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DESCRIPTION OF ZERO SEQUENCES FOR HOLOMORPHIC AND MEROMORPHIC FUNCTIONS OF FINITE λ -TYPE IN A CLOSED HALF-STRIP

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Abstract. We describe the zero sets of holomorphic and meromorphic functions f of finite λ -type in a closed half-strip satisfying $f(\sigma) = f(\sigma + 2\pi i)$ on the boundary.

Keywords: holomorphic function, meromorphic function, function of finite λ - type, sequence of finite λ -density, λ -admissible sequence

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1. INTRODUCTION

Let f be a meromorphic function in the closure of the half-strip

$$S = \{ s = \sigma + it : \sigma > 0, \quad 0 < t < 2\pi \}.$$

Suppose f has neither zeros nor poles on ∂S , and $f(\sigma) = f(\sigma + 2\pi i), \sigma \geq 0$. Denote by $\{s_j\}$ the zero sequence of function f in S, $s_j = \sigma_j + it_j$, by $\{p_j\}$ the sequence of its poles in S.

Let S^* be the strip S with the straight slits $\{\tau \sigma_j + it_j\}$, $\{\tau \operatorname{Re} p_j + i \operatorname{Im} p_j\}$, $1 \leq \tau < \infty$. Given $s_0 \in S^*$, suppose log $f(s_0)$ is well-defined and let

$$\log f(s) = \log f(s_0) + \int_{s_0}^s \frac{f'(\zeta)}{f(\zeta)} d\zeta, \tag{1}$$

where integral is taken along a piecewise-smooth path in $S^* \cup \partial S$, which connects s_0 and s.

By $n(\eta, f)$ we denote the counting function of poles of f in the rectangle $R_{\eta} = \{\sigma + it : 0 < \sigma \leq \eta, 0 \leq t < 2\pi\}$. We let

$$N(\sigma, f) = \int_{0}^{\sigma} n(\eta, f) d\eta.$$
⁽²⁾

and

$$c_0(\sigma, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(\sigma + it)| dt.$$
 (3)

The following Lemma is a counterpart of Jensen-Littlewood Theorem ([1]).

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Lemma 1. [2] Let f be a meromorphic function in the closure of half-strip S, $f(\sigma) = f(\sigma + 2\pi i), \sigma \geq 0$. Then

$$N(\sigma, \frac{1}{f}) - N(\sigma, f) = c_0(\sigma, f) - \frac{\sigma}{\sigma_0} c_0(\sigma_0, f) + (\frac{\sigma}{\sigma_0} - 1) c_0(0, f),$$

$$\sigma \ge \sigma_0 > 0.$$
(4)

The Nevanlinna characteristic of such functions was defined in [2] as

$$T(\sigma, f) = m_0(\sigma, f) - \frac{\sigma}{\sigma_0} m_0(\sigma_0, f) + \left(\frac{\sigma}{\sigma_0} - 1\right) m_0(0, f) + N(\sigma, f), \quad \sigma \ge \sigma_0 > 0,$$

where

$$m_0(\sigma, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(\sigma + it)| dt.$$

Definition 1. A positive non-decreasing continuous unbounded function $\lambda(\sigma)$ defined for all $\sigma \geq \sigma_0 > 0$ is said to be a growth function.

Definition 2. Let $\lambda(\sigma)$ be a growth function and f be a meromorphic function in \overline{S} , such that $f(\sigma + 2\pi i) = f(\sigma), \sigma \ge \sigma_0 > 0$. We say that f is of finite λ -type if $T(\sigma, f) \le A\lambda(\sigma + B)$, $\sigma \ge \sigma_0$ for some constants A > 0, B > 0 and all $\sigma, \sigma \ge \sigma_0 > 0$.

We denote by Λ the class of meromorphic functions of finite λ -type in \overline{S} and Λ_H the class of holomorphic functions of finite λ -type in \overline{S} .

In this paper we describe the zero sequences of holomorphic functions in Λ_H , as well as zero and pole sequences of meromorphic functions in Λ .

For entire and meromorphic in \mathbb{C} functions similar problems were solved by L. Rubel and B. Taylor ([3]), for holomorphic and meromorphic functions in a punctured plane the same was done by A. Kondratyuk and I. Laine ([4]).

2. Description of zero sequences of holomorphic and meromorphic functions of finite λ -type in a half-strip

Let $Q = \{s_j\}$ be a sequence of complex numbers in \overline{S} . By $n(\eta, Q)$ we indicate the counting function of Q in the rectangle R_{η} and we let

$$N(\sigma, Q) = \int_{0}^{\sigma} n(\eta, Q) d\eta$$

Definition 3. A sequence $Q = \{s_j\}$ from \overline{S} has a finite λ - density if

 $N(\sigma, Q) \leqslant A\lambda(\sigma + B)$

for some positive constants A, B and all σ , $\sigma \geq \sigma_0 > 0$.

Definition 4. A sequence $Q = \{s_j\}$ from \overline{S} is said to be λ -admissible if it has finite λ -density and there are positive constants A, B such that

$$\frac{1}{k} \left| \sum_{\sigma_1 < \operatorname{Re}_{s_j \leqslant \sigma_2}} \left(\frac{1}{e^{s_j}} \right)^k \right| \leqslant \frac{A\lambda(\sigma_1 + B)}{e^{k\sigma_1}} + \frac{A\lambda(\sigma_2 + B)}{e^{k\sigma_2}},$$

for all $\sigma_1, \sigma_2, \sigma_0 \leq \sigma_1 < \sigma_2$ and each $k \in \mathbb{N}$.

Denote

$$c_k(\sigma, f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \log |f(\sigma + it)| dt, \quad k \in \mathbb{Z}.$$
(5)

For a meromorphic in \overline{R}_{σ} function f such that $f(\sigma) = f(\sigma + 2\pi i)$ the following relations hold true (see [2]):

$$c_{k}(\sigma, f) = \frac{e^{k\sigma}}{2k} \alpha_{k}(f) - \frac{e^{-k\sigma}}{2k} \overline{\alpha_{-k}}(f) + \frac{1}{2k} \sum_{s_{j} \in R_{\sigma}} \left[\left(\frac{e^{\sigma}}{e^{s_{j}}} \right)^{k} - \left(\frac{e^{\overline{s_{j}}}}{e^{\sigma}} \right)^{k} \right] - \frac{1}{2k} \sum_{p_{j} \in R_{\sigma}} \left[\left(\frac{e^{\sigma}}{e^{p_{j}}} \right)^{k} - \left(\frac{e^{\overline{p_{j}}}}{e^{\sigma}} \right)^{k} \right], \quad (6)$$

$$c_{-k}(\sigma, f) = \overline{c}_{k}(\sigma, f) \quad k \in \mathbb{N},$$

where s_j , p_j are its zeroes and poles in \overline{R}_{σ} respectively, and

$$\alpha(f) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ikt} \frac{f'(it)}{f(it)} dt, \quad k \in \mathbb{N}.$$

Theorem 1. A sequence Q in \overline{S} is the zero sequence of the function in Λ_H if and only if it is λ -admissible.

Proof. Let $Q = \{s_j\}$ be the zero sequence of a function f from Λ_H . Then by (6)

$$\begin{aligned} \frac{c_k(\sigma_2, f)}{e^{k\sigma_2}} &- \frac{c_k(\sigma_1, f)}{e^{k\sigma_1}} = \frac{\alpha_k e^{k\sigma_2} - \overline{\alpha_{-k}} e^{-k\sigma_2}}{2ke^{k\sigma_2}} + \frac{1}{2ke^{k\sigma_2}} \left[\sum_{s_j \in R_{\sigma_2}} \left(\frac{e^{\sigma_2}}{e^{s_j}} \right)^k - \sum_{s_j \in R_{\sigma_2}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}} \right)^k \right] \\ &- \frac{\alpha_k e^{k\sigma_1} - \overline{\alpha_{-k}} e^{-k\sigma_1}}{2ke^{k\sigma_1}} - \frac{1}{2ke^{k\sigma_1}} \left[\sum_{s_j \in R_{\sigma_1}} \left(\frac{e^{\sigma_1}}{e^{s_j}} \right)^k - \sum_{s_j \in R_{\sigma_1}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}} \right)^k \right] \\ &= \frac{\overline{\alpha_{-k}}}{2k} \left[\frac{1}{e^{2k\sigma_1}} - \frac{1}{e^{2k\sigma_2}} \right] + \frac{1}{2k} \left[\sum_{s_j \in R_{\sigma_2}} \frac{1}{(e^{s_j})^k} - \sum_{s_j \in R_{\sigma_1}} \frac{1}{(e^{s_j})^k} \right] \\ &+ \frac{1}{2ke^{k\sigma_1}} \sum_{s_j \in R_{\sigma_1}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}} \right)^k - \frac{1}{2ke^{k\sigma_2}} \sum_{s_j \in R_{\sigma_2}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}} \right)^k, \end{aligned}$$

where $0 \leq \sigma_1 < \sigma_2$.

Then we obtain

$$\frac{1}{k} \sum_{\sigma_1 < \operatorname{Re} s_j \leqslant \sigma_2} \frac{1}{(e^{s_j})^k} = \frac{2c_k(\sigma_2, f)}{e^{k\sigma_2}} - \frac{2c_k(\sigma_1, f)}{e^{k\sigma_1}} + \frac{\overline{\alpha_{-k}}}{k} \left[\frac{1}{e^{2k\sigma_2}} - \frac{1}{e^{2k\sigma_1}} \right] + \frac{1}{ke^{k\sigma_2}} \sum_{s_j \in R_{\sigma_2}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}} \right)^k - \frac{1}{ke^{k\sigma_1}} \sum_{s_j \in R_{\sigma_1}} \left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}} \right)^k.$$
(7)

We have

$$\sum_{s_j \in R_{\sigma_i}} \left| \frac{e^{\overline{s_j}}}{e^{\sigma_i}} \right|^k \leq \sum_{s_j \in R_{\sigma_i}} 1 \leq n(\sigma_i + 1, \frac{1}{f}) \leq N(\sigma_i + 1, \frac{1}{f}) \leq A_1 \lambda(\sigma_i + 1 + B_1), \quad \sigma_i \in R_{\sigma_i}, i = 1, 2,$$

for some constants $A_1, B_1 > 0$.

We also get the estimate for the left-hand side of identity (7):

$$\frac{1}{k} \left| \sum_{\sigma_1 < \operatorname{Re} s_j \leqslant \sigma_2} \frac{1}{e^{ks_j}} \right| \leqslant \frac{A_2 \lambda(\sigma_2 + B_2)}{e^{k\sigma_2}} + \frac{A_2 \lambda(\sigma_1 + B_2)}{e^{k\sigma_1}} + \frac{|\overline{\alpha_{-k}}|}{k} \left[\frac{1}{e^{2k\sigma_2}} + \frac{1}{e^{2k\sigma_1}} \right] \\
+ \frac{1}{ke^{k\sigma_2}} \sum_{s_j \in R_{\sigma_2}} \left| \frac{e^{\overline{s_j}}}{e^{\sigma_2}} \right|^k + \frac{1}{ke^{k\sigma_1}} \sum_{s_j \in R_{\sigma_1}} \left| \frac{e^{\overline{s_j}}}{e^{\sigma_1}} \right|^k \\
\leqslant \frac{A_2 \lambda(\sigma_2 + B_2)}{e^{k\sigma_2}} + \frac{A_2 \lambda(\sigma_1 + B_2)}{e^{k\sigma_1}} + C \left[\frac{1}{e^{2k\sigma_2}} + \frac{1}{e^{2k\sigma_1}} \right] \\
+ \frac{1}{ke^{k\sigma_2}} N \left(\sigma_2 + 1, \frac{1}{f} \right) + \frac{1}{ke^{k\sigma_1}} N \left(\sigma_1 + 1, \frac{1}{f} \right) \\
\leqslant \frac{A \lambda(\sigma_2 + B)}{e^{k\sigma_2}} + \frac{A \lambda(\sigma_1 + B)}{e^{k\sigma_1}}, \qquad k \in \mathbb{N}, \quad \sigma_2 > \sigma_1 \geqslant \sigma_0,$$
(8)

where $A = \max\{A_1, A_2, C\}, B = \max\{B_1 + 1, B_2\}.$

Theorem 2 in [2] implies that the sequence Q has a finite λ -density. Hence, it is λ -admissible. Let now $Q = \{s_j\}$ be λ -admissible. Then the sequence $Z = \{z_j\}, z_j = e^{s_j} \in \mathbb{C}$, is λ_1 admissible in \mathbb{C} , where $\lambda_1(r) = \lambda(\log r)$. By the Rubel-Taylor Theorem [3, p. 84], (see also [5, p. 29]), there exists an entire function F(z) of finite λ_1 -type with zero sequence $Z = \{z_j\}$. Therefore, the function $f(s) = F(e^s)$ is holomorphic of finite λ -type in \overline{S} with the zero sequence $\{s_j\}$.

Theorem 2. A sequence Q in \overline{S} is the zero sequence of a function in Λ if and only if it has finite λ -density.

Proof. If $Q = \{s_j\}$ is the zero sequence of a function $f, f \in \Lambda$, then from [2], we have

$$N(\sigma, Q) = N(\sigma, \frac{1}{f}) \leqslant T(\sigma, f) \leqslant B\lambda(\sigma + C),$$

for all $\sigma \ge \sigma_0 > 0$ and some B, C > 0.

Let now $Q = \{s_j\}$ be a sequence of finite λ -density. Then the sequence $Z = \{z_j\}, \quad z_j = e^{s_j},$ has the finite λ_1 -density if $\lambda_1(r) = \lambda(\log r)$. By the Rubel-Taylor Theorem [3, p. 88] (see also [5, p. 35]) there exist a meromorphic function F of finite λ_1 -type with zero sequence Z. The function $f(s) = F(e^s)$ is the meromorphic of finite λ -type in \overline{S} with zero sequence $\{s_j\}$. \Box

Corollary 1. A sequence $P = \{p_j\}$ is the pole sequence of a function f from Λ if and only if it has finite λ -density.

Proof. Apply Theorem 2 to the function $\frac{1}{f}$.

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BIBLIOGRAPHY

- J.E. Littlewood On the zeros of the Riemann zeta-function, Proc. Camb. Philos. Soc. 22 (1924), 295-318.
- N.B. Sokul's'ka Meromorphic functions of finite λ-type in half-strip // Carpathian Mathematical Publications 2012. V.4, 2. P. 328–339 (in Ukrainian).
- 3. L.A. Rubel, B.A. Taylor Fourier series method for meromorphic functions // Bull. Soc. Math. France 1968. 96. P. 53–96.
- A. Kondratyuk, I. Laine Meromorphic functions in multiply connected domains // Fourier series method in complex analysis (Merkrijärvi, 2005), Univ. Joensuu Dept. Math. Rep. Ser., 10 (2006), P. 9–111.
- A.A. Kondratyuk Fourier Series and Meromorphic Functions. L'viv.: Izdat. L'viv Univ., 1988. 196 p (in Russian).

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