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ON UNIFORM CONVERGENCE OF PIECEWISE-LINEAR SOLUTIONS TO MINIMAL SURFACE EQUATION

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Abstract. In the paper we consider piecewise-linear solutions of the minimal surface equation over a given triangulation of a polyhedral domain. It is shown that under certain conditions, the gradients of these functions are bounded as the maximal diameter of the triangles of the triangulation tends to zero. It is stressed that this property holds if the piecewise-linear function approximates the area of the graph of a smooth function with a required accuracy. An implication of the obtained properties is the uniform convergence of piecewise linear solutions to the exact solution of the minimal surface equation.

Keywords: : piecewise-linear functions, minimal surface equation, the approximation of the area functional

Mathematics Subject Classification: 35J25, 35J93, 65N30

1. Introduction

Some problems appearing in designing architectural structures are reduced to constructing surfaces of minimal area. This was reflected quite in detail in book [1] and work [2], where the problem of developing of awning cloth construction was studied. The detailed analysis of the provided results leads one to the problem on developing effective methods for approximative solving the minimal surface equations and mathematical justification of the found methods in the sense of stability and convergence of approximate solutions. The main difficulty in studying such issues is that the minimal surface equation is non-linear and this is way classical methods employed for linear equations are not applicable.

Our approach is based on introducing the notion of a piecewise-linear solutions to the minimal surface equation over a given triangulation of a given domain and we establish needed properties for these solutions. Namely, we show that the accuracy order of approximation for area functional with respect to the triangles diameters equals two; we establish that the partial derivatives are bounded by a constant independent of the fineness of partition under the sufficient approximation of the area functional and so forth. The proved statements allow us to establish, in particular, the uniform convergence as the diameters of triangulation triangles tend to zero.

2. Piece-wise linear solution to minimal surface equation

Suppose we are given a polyhedral bounded domain $\Omega \subset \mathbb{R}^n$. Consider a partition of this polyhedron into non-degenerate tetrahedrons T_1, T_2, ..., T_N. Let M_1, M_2, …, M_m be all the vertices of these tetrahedrons. We shall assume that each of points M_i is neither internal for any side non for any edge of the tetrahedrons.

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For an arbitrary set of values $u_1, u_2, \dots, u_m$ we define a piece-wise linear function $u : \Omega \to \mathbb{R}$ so that $u(M_i) = u_i, i = 1, \dots, m$ and we introduce the function $u(x) = p_1^k x_1 + \dots + p_n^k x_n + b^k$ on each tetrahedron $T_k, k = 1, \dots, N$. This function is continuous in Ω and in each tetrahedrons T_k the gradient $\nabla u \equiv p^k = (p_1^k, \dots, p_n^k)$ is well-defined. Therefore, the area under the graph of function u is given by the sum

$$S(p) = S(p^1, \dots, p^N) = \sum_{k=1}^{N} \int_{T_k} \sqrt{1 + |\nabla u|^2}\, dx = \sum_{k=1}^{N} \sqrt{1 + |p^k|^2}\; v(T_k),$$

where $v(T_k)$ is the n-dimensional volume of tetrahedron T_k.

Since vectors $p^1, \dots, p^N$ are uniquely defined by values $u_1, \dots, u_m$, we can write the value of the area $S(p)$ in terms of the variables $u = (u_1, \dots, u_m)$: $S(u) = S(u_1, \dots, u_m)$. Indeed, variables $p^1, \dots, p^N$ are expressed linearly in terms of variables $u_1, \dots, u_m$. Then there exist numbers a_{li}^k such that

$$p_l^k = \sum_{i=1}^{m} a_{li}^k u_i,\ \ k = 1, \dots, N,\ \ l = 1, \dots, n.$$

Coefficients a_{li}^k are uniquely defined by the partition of domain Ω into tetrahedrons $T_1, \dots, T_N$. Thus,

$$S(u) = S(u_1, \dots, u_m) = \sum_{k=1}^{N} \sqrt{1 + \sum_{l=1}^{n} (\sum_{i=1}^{m} a_{li}^k u_i)^2} \cdot v(T_k).$$

Suppose that in vertices $M_1, \dots, M_m$ we are given values $\varphi_1, \dots, \varphi_m$. We denote by φ the associated piecewise-linear function constructed by these values. Let us state the problem on finding a piecewise-linear function u providing the minimum of area $S(u)$ and satisfying the boundary condition, i.e.,

$$S(u_1, \dots, u_m) \to \min, \qquad u(M_i) = \varphi_i, \qquad \forall\ M_i \in \partial\Omega. \tag{1}$$

Remark. *Let $u^* = (u_1^*, \dots, u_m^*)$ be a solution to problem* (1)*. By u^* we shall denote the associated piecewise-linear function. Suppose that $h(x)$ is an arbitrary piecewise-linear function satisfying the condition $h(M_i) = 0$ for each point $M_i \in \partial\Omega$. Then the function $\sigma(t) = S(u^* + th)$ achieves the minimum at the point $t = 0$. Hence, $\sigma'(0) = 0$ that is equivalent to the identity*

$$\sum_{k=1}^{N} \int_{T_k} \frac{\langle \nabla u^*, \nabla h \rangle}{\sqrt{1 + |\nabla u^*|^2}} dx = 0. \tag{2}$$

Theorem 1. *Problem* (1) *is uniquely solvable.*

Proof. We observe that function $S(u_1, \dots, u_m)$ is convex with respect to variables $u_1, \dots, u_m$. And since in the boundary points the values of function u are fixed,

$$\lim_{|u| \to +\infty} S(u) = +\infty,$$

where $|u| = \max\limits_{1 \leqslant i \leqslant m} |u_i|$. Hence, function $S(u)$ achieves its minimum at some point u^*.

Let us prove the uniqueness. We assume the opposite, i.e., there exists one more solution v^* to problem (1). Then piecewise-linear function v^* satisfies condition (2) as well. Letting $h = v^* - u^*$, we arrive at the identity

$$\sum_{k=1}^{N} \int_{T_k} \left(\frac{\langle \nabla v^*, \nabla (v^* - u^*) \rangle}{\sqrt{1 + |\nabla v^*|^2}} - \frac{\langle \nabla u^*, \nabla (v^* - u^*) \rangle}{\sqrt{1 + |\nabla u^*|^2}} \right) dx = 0. \tag{3}$$

In what follows we shall make use of the inequality

$$\left\langle \frac{\xi}{\sqrt{1+|\xi|^2}} - \frac{\eta}{\sqrt{1+|\eta|^2}}, \xi-\eta \right\rangle \geqslant \frac{|\xi-\eta|^2}{\sqrt{1+|\xi|^2}(\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2}+|\xi||\eta|+1)}, \tag{4}$$

satisfied for all vectors $\xi, \eta \in \mathbb{R}^n$. We note that similar inequalities were obtained in works [3], [4], [5] and were also employed for studying the issues on uniqueness of solution to the minimal surface equation. Inequality (4) by means of which we shall obtain the uniqueness will be also employed for estimating the gradient of piecewise-linear solution u^*. And we can not apply the inequalities from the above cited works for this estimate. This is the reason why we employ inequality (4). We prove it as follows.

First we note that

$$\sqrt{1+|\xi|^2} \geqslant \sqrt{1+|\eta|^2} + \frac{\langle \eta, \xi-\eta\rangle}{\sqrt{1+|\eta|^2}}.$$

Then

$$\begin{aligned}
\left\langle \frac{\xi}{\sqrt{1+|\xi|^2}} - \frac{\eta}{\sqrt{1+|\eta|^2}}, \xi-\eta \right\rangle &= -\frac{\langle \xi, \eta-\xi\rangle}{\sqrt{1+|\xi|^2}} - \frac{\langle \eta, \xi-\eta\rangle}{\sqrt{1+|\eta|^2}} \\
&\geqslant \sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta-\xi\rangle}{\sqrt{1+|\xi|^2}} \\
&= \frac{\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2} - \langle \xi, \eta\rangle - 1}{\sqrt{1+|\xi|^2}} \\
&\geqslant \frac{|\xi-\eta|^2}{\sqrt{1+|\xi|^2}(\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2}+|\xi||\eta|+1)}.
\end{aligned} \tag{5}$$

Letting $\xi = \nabla u^*$ and $\eta = \nabla v^*$ in inequality (4), by (3) we obtain that $\nabla u^* \equiv \nabla v^*$. Since on boundary $\partial\Omega$ functions u^* and v^* coincide, we obtain the desired identity $u^* \equiv v^*$. □

3. Estimate for modulus of gradient

Let f be a solution to the minimal surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{\sqrt{1+|\nabla f|^2}} \right) = 0 \tag{6}$$

in domain Ω continuous in $\overline{\Omega}$ and $f|_{\partial\Omega} = \varphi|_{\partial\Omega}$, where $\varphi(x)$ is a continuous function defined on the boundary of domain Ω. It is worth mentioning that the corresponding Dirichlet problem for an arbitrary domain (even with a smooth boundary) is not always solvable. For planar domains the solvability criteria of the Dirichlet problem for an arbitrary continuous boundary function $\varphi(x)$ is the convexity of this domains. In the space of the dimension greater than two such condition is the nonnegativity of the mean curvature with respect to the outward normal for the boundary of the domain. One can find the precise formulations and the proofs of these results in works [6]–[13]. In the present paper we impose no conditions for domain Ω but we assume that for a given boundary function $\varphi(x)$ the solution to the Dirichlet problem exists. It is clear that such function $\varphi(x)$ exists for an arbitrary domain Ω.

In what follows we denote by u^* the unique solution to problem (1) with boundary data $\varphi_i = \varphi(M_i)$, $M_i \in \partial\Omega$.

For arbitrary $\xi, \eta \in \mathbb{R}^n$ we let

$$\delta(\xi,\eta) = \sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta-\xi\rangle}{\sqrt{1+|\xi|^2}}.$$

It follows from inequality (5) that $\delta(\xi,\eta) > 0$ for each $\xi \neq \eta$. Letting $\xi = \nabla f$, $\eta = \nabla u^*$ and employing equation (6), we obtain

$$\sum_{k=1}^{N} \int\limits_{T_k} \delta(\nabla f, \nabla u^*)\, dx = S(u^*) - S(f) + \int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^* - f)}{\sqrt{1+|\nabla f|^2}}\, dS,$$

where ν is the outward normal to $\partial\Omega$ in the points where it exists. Employing inequality (see the proof of Theorem 1)

$$\sqrt{1+|\eta|^2} - \sqrt{1+|\xi|^2} - \frac{\langle \xi, \eta-\xi\rangle}{\sqrt{1+|\xi|^2}} \geqslant \frac{|\xi-\eta|^2}{\sqrt{1+|\xi|^2}(\sqrt{1+|\xi|^2}\sqrt{1+|\eta|^2} + |\xi||\eta| + 1)},$$

we conclude that

$$\sum_{k=1}^{N} \int\limits_{T_k} \frac{|\nabla f - \nabla u^*|^2\, dx}{\sqrt{1+|\nabla f|^2}(\sqrt{1+|\nabla f|^2}\sqrt{1+|\nabla u^*|^2} + |\nabla f||\nabla u^*| + 1)}$$
$$\leqslant S(u^*) - S(f) + \int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^* - f)}{\sqrt{1+|\nabla f|^2}}\, dS.$$

We fix arbitrary $k = 1, \ldots, N$. Then

$$\begin{aligned} &\int\limits_{T^k} \frac{|\nabla f - \nabla u^*|^2\, dx}{\sqrt{1+|\nabla f|^2}(\sqrt{1+|\nabla f|^2}\sqrt{1+|\nabla u^*|^2} + |\nabla f||\nabla u^*| + 1)} \\ &\qquad \leqslant S(u^*) - S(f) + \int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^* - f)}{\sqrt{1+|\nabla f|^2}}\, dS \equiv B. \end{aligned} \tag{7}$$

In what follows we assume that $|\nabla f| \leqslant P_0$ in domain Ω. Then inequality (7) yields

$$\int\limits_{T^k} \frac{|\nabla f - \nabla u^*|^2}{\sqrt{1+|\nabla u^*|^2}} dx \leqslant 3(1+P_0^2)B. \tag{8}$$

It implies that

$$\int\limits_{T^k} \frac{|\nabla u^*|^2}{\sqrt{1+|\nabla u^*|^2}} dx \leqslant 3(1+P_0^2)B + 2P_0 v(T^k)$$

or

$$\int\limits_{T^k} \sqrt{1+|\nabla u^*|^2} dx \leqslant 3(1+P_0^2)B + (2P_0+1)v(T^k).$$

Then by (8) and Hölder inequality we arrive at the estimate

$$\int\limits_{T^k} |\nabla f - \nabla u^*| dx \leqslant 3(1+P_0^2)\left(\left(B + v(T^k)\right) B\right)^{1/2}.$$

Hence,

$$\int\limits_{T^k} |\nabla u^*| dx \leqslant P_0 v(T^k) + 3(1+P_0^2)\left(\left(B + v(T^k)\right) B\right)^{1/2}.$$

Since the gradient ∇u^* is constant in T_k, we obtain the inequality

$$v(T_k)|\nabla u^*(x)| \leqslant P_0 v(T_k) + 3(1+P_0^2)\left(\left(B + v(T_k)\right) B\right)^{1/2}, \quad x \in T_k.$$

Dividing by $v(T_k)$, we arrive at the estimate of the gradient

$$|\nabla u^*(x)| \leqslant P_0 + 3(1+P_0^2)\sqrt{(\alpha_k + 1)\,\alpha_k}\,, \tag{9}$$

where

$$\alpha_k = \frac{1}{v(T_k)}\left(S(u^*) - S(f) + \int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^* - f)}{\sqrt{1+|\nabla f|^2}}\, dS\right).$$

Theorem 2. *Let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to equation (6) such that $f|_{\partial\Omega} = \varphi|_{\partial\Omega}$ and $P_0 = \sup\limits_{\Omega} |\nabla f| < +\infty$. Suppose that u^* is a solution to problem (1) subject to the condition $u^*(M_i) = \varphi(M_i)$, $M_i \in \partial\Omega$. Then inequality (9) holds true for each point $x \in \Omega$.*

Remark. *Denote by f^L the piecewise-linear function constructed by the values of functionf at points $M_i, i = 1\ldots, m$. If the quantity*

$$A(f) = \frac{1}{\min\limits_{1\leqslant k\leqslant N} v(T_k)}\left(S(f^L) - S(f) + \int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^* - f)}{\sqrt{1+|\nabla f|^2}}\, dS\right)$$

is bounded for sufficiently smooth functions f as the fineness of partition $\mu = \max\limits_{1\leqslant k\leqslant N} \operatorname{diam} T_k$ of domain Ω tends to zero in a certain way, then by Theorem 2 and the inequality $S(u^*) \leqslant S(f^L)$ we conclude that the approximate solution u^* has a gradient bounded by a constant independent of the partition fineness.

4. Approximation of area functional

Let us study the quantity $S(f^L) - S(f)$ for functions $f \in C^3(\Omega)$ as $n = 2$, where Ω is a closed rectangle with the sides parallel to the coordinate axes. Consider the surface defined as the graph of a function $z = f(x, y)$ over set Ω.

Let $\Omega = [a, b] \times [c, d]$ and $a = x_0 < x_1 < \cdots < x_n = b$, $c = y_0 < y_1 < \cdots < y_m = d$. We partition Ω into rectangles $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $0 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m-1$. Then we split each of these rectangles by the right or left diagonal. All the following arguments are made for the rectangle $[x_0, x_1] \times [y_0, y_1]$ since the estimate for error of calculating the area for other rectangles is similar.

If the splitting is made by the right diagonal, the area under the graph of piecewise-linear function over the diagonal is

$$S_U^r = \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\sqrt{1 + \left(\frac{f(x_1, y_1) - f(x_0, y_1)}{x_1 - x_0}\right)^2 + \left(\frac{f(x_0, y_1) - f(x_0, y_0)}{y_1 - y_0}\right)^2},$$

and for the triangle under the diagonal

$$S_D^l = \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\sqrt{1 + \left(\frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}\right)^2 + \left(\frac{f(x_0, y_1) - f(x_1, y_1)}{y_1 - y_0}\right)^2}.$$

If the triangle is split by the left diagonal, the area under the graph of the piecewise-linear function over the diagonal equals

$$S_U^l = \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\sqrt{1 + \left(\frac{f(x_0, y_1) - f(x_1, y_1)}{x_1 - x_0}\right)^2 + \left(\frac{f(x_1, y_0) - f(x_1, y_1)}{y_1 - y_0}\right)^2},$$

and for the triangle under the diagonal

$$S_D^l = \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\sqrt{1 + \left(\frac{f(x_1, y_0) - f(x_0, y_1)}{x_1 - x_0}\right)^2 + \left(\frac{f(x_0, y_1) - f(x_0, y_0)}{y_1 - y_0}\right)^2}.$$

We denote by S the area of the obtained piecewise-linear surface. We have

Theorem 3. *Let $f(x,y) \in C^3(\Omega)$ and*

$$M_2 = \sup_{\Omega} \max\{|f''_{xx}|, |f''_{xy}|, |f''_{yy}|\}, \quad M_3 = \sup_{\Omega} \max\{|f'''_{xxx}|, |f'''_{xxy}|, |f'''_{xyy}|, |f'''_{yyy}|\},$$

$$h_1 = \max_{1\leqslant i\leqslant n}(x_i - x_{i-1}), \quad h_2 = \max_{1\leqslant j\leqslant m}(y_j - y_{j-1}), \quad h = \max(h_1, h_2).$$

Then

$$\left| \iint_{\Omega} \sqrt{1 + |\nabla f(x,y)|^2}\, dxdy - S \right|$$

$$\leqslant |\Omega| \left(2M_3 + 12\left(3 + \frac{h}{\min\{h_1, h_2\}}\right) M_2^2 + \frac{5}{2} M_3^2 h^2 \right) h^2.$$

Proof. For the sake of brevity we assume that the rectangles in the partition of set Ω are split by the right diagonal. The results for this case and for the general case coincide.

Consider the rectangle $[x_0, x_1] \times [y_0, y_1]$. By means of Newton interpolation formula (see [14, Ch. 3, Sect. 12]) we obtain a linear approximation l_U and l_D for function $f(x,y)$ over upper and lower partition triangles of this rectangle

$$\begin{aligned} f(x,y) =& l_U(x,y) + R_U(x,y) = f(x_0,y_0) + (x-x_0) f(x_0;x_1,y_0) \\ &+ (y-y_0) f(x_0,y_0;y_1) + (y-y_0)(y-y_1) f(x,y_0;y_1;y) \\ &+ (x-x_0)(y-y_1) f(x_0;x,y_0;y_1) + (x-x_0)(x-x_1) f(x_0;x_1;x,y_0), \\ f(x,y) =& l_D(x,y) + R_D(x,y) = f(x_1,y_1) + (x-x_1) f(x_1;x_0,y_1) \\ &+ (y-y_1) f(x_1,y_1;y_0) + (y-y_1)(y-y_0) f(x,y;y_1;y_0) \\ &+ (x-x_1)(y-y_0) f(x_0;x,y_1;y_0) + (x-x_0)(x-x_1) f(x_0;x_1;x,y_1), \end{aligned}$$

where $f(\alpha_1;\alpha_2;\alpha_3;\cdots;\alpha_n,\beta_1;\beta_2;\beta_3;\cdots;\beta_m)$ are separated differences of function $f(x,y)$ (see [14, Ch. 2, Sect. 5]). In particular,

$$\begin{aligned} f(x_0;x_1,y_0) =& \frac{f(x_1,y_0) - f(x_0,y_0)}{x_1 - x_0}, \qquad & f(x_0,y_0;y_1) &= \frac{f(x_0,y_1) - f(x_0,y_0)}{y_1 - y_0}, \\ f(x_0;x_1,y_1) =& \frac{f(x_1,y_1) - f(x_0,y_1)}{x_1 - x_0}, \qquad & f(x_1;y_0,y_1) &= \frac{f(x_1,y_1) - f(x_1,y_0)}{y_1 - y_0}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{f(x_1,y_0) - f(x_0,y_0)}{x_1 - x_0} + \frac{\partial R_U}{\partial x}(x,y), \\ \frac{\partial f}{\partial y}(x,y) &= \frac{f(x_0,y_1) - f(x_0,y_0)}{y_1 - y_0} + \frac{\partial R_U}{\partial y}(x,y), \\ \frac{\partial f}{\partial x}(x,y) &= \frac{f(x_1,y_1) - f(x_0,y_1)}{x_1 - x_0} + \frac{\partial R_D}{\partial x}(x,y), \\ \frac{\partial f}{\partial y}(x,y) &= \frac{f(x_1,y_1) - f(x_1,y_0)}{y_1 - y_0} + \frac{\partial R_D}{\partial y}(x,y). \end{aligned}$$

Consider the difference of areas under the graph of the given function and the graph of the obtained interpolation by the linear function over the upper partition triangle

$$\iint_U \sqrt{1 + |\nabla f|^2}\, dxdy - S_U^r = \iint_U \sqrt{1 + |\nabla f|^2}\, dxdy$$

$$- \iint_U \sqrt{1 + \left(\frac{f(x_1,y_1) - f(x_0,y_1)}{x_1 - x_0}\right)^2 + \left(\frac{f(x_0,y_1) - f(x_0,y_0)}{y_1 - y_0}\right)^2}\, dxdy$$

$$= \iint\limits_U \sqrt{1+|\nabla f|^2}\,dxdy - \iint\limits_U \sqrt{1+\left(\frac{\partial f}{\partial x}(x,y)-\frac{\partial R_U}{\partial x}(x,y)\right)^2+\left(\frac{\partial f}{\partial y}(x,y)-\frac{\partial R_U}{\partial y}(x,y)\right)^2}$$

$$= \iint\limits_U C_U(x,y)\Big(2\frac{\partial f}{\partial x}(x,y)\frac{\partial R_U}{\partial x}(x,y)+2\frac{\partial f}{\partial y}(x,y)\frac{\partial R_U}{\partial y}(x,y)$$

$$-\left(\frac{\partial R_U}{\partial x}(x,y)\right)^2-\left(\frac{\partial R_U}{\partial y}(x,y)\right)^2\Big)\,dxdy,$$

where

$$C_U(x,y) = \left(\sqrt{1+|\nabla f|^2}+\sqrt{1+\left(\frac{\partial f}{\partial x}-\frac{\partial R_U}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}-\frac{\partial R_U}{\partial y}\right)^2}\right)^{-1}.$$

Let $y=\Gamma_1(x)$ define the hypotenuse of the partition triangle and $x=\Gamma_2(y)$ be the inverse function. We integrate by parts in the integral in the right hand side of the identity

$$\iint\limits_U \sqrt{1+|\nabla f|^2}\,dxdy - S^r_U = 2\int\limits_{y_0}^{y_1}\Big(R_U(x,y)\frac{\partial f}{\partial x}(x,y)C_U(x,y)\Big|_{x=x_0}^{x=\Gamma_2(y)}$$

$$-\int\limits_{x_0}^{\Gamma_2(y)} R_U(x,y)\left[\frac{\partial f}{\partial x}(x,y)\frac{\partial C_U}{\partial x}(x,y)+\frac{\partial^2 f}{\partial x^2}(x,y)C_U(x,y)\right]\,dx\Big)\,dy$$

$$+2\int\limits_{x_0}^{x_1}\Big(R_U(x,y)\frac{\partial f}{\partial y}(x,y)C_U(x,y)\Big|_{y=\Gamma_1(x)}^{y=y_1}$$

$$-\int\limits_{\Gamma_1(x)}^{y_1} R_U(x,y)\left[\frac{\partial f}{\partial y}(x,y)\frac{\partial C_U}{\partial y}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)C_U(x,y)\right]\,dy\Big)\,dx - \iint\limits_U |\nabla R_U(x,y)|^2 dxdy.$$

We indicate by $C_D(x,y)$ the function on D similar to function $C_U(x,y)$. In the same way on D we obtain the identity

$$\iint\limits_D \sqrt{1+|\nabla f|^2}\,dxdy - S^r_D = 2\int\limits_{y_0}^{y_1}\Big(R_D(x,y)\frac{\partial f}{\partial x}(x,y)C_D(x,y)\Big|_{x=\Gamma_2(y)}^{x=x_1}$$

$$-\int\limits_{\Gamma_2(y)}^{x_1} R_D(x,y)\left[\frac{\partial f}{\partial x}(x,y)\frac{\partial}{\partial C_D x}(x,y)+\frac{\partial^2 f}{\partial x^2}(x,y)C_D(x,y)\right]\,dx\Big)\,dy$$

$$+2\int\limits_{x_0}^{x_1}\Big(R_D(x,y)\frac{\partial f}{\partial y}(x,y)C_D(x,y)\Big|_{y=y_0}^{y=\Gamma_1(x)}$$

$$-\int\limits_{y=y_0}^{\Gamma_1(x)} R_D(x,y)\left[\frac{\partial f}{\partial y}(x,y)\frac{\partial C_D}{\partial y}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)C_D(x,y)\right]\,dy\Big)\,dx - \iint\limits_D |\nabla R_D(x,y)|^2 dxdy.$$

We sum up the obtained identities taking into account that on Γ the values of continuous functions coincide

$$\begin{aligned}
\iint\limits_{U\cup D} &\sqrt{1+|\nabla f|^2}\,dxdy - \Big(S^r_U + S^r_D\Big) \\
&= 2\int\limits_{x_0}^{x_1}\Big(R_U(x,y_1)f'_y(x,y_1)C_U(x,y_1) - R_D(x,y_0)f'_y(x,y_0)C_D(x,y_0)\Big)dx \\
&+ 2\int\limits_{y_0}^{y_1}\Big(R_D(x_1,y)f'_x(x_1,y)C_D(x_1,y) - R_U(x_0,y)f'_x(x_0,y)C_U(x_0,y)\Big)dy \\
&+ 2\int\limits_{x_0}^{x_1} R_U(\Gamma_1(x))f'_y(\Gamma_1(x))\Big(C_U(\Gamma_1(x)) - C_D(\Gamma_1(x))\Big)dx \\
&- 2\int\limits_{y_0}^{y_1} R_U(\Gamma_2(y))f'_x(\Gamma_2(y))\Big(C_D(\Gamma_2(y)) - C_U(\Gamma_2(y))\Big)dy \\
&- 2\iint\limits_{U}\Big(R_U\,\mathrm{div}(C_U(x,y)\nabla f(x,y)) + \frac{1}{2}|\nabla R_U(x,y)|^2\Big)\,dxdy \\
&- 2\iint\limits_{D}\Big(R_D\,\mathrm{div}(C_D(x,y)\nabla f(x,y)) + \frac{1}{2}|\nabla R_D(x,y)|^2\Big)\,dxdy.
\end{aligned} \tag{10}$$

It is easy to make sure that

$$\begin{aligned}
&|C_U(x,y) - C_D(x,y)| \leqslant 2C_U(x,y)C_D(x,y)M_2h, \\
&|C_U(x_1,y) - C_D(x_0,y)| \leqslant 4C_U(x,y)C_D(x,y)M_2h_1, \\
&|C_U(x,y_1) - C_D(x,y_0)| \leqslant 4C_U(x,y)C_D(x,y)M_2h_2.
\end{aligned} \tag{11}$$

Then, employing that the separated differences are equal to the values of the derivatives at some point of the domain, we obtain

$$\begin{aligned}
&\left|\int\limits_{x_0}^{x_1}\Big(R_U(x,y_1)\frac{\partial f}{\partial y}(x,y_1)C_U(x,y_1) - R_D(x,y_0)\frac{\partial f}{\partial y}(x,y_0)C_D(x,y_0)\Big)dx\right| \\
&= \left|\int\limits_{x_0}^{x_1}(x-x_0)(x-x_1)\left[\frac{\partial f}{\partial y}(x,y_1)C_U(x,y_1)\frac{\partial^3 f}{\partial x^2\partial y}(\xi_1,\eta_1)(y_1-y_0)\right.\right. \\
&\quad + \frac{\partial^2 f}{\partial y^2}(\xi_2,\eta_2)C_U(x,y_1)\frac{\partial^2 f}{\partial x^2}(\xi_3,y_0)(y_1-y_0) \\
&\quad \left.\left. + \frac{\partial^2 f}{\partial x^2}(\xi_3,y_0)f_y(x,y_0)\Big(C_U(x,y_1) - C_D(x,y_0)\Big)\right]dx\right| \leqslant \frac{1}{4}h_1h_2\Big(M_3 + 5M_2^2\Big)h_1^2.
\end{aligned} \tag{12}$$

In the same way,

$$\left|\int_{y_0}^{y_1}\Big(R_D(x_1,y)f_x(x_1,y)C_U(x_1,y)-R_U(x_0,y)f_x(x_0,y)C_U(x_0,y)\Big)dy\right| \leqslant \frac{1}{4}h_1h_2\Big(M_3+5M_2^2\Big)h_2^2. \tag{13}$$

Let us estimate the modules of interpolation error for each of the triangles:

$$|R_U(x,y)|\leqslant M_2(\frac{1}{4}h_2^2+h_1h_2+\frac{1}{4}h_1^2),\quad |R_D(x,y)|\leqslant M_2(\frac{1}{4}h_2^2+h_1h_2+\frac{1}{4}h_1^2). \tag{14}$$

These inequalities can be employed as the estimates for the errors $R_U(\Gamma)$ and $R_D(\Gamma)$ on diagonal Γ

$$y=y_0+\frac{y_1-y_0}{x_1-x_0}(x-x_0)$$

of the considered rectangle. Then the relations (11) yield

$$\left|\int_{x_0}^{x_1}R_U(\Gamma_1(x))f'_y(\Gamma_1(x))\Big(C_U(\Gamma_1(x))-C_D(\Gamma_1(x))\Big)dx\right|\leqslant 2M_2^2\left(\frac{1}{4}h_2^2+h_1h_2+\frac{1}{4}h_1^2\right)h_1h, \tag{15}$$

$$\left|\int_{y_0}^{y_1}R_U(\Gamma_2(y))f'_x(\Gamma_2(y))\Big(C_D(\Gamma_2(y))-C_U(\Gamma_2(y))\Big)dy\right|\leqslant 2M_2^2\left(\frac{1}{4}h_2^2+h_1h_2+\frac{1}{4}h_1^2\right)h_2h. \tag{16}$$

Let us estimates the squares of moduli of the errors gradient

$$\begin{aligned}
\frac{\partial R_U}{\partial x}=&(y-y_0)(y-y_1)f(x;x,y_0;y_1;y)+(y-y_1)f(x_0;x,y_0;y_1)\\
&+(x-x_0)(y-y_1)f(x_0;x;x,y_0;y_1)+(x-x_0)(x-x_1)f(x_0;x_1;x;x,y_0)\\
&+2\left(x-\frac{x_0+x_1}{2}\right)f(x_0;x_1;x,y_0),\\
\frac{\partial R_U}{\partial y}=&(y-y_0)(y-y_1)f(x,y_0;y_1;y;y)+2\left(y-\frac{y_0+y_1}{2}\right)f(x,y_0;y_1;y)\\
&+(x-x_0)f(x_0;x,y_0;y_1),\\
\frac{\partial R_D}{\partial x}=&(y-y_1)(y-y_0)f(x;x,y_1;y_0;y)+(y-y_1)f(x_0;x,y_1;y_0)\\
&+(x-x_0)(x-x_1)f(x_0;x_1;x;x,y_1)+2\left(x-\frac{x_0+x_1}{2}\right)f(x_0;x_1;x;x,y_1)\\
&+(x-x_1)(y-y_1)f(x_0;x;x,y_1;y_0),\\
\frac{\partial R_D}{\partial y}=&(y-y_1)(y-y_0)f(x,y;y;y_1;y_0)+2\left(y-\frac{y_0+y_1}{2}\right)f(x,y;y_1;y_0)\\
&+(x-x_1)f(x_0;x,y_1;y_0).
\end{aligned}$$

By Cauchy-Schwarz inequality it implies

$$\left|\frac{\partial R_U}{\partial x}\right|^2\leqslant\frac{1}{16}M_3^2h_2^4+M_2^2h_2^2+M_3^2h_1^2h_2^2+\frac{1}{16}M_3^2h_1^4+M_2^2h_1^2,$$

$$\left|\frac{\partial R_U}{\partial y}\right|^2\leqslant\frac{1}{16}M_3^2h_2^4+M_2^2h_2^2+M_2^2h_1^2,$$

$$\left|\nabla R_U\right|^2 \leqslant \frac{1}{8}M_3^2h_2^4 + 2M_2^2h_2^2 + M_3^2h_1^2h_2^2 + \frac{1}{16}M_3^2h_1^2 + 2M_2^2h_1^2. \tag{17}$$

In the same way,

$$\left|\nabla R_D\right|^2 \leqslant \frac{1}{8}M_3^2h_2^4 + 2M_2^2h_2^2 + M_3^2h_1^2h_2^2 + \frac{1}{16}M_3^2h_1^2 + 2M_2^2h_1^2. \tag{18}$$

Then we observe that U satisfies

$$\begin{aligned}
&\mathrm{div}(C_U\nabla f) = C_U\Delta f + \langle\nabla C_U, \nabla f\rangle,\\
&\frac{\partial C_U}{\partial x} = -(C_U)^2\frac{\partial}{\partial x}\sqrt{1+|\nabla f|^2} = -(C_U)^2\frac{f'_x f''_{xx} + f'_y f''_{xy}}{\sqrt{1+|\nabla f|^2}},\\
&\frac{\partial C_U}{\partial y} = -(C_U)^2\frac{\partial}{\partial y}\sqrt{1+|\nabla f|^2} = -(C_U)^2\frac{f'_x f''_{xy} + f'_y f''_{yy}}{\sqrt{1+|\nabla f|^2}}.
\end{aligned}$$

Hence,

$$\left|\langle\nabla f, \nabla C_U\rangle\right| = \left|\frac{C_U^2}{\sqrt{1+|\nabla f|^2}}\Big((f'_x)^2 f''_{xx} + 2f'_x f'_y f''_{xy} + (f'_y)^2 f''_{yy}\Big)\right| \leqslant 4M_2.$$

And since

$$|C_U\Delta f| \leqslant 2M_2,$$

then

$$|\mathrm{div}(C_U\nabla f)| \leqslant 6M_2. \tag{19}$$

In the same way for D we have

$$|\mathrm{div}(C_D\nabla f)| \leqslant 6M_2. \tag{20}$$

Let us estimate the modulus of the sum in the right hand side of identity (10). Employing inequalities (14), (17)–(20), we obtain

$$\begin{aligned}
&\left|\iint\limits_U\left(R_U\,\mathrm{div}(C_U(x,y)\nabla f(x,y)) + \frac{1}{2}|\nabla R_U(x,y)|^2\right)dxdy\right.\\
&\left.+\iint\limits_D\left(R_D\,\mathrm{div}(C_D(x,y)\nabla f(x,y)) + \frac{1}{2}|\nabla R_D(x,y)|^2\right)dxdy\right|\\
&\leqslant\left(3M_2^2(h_2^2+4h_1h_2+h_1^2) + \frac{1}{8}M_3^2h_2^4 + 2M_2^2h_2^2 + M_3^2h_1^2h_2^2 + \frac{1}{16}M_3^2h_1^2 + 2M_2^2h_1^2\right)|P|\\
&\leqslant\left(\frac{1}{8}M_3^2h_2^4 + 5M_2^2h_2^2 + 3M_2^2h_1h_2 + M_3^2h_1^2h_2^2 + 5M_2^2h_1^2 + \frac{1}{16}M_3^2h_1^4\right)|P|,
\end{aligned} \tag{21}$$

where $|P|$ is the area of the rectangle $P = [x_0, x_1]\times[y_0, y_1]$. Now by (10), (12), (13), (15), (16), (21), and the fact that the area of rectangle P does not exceeds h_1h_2, we get

$$\begin{aligned}
&\left|\iint\limits_P\sqrt{1+|\nabla f|^2}dxdy - \Big(S_U^r + S_D^r\Big)\right| \leqslant \frac{1}{2}h_1h_2\Big(M_3 + 5M_2^2\Big)(h_1^2+h_2^2)\\
&+4M_2^2\left(\frac{1}{4}h_2^2 + h_1h_2 + \frac{1}{4}h_1^2\right)h(h_1+h_2)\\
&+\left(\frac{1}{8}M_3^2h_2^4 + 5M_2^2h_2^2 + 3M_2^2h_1h_2 + M_3^2h_1^2h_2^2 + 5M_2^2h_1^2 + \frac{1}{16}M_3^2h_1^4\right)h_1h_2
\end{aligned}$$

$$\leqslant h_1h_2\left(2M_3h^2+\left(36+12\frac{h}{\min\{h_1,h_2\}}\right)M_2^2h^2+\frac{5}{2}M_3^2h^4\right).$$

Summing up the inequality over all the partition rectangles for set Ω, we finally obtain

$$\Big|\iint\limits_{\Omega}\sqrt{1+|\nabla f|^2}dxdy-S\Big|\leqslant|\Omega|\left(2M_3+12\left(3+\frac{h}{\min\{h_1,h_2\}}\right)M_2^2+\frac{5}{2}M_3^2h^2\right)h^2.$$

□

5. Uniform convergence of approximate solutions

Now we are going to obtain the uniform estimate for piecewise-linear solutions of minimal surface equation. Let f be a solution to equation (6) in domain Ω. We assume that

$$\sup_{\Omega}|\nabla f|=P_0<+\infty.$$

In what follows we shall argue as in works [15], [16]. We denote by f^L the piecewise-linear function such that $f^L(M_i)=f(M_i)$. We let $f^t(x)=u^*(x)+t(f^L(x)-u^*(x))$ and $P_1=\sup\limits_{\Omega}|\nabla u^*|$, $P=\max\{1,P_0,P_1\}$. It is clear that $u^*|_{\partial\Omega}=f^L|_{\partial\Omega}$. For each $t\in\mathbb{R}$ function $f^t(x)$ is piecewise-linear and one can calculate the area under its graph

$$\sigma(t)=\int\limits_{\Omega}\sqrt{1+|\nabla f^t|^2}dx.$$

Since as $t=0$ function $\sigma(t)$ achieves its minimum, then $\sigma'(0)=0$. Employing this identity, we obtain

$$\begin{aligned}S(f^L)-S(u^*)&=\int\limits_0^1 ds\int\limits_0^s\sigma''(t)dt\\&=\int\limits_0^1 ds\int\limits_0^s dt\int\limits_{\Omega}\frac{(1+|\nabla f^t|^2)|\nabla f^L-\nabla u^*|^2-\langle\nabla f^t,\nabla f^L-\nabla u^*\rangle^2}{(1+|\nabla f^t|^2)^{3/2}}dx\\&\geqslant\int\limits_0^1 ds\int\limits_0^s dt\int\limits_{\Omega}\frac{|\nabla f^L-\nabla u^*|^2}{(1+|\nabla f^t|^2)^{3/2}}dx\geqslant\frac{1}{\sqrt{(1+P^2)^3}}\int\limits_{\Omega}|\nabla f^L-\nabla u^*|^2dx.\end{aligned}\tag{22}$$

We employ Poincaré inequality (see, for instance, [17, Sect. 7.8]) for function $h(x)=f^L(x)-u^*(x)$, $h|_{\partial\Omega}=0$. By (22) we get

$$S(f^L)-S(u^*)\geqslant\frac{\lambda(\Omega)}{\sqrt{(1+P^2)^3}}\int\limits_{\Omega}|h(x)|^2dx,$$

where $\lambda(\Omega)=(\omega_n/|\Omega|)^{2/n}$ and $\omega_n,|\Omega|$ are n-dimensional volumes of the unit ball and domain Ω, respectively. Then we let $M=\sup\limits_{\Omega}|h|$ and without loss of generality we can assume that there exists a point $x_0\in\Omega$ satisfying $h(x_0)=M$. Let us show that $B_{M/4P}(x_0)\subset\Omega$. Indeed, let $x'\in\partial\Omega$ be such that $|x_0-x'|=\operatorname{dist}(x,\partial\Omega)$. Hence,

$$2P|x_0-x'|\geqslant h(x_0)-h(x')=M-h(x')\geqslant M-M'\geqslant M/2.$$

Thus, the distance from point x_0 to boundary $\partial\Omega$ is greater than $M/4P$. Therefore, $B_{M/4P}(x_0) \subset \Omega$. Suppose that $x \in B_{M/4P}(x_0)$. Then

$$h(x) \geqslant h(x_0) - 2P|x - x_0| > M - 2P\frac{M}{4P} = M/2.$$

Hence, $B_{M/4P}(x_0) \subset D_M$, where

$$D_M = \{x \in \Omega : |h| > M/2\} \subset\subset \Omega.$$

Therefore,

$$\int\limits_{\Omega} |h(x)|^2 dx \geqslant \int\limits_{D_M} |h|^2 dx \geqslant \int\limits_{B_{M/4P}(x_0)} \left(\frac{M}{2}\right)^2 dx = \frac{M^2}{4}\left(\frac{M}{4P}\right)^n \omega_n = \frac{M^{n+2}}{4^{n+1}P^n}\omega_n.$$

Hence,

$$\max_{\Omega} |f^L - u^*| \leqslant 4P^{4/3}\left(\frac{S(f^L) - S(u^*)}{\lambda(\Omega)\omega_n}\right)^{\frac{1}{n+2}}.$$

Theorem 4. *Let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to minimal surface equation (6) and u^* be a piecewise-linear function solving problem (1) with $\varphi_i = f(M_i)$ for each $M_i \in \partial\Omega$. Suppose that $P_0 = \sup\limits_{\Omega} |\nabla f| < +\infty$ and $P_1 = \sup\limits_{\Omega} |\nabla u^*|$. Then*

$$\max_{\Omega} |f^L - u^*| \leqslant 4P^{4/3}\left(\frac{S(f^L) - S(u^*)}{\lambda(\Omega)\omega_n}\right)^{\frac{1}{n+2}},$$

where $P = \max\{1, P_0, P_1\}$.

Now let Ω be a rectangle $[a, b] \times [c, d]$. We fix a natural number m and consider the partition of the rectangle defined by the points $x_i = a + \frac{i}{m}(b - a)$, $y_j = c + \frac{j}{m}(d - c)$, $i, j = 0, 1, \ldots, m$. We partition each of the rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i, j = 0, 1, \ldots, m - 1$ by the diagonal connecting vertices (x_i, y_j) and (x_{i+1}, y_{j+1}) into two triangles. Suppose in rectangle Ω we are given a solution f to the minimal surface equation, $f \in C^3(\overline{\Omega})$. Let u^*_m be the solution to problem (1) associated with this partition and satisfying the boundary conditions

$$u^*_m(x_i, c) = f(x_i, c),\ u^*_m(x_i, d) = f(x_i, d), \quad i = 0, \ldots, m,$$
$$u^*_m(a, y_j) = f(a, y_j),\ u^*_m(b, y_j) = f(b, y_j), \quad j = 0, \ldots, m.$$

Corollary. *Sequence u^*_m converges uniformly in Ω to solution f, at that*

$$\sup_{\Omega} |f(x, y) - u^*_m(x, y)| = O\left(\frac{1}{\sqrt{m}}\right)$$

as $m \to \infty$.

Proof. We denote by f^L the piecewise-linear function such that $f^L(x_i, y_j) = f(x_i, y_j)$, $i, j = 0, \ldots, m$. First let us show that the gradients of functions u^*_m are bounded by a constant independent of m. In order to do it, we employ inequality (9). It follows from Theorem 3 that for some constant C_1 independent of m we have

$$|S(f) - S(f^L)| \leqslant \frac{C_1}{m^2},$$

and applying the trapezium formula of numerical integration (see [14, Ch. 3]), we obtain

$$\left|\int\limits_{\partial\Omega} \frac{\langle \nabla f, \nu\rangle (u^*_m - f)}{\sqrt{1 + |\nabla f|^2}} dS\right| \leqslant \frac{C_2}{m^2}.$$

Thus,

$$|A(f)| \leqslant \frac{2}{(b-a)(d-c)}(C_1 + C_2) \equiv C_3.$$

If $P_0 = \sup\limits_{\Omega} |\nabla f|$, it follows from inequality (9) that

$$|\nabla u_m^*| \leqslant P_0 + 3(1 + P_0^2)\sqrt{C_3(1 + C_3)} \equiv P_1.$$

Let us estimate the quantity $S(f^L) - S(u_m^*)$. We construct arbitrary function $\tilde{u}_m$ such that $\tilde{u}_m = u_m^*$ in rectangle $\Omega_m = [x_1, x_{m-1}] \times [y_1, y_{m-1}]$, $\tilde{u}_m = f$ on $\partial\Omega$ and $|\nabla \tilde{u}_m - \nabla u_m^*| \leqslant C_4/m$, where constant C_4 is independent of m. Then

$$\begin{aligned}
0 \leqslant & S(f^L) - S(u_m^*) = S(f^L) - S(f) + S(f) - S(\tilde{u}_m) + S(\tilde{u}_m) - S(u_m^*) \\
\leqslant & S(f^L) - S(f) + S(\tilde{u}_m) - S(u_m^*) \leqslant \frac{C_1}{m^2} + \iint\limits_{\Omega\setminus\Omega_m} (\sqrt{1 + |\nabla \tilde{u}_m|^2} - \sqrt{1 + |\nabla u_m^*|^2}) dxdy \\
\leqslant & \frac{C_1}{m^2} + \frac{C_4}{m}(b-a)(d-c)(1 - (1 - 2/m)^2) = \frac{C_1}{m^2} + \frac{C_5}{m}(1 - (1 - 2/m)^2),
\end{aligned}$$

where $C_5 = C_4(b-a)(d-c)$. Therefore, it follows from Theorem 4 that

$$|f - u_m^*| \leqslant |f - f^L| + |f^L - u_m^*| \leqslant |f - f^L| + 4P^{4/3}\left(\frac{S(f^L) - S(u_m^*)}{\lambda(\Omega)\pi}\right)^{\frac{1}{4}},$$

where $P = \max\{1, P_0, P_1\}$. Applying the previous inequality, we obtain

$$\begin{aligned}
\sup_{\Omega} |f(x,y) - u_m^*(x,y)| & \leqslant P_0\sqrt{(b-a)^2 + (d-c)^2}\frac{1}{m} + 4P^{4/3}\left(\frac{\frac{C_1}{m^2} + \frac{C_5}{m}(1 - (1 - 2/m)^2)}{\lambda(\Omega)\pi}\right)^{\frac{1}{4}} \\
& \leqslant \frac{1}{m}P_0\sqrt{(b-a)^2 + (d-c)^2} + 4P^{4/3}\left(\frac{C_1 + 4C_5}{m^2\lambda(\Omega)\pi}\right)^{\frac{1}{4}} \to 0
\end{aligned}$$

as $m \to \infty$. □

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