform

$$\mathcal{F}(\varphi)(z) := \varphi(e_z),\ z \in \mathbb{C},\ \varphi \in A(G)',$$

is a topological isomorphism of strongly dual space $A(G)'_b$ for $A(G)$ on $E$ [16, Thm. 4.5.3]. The bilinear form

$$\langle x, f \rangle := \mathcal{F}^{-1}(f)(x),\ x \in F,\ f \in E,$$

defines the duality between $F$ and $E$. As in 1), conditions (F1)-(F3) hold true.

Let $Q$ be an entire in $\mathbb{C}^2$ function such that $Q(\cdot, z) \in E$ for each $z \in \mathbb{C}$. In accordance with [2, Sect. 3, Def. 3.1], $Q$-interpolating functional is introduced as follows:

$$\tilde{\Omega}_Q(\mu, z, x) := \mathcal{F}^{-1}(Q(\cdot, z))_t \Big( \int_0^t x(t - \xi) \exp(\mu \xi) d\xi \Big),\ \mu, z \in \mathbb{C},\ x \in A(G).$$

To distinguish it from our functional, we have denoted it slightly different than in [2]. In the present case we can also take $g_0 \equiv 1$. The continuity of functionals $\tilde{\Omega}_Q(\mu, z, \cdot)$ and $\Omega_Q(\mu, z, \cdot)$ on $A(G) = F$ ([2, Sect. 3, Lm. 3.2 (b)] and Statement (ii) of Theorem 11, respectively), the completeness of the system $\{e_\lambda : \lambda \in \mathbb{C}\}$ in $A(G)$ and the identity $\Omega_Q(\mu, z, e_\lambda) = \tilde{\Omega}_Q(\mu, z, e_\lambda)$ for each $\mu, z, \lambda \in \mathbb{C}$ yield that $\Omega_Q = \tilde{\Omega}_Q$ on $\mathbb{C}^2 \times F$. We note that the identity $\Omega_Q(\mu, z, e_\lambda) = \frac{Q(\lambda,z) - Q(\mu,z)}{\lambda - \mu}$ was established in [2, Lm. 3.2 (b)] for $\mu = z$; it is obviously true for each $\mu, z \in \mathbb{C}$.

**3)** Let $G$ be a bounded convex set in $\mathbb{C}$ containing the origin. Suppose that $G$ is locally closed, i.e., it has a countable fundamental system of compact subsets $G_n \subseteq G$, $n \in \mathbb{N}$. We can assume that all the compact sets $G_n$ are convex and $G_n \subseteq G_{n+1}$, $n \in \mathbb{N}$ (see, for instance, [4], [17], [18]). Let $F := A(G) := \operatorname{proj}_{\leftarrow n} A(G_n)$ be the space of the germs of functions analytic in $G$ with the topology of projective limit of $(LB)$-spaces $A(G_n)$, $n \in \mathbb{N}$. We introduce a weighted $(LF)$-space $E := \operatorname{ind}_{n \rightarrow} \operatorname{proj}_{\leftarrow k} A_{nk}$, where a Banach space $A_{nk}$ is defined as

$$A_{nk} := \left\{ f \in A(\mathbb{C}) : \|f\|_{nk} := \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp(H_{G_n}(z) + |z|/k)} < +\infty \right\},$$

$H_{G_n}$ is the support function of $G_n$. In the present case

$$v_{n,k}(z) := H_{G_n}(z) + |z|/k,\ z \in \mathbb{C},\ n,k \in \mathbb{N};$$

the family $(v_{n,k})_{n,k\in\mathbb{N}}$ obeys conditions (1) and (2).

As above, $e_\lambda(z) := \exp(\lambda z)$, $\lambda, z \in \mathbb{C}$. The Laplace transform

$$\mathcal{F}(\varphi)(z) := \varphi(e_z),\ z \in \mathbb{C},\ \varphi \in A(G)',$$

defines a topological isomorphism of strongly dual space $A(G)'_b$ for $A(G)$ on $E$ [17, Lm. 1.10]. The bilinear form

$$< x, f >:= \mathcal{F}^{-1}(f)(x),\ x \in F,\ f \in E,$$

defines the natural duality between $F$ and $E$; properties (F1)-(F3) are satisfied.

Let $\tilde{L}$ be an entire in $\mathbb{C}^2$ function such that $\tilde{L}(\cdot, z) \in E$ for each $z \in \mathbb{C}$. In [4, Sect. 3] $\tilde{L}$-interpolating functional $\Omega_{\tilde{L}}$ was introduced as

$$\Omega_{\tilde{L}}(\mu, z, x) := \mathcal{F}^{-1}(\tilde{L}(\cdot, z))_t\Big(\int\limits_0^t x(t-\xi)\exp(\mu\xi)d\xi\Big),\ \mu, z \in \mathbb{C},\ x \in F.$$

And in this case we again can take $g_0 \equiv 1$. We let $Q := \tilde{L}$. As in 1) and 2), due to the continuity of the functionals $\Omega_{\tilde{L}}(\mu, z, \cdot)$ and $\Omega_Q(\mu, z, \cdot)$ on $F$ ([4, Lm. 3.3] and Statement (ii) of Theorem 11, respectively), the completeness of system $\{e_\lambda : \lambda \in \mathbb{C}\}$ in $F$ and the identity $\Omega_Q(\mu, z, e_\lambda) = \Omega_{\tilde{L}}(\mu, z, e_\lambda)$ for each $\mu, z \in \mathbb{C}$, the identity $\Omega_{\tilde{L}} = \Omega_Q$ holds true on $\mathbb{C}^2 \times F$. We observe that identity $\Omega_{\tilde{L}}(\mu, z, e_\lambda) = \frac{\tilde{L}(\lambda,z)-\tilde{L}(\mu,z)}{\lambda-\mu}$ was established in [4, Lm. 3.3] for $\mu = z$; it is obvious that it holds true for each $\mu, z \in \mathbb{C}$.

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