ON CONTROLLABILITY OF DEGENERATE DISTRIBUTED SYSTEMS

M.V. PLEKHANOVA, V.E. FEDOROV

Abstract. We study the controllability of linear distributed control systems described by differential equations in Banach spaces with a degenerate operator at the derivative. A homogeneous part of equations has a degenerate strongly continuous resolving semigroup. For such system with, generally speaking, a time-dependent bounded operator at the control function we find the criteria of ε -controllability in time T and of ε -controllability in a free time in terms of the operators involved in the equation. General results are used for studying of ε -controllability of the considered systems with a finite-dimensional input. The obtained conditions are demonstrated by examples of control systems described by partial differential equations and systems of equations unsolved with respect to the time derivative.

Keywords: control system, degenerate evolution equation, Sobolev type equation, controllability

Mathematics Subject Classification: 93B05, 34G10, 47N70, 47F05

1. INTRODUCTION

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$ be Banach spaces, $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ be linear continuous operators from $\mathcal{X}$ into $\mathcal{Y}$, $\ker L \neq \{0\}$, $M \in \mathcal{Cl}(\mathcal{X}; \mathcal{Y})$ be a linear closed densely defined operator in $\mathcal{X}$ acting in $\mathcal{Y}$, $B : [0, T] \rightarrow \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $y : [0, T] \rightarrow \mathcal{Y}$. We consider the problem on ε -controllability of distributed control systems whose dynamics is described by the equation

$$L\dot{x}(t) = Mx(t) + B(t)u(t) + y(t), \quad (1)$$

i.e., we study the possibility of moving the trajectory of a solution to equation (1) from an arbitrary initial state x_0 into the ε -neighborhood of a prescribed point for each $\varepsilon > 0$ by a choice of control function $u(\cdot)$.

In what follows, meaning the control system described, say, by equation, we shall call it shortly either control system (1) or system (1).

If operator L has a continuous inverse, equation (1) can be solved explicitly w.r.t. the derivative as $\dot{x}(t) = Sx(t) + B_1(t)u(t) + y_1(t)$. The controllability (ε -controllability) of control system described by an explicitly solved equation, generally speaking, in Banach spaces, was studied by N.N. Krasovski [1], R.E. Kalman, Y.C. Ho, K.S. Narendra [2], H.O. Fattorini [3], F.A. Sholokhov [4], A.B. Kurzhanski [5], R. Triggiani [6], B. Shklyar [7, 8] and many other (see also surveys [9, 10, 11] and works [12, 13]).

As a solution to equation (1), we regard a function $x \in W_q^1(0, T; \mathcal{X})$, $q \geq 1$, satisfying the equation a.e. in $(0, T)$; it is a so-called strong solution [14]. We shall be interested in the case $\ker L \neq \{0\}$ under the assumption of strong (L, p) -radiality for operator M [15, 16]. In this case

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equation (1) is reduced to a system of two equations on mutually complementing subspaces. One of these equations is

$$\dot{x}^1(t) = S_1 x^1(t) + B_1(t)u(t) + y^1(t);$$

it is solved explicitly w.r.t. the derivative, while the other, on the kernel of the semi-group of the homogeneous part of original equation (1), has a nilpotent operator G at the derivative:

$$G\dot{x}^0(t) = x^0(t) + B_0(t)u(t) + y^0(t). \quad (2)$$

A specific character of equation (1) with degenerate operator L is due to features of equation (2) on the semi-group kernel and the matter is that we have to deal with control functions smoother than that in Lebesgue spaces. Moreover, this restriction is essential; it was shown in [14, Thm. 3] that necessary and sufficient conditions for the solvability of equation (2) are very close to the belonging to the Sobolev space $W_q^{p+1}(0, T; \mathcal{U})$, where $p \in \{0\} \cup \mathbb{N}$ is a parameter characterizing the degeneracy of the system. At that, in accordance with [14, Cor. 3], if the control is in the space $L_2(0, T; \mathcal{U})$, equation (2) is, generally speaking, unsolvable. Other specific features of degenerate equation (2) are as follows:

- its unique solvability under the absence of the initial condition, and thus, the need of matching of control functions and its derivatives at the initial moment with initial condition x_0 once the latter is prescribed;
- the independence of solution of the initial condition x_0 for the time $t > 0$;
- not integral but a differential form of equation involving the derivatives up to the order p for the control function at a current time.

In the present work we study the properties of ε -controllability in time T and ε -controllability in a free time for the system described by equation (1). The main result is the necessary and sufficient conditions for ε -controllability of equation (1) in the sense of strong solutions; these conditions are written in terms of the operators involved in the equation. These conditions are simple enough for the possibility to check them for particular distributed systems described by partial differential equations and systems of such equations. This is demonstrated by examples.

As a particular case of the general situation we obtain the criteria of ε -controllability of system (1) in the case $B(t) \equiv B_1$ for each $t \geq 0$ as well as in the case $\mathcal{U} = \mathbb{R}^m$, $m \in \mathbb{N}$, $u = (u_1, u_2, \dots, u_m)$, $b_i : [0, T] \rightarrow \mathcal{Y}$, $i = 1, 2, \dots, m$, $B(t)u(t) = \sum_{i=1}^m b_i(t)u_i(t)$, i.e., for the systems

$$L\dot{x}(t) = Mx(t) + \sum_{i=1}^m b_i(t)u_i(t) + y(t). \quad (3)$$

It is called a control system with a finite-dimensional input. Here an essential fact is that in the case of control systems with a finite-dimensional input a necessary condition for ε -controllability is a finite dimension of the kernel of the semi-group on which equation (2) is defined. We provide examples of ε -controllable distributed systems with a finite-dimensional input. These systems are described by partial differential equations or systems of such equations not solved explicitly w.r.t. the time derivative. We also provide examples of distributed systems with a finite-dimensional input not being ε -controllable (in time T or in a free time).

The issues of ε -controllability in time T for a degenerate equation (1) with a strongly (L, p) -radial operator M were studied earlier in works [17, 18]. At that, there was considered only the case $B(t) \equiv B_1$ for each $t \geq 0$ and $y \equiv 0$ and only the notion of the classical solution (in the space $C^1([0, T]; \mathcal{U})$) to the equation was employed. However, while constructing a general theory of optimal control for the systems described by operator-differential equations (1) in Banach spaces, it is more convenient to employ strong solutions in $W_q^1(0, T; \mathcal{U})$ (see [14]). For a constant operator B and $y \equiv 0$, some particular examples of ε -controllability of equation (1) in the sense of strong solutions were obtained in [19] and are generalized in the present work.

The controllability and ε -controllability in the sense of classical solutions to degenerate ($\ker L \neq \{0\}$) systems (3) with the one-dimensional or two-dimensional input were considered in works [20, 21] in the case of much more restrictive than in the present works conditions for the parameters in the equation; it was assumed that operator M is (L, p) -bounded, $b_i(t) \equiv b_{i1}$ as $t \geq 0$, $y \equiv 0$.

We also mention work [22] dealing with distributed control system (1) involving a strongly (L, p) -radial operator M . Here the criterion of complete (or exact) controllability system in a Hilbert space was formulated by means of notion of strongly minimal sequence of generalized exponentials.

2. CAUCHY PROBLEM FOR DEGENERATE EVOLUTION EQUATION

In the present section we provide some preliminary results on existence and properties of strongly continuous resolving semi-group for a linear first order differential equation with a degenerate operator at the derivative as well as on solvability of Cauchy problem for an appropriate inhomogeneous equation in the sense of strong solutions. These results will be used in what follows; their proofs can be found in [14, 16].

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, and $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $\ker L \neq \{0\}$, $M \in \mathcal{Cl}(\mathcal{X}; \mathcal{Y})$. We introduce the notations $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\bar{\mathbb{R}}_+ = \{0\} \cup \mathbb{R}_+$, $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})\}$, $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L(M) = L(\mu L - M)^{-1}$.

Let $p \in \mathbb{N}_0$. An operator M is called *strongly (L, p) -radial*, if

- (i) $\exists a \in \mathbb{R} \ (a, +\infty) \subset \rho^L(M)$;
- (ii) $\exists K > 0 \ \forall \mu \in (a, +\infty) \ \forall n \in \mathbb{N}$

$$\max\{\|(R_\mu^L(M))^{n(p+1)}\|_{\mathcal{L}(\mathcal{X})}, \|(L_\mu^L(M))^{n(p+1)}\|_{\mathcal{L}(\mathcal{Y})}\} \leq \frac{K}{(\mu - a)^{n(p+1)}};$$

- (iii) there exists a lineal $\overset{\circ}{\mathcal{Y}}$ dense in $\mathcal{Y}$ such that for each $y \in \overset{\circ}{\mathcal{Y}}$, $\mu \in (a, +\infty)$

$$\|M(\mu L - M)^{-1}(L_\mu^L(M))^{p+1}y\|_{\mathcal{Y}} \leq \frac{\text{const}(y)}{(\mu - a)^{p+2}};$$

- (iv) for each $\mu \in (a, +\infty)$

$$\|(R_\mu^L(M))^{p+1}(\mu L - M)^{-1}\|_{\mathcal{L}(\mathcal{Y}; \mathcal{X})} \leq \frac{K}{(\mu - a)^{p+2}}.$$

By $\mathcal{X}^0$ ($\mathcal{Y}^0$) we indicate the kernel $\ker(R_\mu^L(M))^{p+1}$ ($\ker(L_\mu^L(M))^{p+1}$), and $\mathcal{X}^1$ ($\mathcal{Y}^1$) stands for the closure of the image $\text{im}(R_\mu^L(M))^{p+1}$ ($\text{im}(L_\mu^L(M))^{p+1}$) in the norm of space $\mathcal{X}$ ($\mathcal{Y}$). By M_k (L_k) we denote the restriction of operator M (L) on $\text{dom } M_k = \mathcal{X}^k \cap \text{dom } M$ ($\mathcal{X}^k$), $k = 0, 1$.

Theorem 1. [16]. *Suppose that operator M is strongly (L, p) -radial. Then*

- (i) $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$;
- (ii) $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $M_k \in \mathcal{Cl}(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$;
- (iii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$;
- (iv) operator $G = M_0^{-1}L_0$ is nilpotent of degree at most p , i.e., $G^{p+1} = \mathbb{O}$;
- (v) there exists a degenerate strongly continuous resolving semi-group of operators $\{X^t \in \mathcal{L}(\mathcal{X}) : t \in \bar{\mathbb{R}}_+\}$ resolving the equation $L\dot{x}(t) = Mx(t)$. At that, for each $t \in \bar{\mathbb{R}}_+$ the inequality $\|X^t\|_{\mathcal{L}(\mathcal{X})} \leq Ke^{at}$ holds with constants K, a in the definition of strong (L, p) -radiality;
- (vi) operator $S_1 = L_1^{-1}M_1 \in \mathcal{Cl}(\mathcal{X}^1)$ is an infinitesimal generator of C_0 -continuous semi-group $\{X_1^t = X^t|_{\mathcal{X}^1} \in \mathcal{L}(\mathcal{X}^1) : t \in \bar{\mathbb{R}}_+\}$.

The unity of the semi-group $X^0 \equiv P = s\text{-}\lim_{\mu \rightarrow +\infty} (\mu R_\mu^L(M))^{p+1}$ is the projector along the subspace $\mathcal{X}^0$ onto $\mathcal{X}^1$, and $Q = s\text{-}\lim_{\mu \rightarrow +\infty} (\mu L_\mu^L(M))^{p+1}$ is the projector along $\mathcal{Y}^0$ onto $\mathcal{Y}^1$.

Consider the Cauchy problem

$$x(0) = x_0 \in \text{dom } M \quad (4)$$

for the equation

$$L\dot{x}(t) = Mx(t) + y(t). \quad (5)$$

A *strong solution* to problem (4), (5) is called a vector-function $x \in W_q^1(0, T; \mathcal{X})$, $q \geq 1$, satisfying condition (4), $x(t) \in \text{dom } M$ a.e. on $(0, T)$, and identity (5) is obeyed. Due to the embedding $W_q^1(0, T; \mathcal{X}) \hookrightarrow C([0, T]; \mathcal{X})$ this definition is well-defined.

Theorem 2. [14]. *Assume that operator M is strongly (L, p) -radial. Then for each $y \in W_q^{p+1}(0, T; \mathcal{Y})$ and*

$$x_0 \in \mathcal{M}_y = \left\{ x \in \text{dom } M : (I - P)x = - \sum_{k=0}^p G^k M_0^{-1} (I - Q)y^{(k)}(0) \right\}$$

there exists the unique strong solution $x \in W_q^1(0, T; \mathcal{X})$ to problem (4), (5). At that,

$$x(t) = X^t x_0 + \int_0^t X^{t-s} L_1^{-1} Q y(s) ds - \sum_{k=0}^p G^k M_0^{-1} (I - Q) y^{(k)}(t).$$

3. ON RELATION BETWEEN ε -CONTROLLABILITY FOR DEGENERATE EVOLUTION SYSTEMS AND ITS SUBSYSTEMS

Under the assumptions of the previous section, in what follows we suppose that operator M is strongly (L, p) -radial. Moreover, we shall often make use of the conditions

$$B \in C^1([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y})), \quad (I - Q)B \in C^{p+1}([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y})), \quad T > 0, \quad (6)$$

$$y \in W_q^1(0, T; \mathcal{Y}), \quad (I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y}), \quad T > 0, \quad q \geq 1. \quad (7)$$

Control functions $u(\cdot)$ for the system described by the Cauchy problem

$$x(0) = x_0 \in \text{dom } M, \quad (8)$$

$$L\dot{x}(t) = Mx(t) + B(t)u(t) + y(t), \quad (9)$$

are chosen in the space $W_q^{p+1}(0, T; \mathcal{U})$. We also suppose that the condition $x_0 \in \mathcal{M}_{Bu+y}$ of Theorem 2 is obeyed. The set of control functions obeying this condition is indicated as

$$\begin{aligned} H_\partial(x_0, y) &\equiv \left\{ u \in W_q^{p+1}(0, T; \mathcal{Y}) : (I - P)x_0 \right. \\ &= \left. - \sum_{k=0}^p G^k M_0^{-1} \left(\sum_{l=0}^k \frac{k!}{l!(k-l)!} B_0^{(k-l)}(0) u^{(l)}(0) + y^{(k)}(0) \right) \right\}, \end{aligned}$$

where $B_0 \equiv (I - Q)B$, $y^0 \equiv (I - Q)y$. By Theorem 1, Cauchy problem (8), (9) can be reduced to two problems

$$x^1(0) = Px_0,$$

$$\dot{x}^1(t) = S_1 x^1(t) + L_1^{-1} Q B(t) u(t) + L_1^{-1} Q y(t) \quad (10)$$

and

$$x^0(0) = (I - P)x_0,$$

$$G\dot{x}^0(t) = x^0(t) + M_0^{-1} (I - Q) B(t) u(t) + M_0^{-1} (I - Q) y(t), \quad (11)$$

considered on subspaces $\mathcal{X}^1$ and $\mathcal{X}^0$, respectively. Here $S_1 = L_1^{-1} M_1 \in \mathcal{C}l(\mathcal{X}^1)$, $G = M_0^{-1} L_0 \in \mathcal{L}(\mathcal{X}^0)$, $x^1(t) = Px(t)$, $x^0(t) = (I - P)x(t)$.

Remark 1. In accordance with Theorem 2 (see also [14, 16]), the unique solution of the Cauchy problem for equation (10) is the function

$$x^1(t) = X^t x_0 + \int_0^t X^{t-s} L_1^{-1} Q(B(s)u(s) + y(s)) ds,$$

and for equation (11), under the condition $u \in H_\partial(x_0, y)$, it is the function

$$x^0(t) = - \sum_{k=0}^p G^k M_0^{-1} \left(\sum_{l=0}^k \frac{k!}{l!(k-l)!} B_0^{(k-l)}(t) u^{(l)}(t) + y^{0(k)}(t) \right).$$

Remark 2. One can weaken conditions (6) for operator function B assuming instead the conditions

$$B \in W_{q'}^1(0, T; \mathcal{L}(\mathcal{U}; \mathcal{Y})), \quad (I - Q)B \in W_{q'}^{p+1}(0, T; \mathcal{L}(\mathcal{U}; \mathcal{Y})), \quad q' = \frac{q}{q-1},$$

as $T > 0$. Then $B(\cdot)u(\cdot) \in W_1^{p+1}(0, T; \mathcal{Y})$ and strong solutions should be sought in $W_1^1(0, T; \mathcal{X})$.

Speaking on ε -controllability of the system described by some equation, by $x(T; x_0; u)$ we denote the value of the strong solution to the Cauchy problem for this equation with an initial condition x_0 and a control function u at time T .

System (9) is called ε -controllable in time $T > 0$, if for each $x_0 \in \text{dom } M$, $\tilde{x} \in \mathcal{X}$, $\varepsilon > 0$ there exists a control $u \in H_\partial(x_0, y)$ such that $\|x(T; x_0; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$.

Let $\mathcal{E}$ be a Banach space, $D \subset \mathcal{E}$. By $\text{span } D$ we indicate the linear span of set D and $\overline{\text{span } D}$ stands for its closure in space $\mathcal{E}$.

Lemma 1. Suppose that operator M is strongly (L, p) -radial. If we have $(I - Q)B \in C^{p+1}([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$, then ε -controllability in time T of system (11) implies the identity

$$\text{span} \left\{ \text{im} \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), \quad l = 0, 1, \dots, p \right\} = \text{dom } M_0. \quad (12)$$

Proof. It follows from ε -controllability of system (11) that for each $x_0 \in \text{dom } M_0$ the inclusion

$$x^0 = x_0 + \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(0) \in \text{span} \left\{ \text{im} \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), \quad l = 0, 1, \dots, p \right\},$$

holds true, otherwise the set of admissible control function $H_\partial(x_0, y)$ is empty. Hence, since element $x_0 \in \text{dom } M_0$ is arbitrary, and thus the same is true for $x^0 \in \text{dom } M_0$, we have

$$\text{span} \left\{ \text{im} \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), \quad l = 0, 1, \dots, p \right\} \supset \text{dom } M_0.$$

The opposite inclusion holds true due to $G = M_0^{-1} L_0$, and therefore $\text{im } G \subset \text{dom } M_0$. The proof is complete. $\square$

We denote by $x^1(T; x_0; u)$ the value of the strong solution to the Cauchy problem $x^1(0) = x_0 \in \text{dom } M_1$ for equation (10) at time T , and $x^0(T; x_0; u)$ indicates the value of the strong solution to the Cauchy problem $x^0(0) = x_0 \in \text{dom } M_0$ for equation (11) at time T . Moreover, by $x^0(T; u)$ we denote the value of the strong solution to equation (11) at time T , which by [14, Th. 3] is determined uniquely without initial condition.

Remark 3. In the case of strong $(L, 0)$ -radiality condition, i.e., as $p = 0$, for $T \geq 0$ the condition

$$\text{span} \left\{ \text{im} \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T), \quad l = 0, 1, \dots, p \right\} = \text{dom } M_0$$

casts into the form $\text{im } M_0^{-1}(I - Q)B(T) = \text{dom } M_0$ and thus it is equivalent to condition $\text{im } (I - Q)B(T) = \mathcal{Y}^0$.

Lemma 2. *Suppose that operator M is strongly (L, p) -radial and condition (12) is satisfied.*

- (i) *If conditions (6), (7) are satisfied and for each $x_0 \in \text{dom } M_1$, $\tilde{x} \in \mathcal{X}^1$, $\varepsilon > 0$ there exists $u \in W_q^{p+1}(0, T; \mathcal{U})$ obeying $\|x^1(T; x_0; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$, then there exists $u_1 \in H_\partial(x_0, y)$ such that $\|x^1(T; x_0; u_1) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$;*
- (ii) *if $(I - Q)B \in C^{p+1}([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$ and for each $\tilde{x} \in \mathcal{X}^0$, $\varepsilon > 0$ there exists $u \in W_q^{p+1}(0, T; \mathcal{U})$ obeying $\|x^0(T; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$, then for each $x_0 \in \text{dom } M_0$, $\tilde{x} \in \mathcal{X}^0$, $\varepsilon > 0$ there exists $u_1 \in H_\partial(x_0, y)$ such that $\|x^0(T; x_0; u_1) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$.*

Proof. Let us prove Item (i). By Remark 1 and the hypothesis of the lemma,

$$\forall x_0 \in \text{dom } M_1 \quad \forall \tilde{x} \in \mathcal{X}^1 \quad \forall \varepsilon > 0 \quad \exists u \in W_q^{p+1}(0, T; \mathcal{U})$$

$$\left\| X^T x_0 + \int_0^T X^{T-s} L_1^{-1} Q (B(s)u(s) + y(s)) ds - \tilde{x} \right\|_{\mathcal{X}} \leq \varepsilon/2.$$

By (12), there exist $u_0, u_1, \dots, u_p \in \mathcal{U}$ such that

$$(I - P)x_0 + \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(0) = - \sum_{l=0}^p \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0) u_l.$$

We change smoothly the given control function $u(\cdot)$ in the right neighborhood of zero to obtain a new control function $u_1 \in W_q^{p+1}(0, T; \mathcal{U})$ obeying $u_1^{(l)}(0) = u_l$, $l = 0, 1, \dots, p$. We seek this function as

$$u_1(t) = \left(\frac{t}{t_0} \right)^{p+1} \sum_{k=0}^p a_k \frac{(t - t_0)^k}{k!} + \sum_{k=0}^p u_k \frac{t^k}{k!}, \quad t \in [0, t_0],$$

$u_1(t) = u(t)$, $t \in [t_0, T]$, for some $t_0 \in (0, T)$. For all coefficients $a_k \in \mathcal{U}$, $k = 0, 1, \dots, p$, such function satisfies the initial conditions. We choose coefficients a_k to satisfy $u_1^{(k)}(t_0) = u^{(k)}(t_0)$ as $k = 0, 1, \dots, p$. Equating the derivatives, we obtain the recurrent formula for the coefficients

$$a_0 = u(t_0) - \sum_{k=0}^p u_k \frac{t_0^k}{k!},$$

$$a_n = u^{(n)}(t_0) - \sum_{m=0}^{n-1} C_n^m \frac{(p+1)p(p-1)\dots(p-n+m+2)}{t_0^{n-m}} a_m - \sum_{k=0}^{p-n} u_{k+n} \frac{t_0^k}{k!}$$

as $n = 1, 2, \dots, p$.

Taking $t_0 < 1$, we obtain

$$\|a_0\|_{\mathcal{U}} \leq \|u\|_{W_q^{p+1}(0, t_0; \mathcal{U})} + \sum_{k=0}^p \|u_k\|_{\mathcal{U}} = c_0,$$

$$\|a_1\|_{\mathcal{U}} \leq \|u\|_{W_q^{p+1}(0, t_0; \mathcal{U})} + \frac{(p+1)c_0}{t_0} + \sum_{k=0}^p \|u_k\|_{\mathcal{U}} \leq \frac{c_1}{t_0}, \quad \dots, \quad \|a_p\|_{\mathcal{U}} \leq \frac{c_p}{t_0^p}.$$

Hence,

$$\begin{aligned} \|u_1\|_{C([0, t_0]; \mathcal{U})} &\leq \sum_{k=0}^p \|a_k\|_{\mathcal{U}} |t - t_0|^k + \sum_{k=0}^p \|u_k\|_{\mathcal{U}} \\ &\leq \sum_{k=0}^p c_k \left| 1 - \frac{t}{t_0} \right|^k + \sum_{k=0}^p \|u_k\|_{\mathcal{U}} \leq \sum_{k=0}^p c_k + \sum_{k=0}^p \|u_k\|_{\mathcal{U}}. \end{aligned}$$

The latter expression is independent of t_0 since constants c_k are independent, too. This is why for sufficiently small $t_0 \in (0, \delta]$,

$$\begin{aligned} \|u - u_1\|_{L_1(0,T;\mathcal{U})} &\leq \left(\|u\|_{C([0,t_0];\mathcal{U})} + \sum_{k=0}^p c_k + \sum_{k=0}^p \|u_k\|_{\mathcal{U}} \right) \delta \\ &\leq \frac{\varepsilon}{2Ke^{a|T}\|L_1^{-1}Q\|_{\mathcal{L}(\mathcal{Y};\mathcal{X})}\|B\|_{C([0,\delta];\mathcal{L}(\mathcal{U};\mathcal{Y}))}}, \end{aligned}$$

where K, a are constants in the definition of the strong (L, p) -radiality for operator M . It yields

$$\begin{aligned} &\left\| X^T x_0 + \int_0^T X^{T-s} L_1^{-1} Q (B(s)u_1(s) + y(s)) ds - \tilde{x} \right\|_{\mathcal{X}} \\ &\leq \left\| X^T x_0 + \int_0^T X^{T-s} L_1^{-1} Q (B(s)u(s) + y(s)) ds - \tilde{x} \right\|_{\mathcal{X}} \\ &\quad + \left\| \int_0^T X^{T-s} L_1^{-1} Q B(s) (u_1(s) - u(s)) ds \right\|_{\mathcal{X}} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

To prove Item (ii) we observe that the aforementioned change of function u by u_1 makes no influence on the value of the solution to equation (11) at time T (cf. the formula for solution in Remark 1). $\square$

Corollary 1. *Let operator M be strongly (L, p) -radial and conditions (6), (7), (12) are satisfied. If for each $x_0 \in \text{dom } M$, $\tilde{x} \in \mathcal{X}$, $\varepsilon > 0$ there exists $u \in W_q^{p+1}(0, T; \mathcal{U})$ obeying*

$$\|x^1(T; Px_0; u) + x^0(T; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon,$$

then there exists $u_1 \in H_{\partial}(x_0, y)$ such that $\|x(T; x_0; u_1) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$.

Proof. Constructing a control function $u_1 \in H_{\partial}(x_0, y)$ for sufficiently small t_0 , as in the proof of Lemma 2 we obtain

$$\begin{aligned} \|x(T; x_0; u_1) - \tilde{x}\|_{\mathcal{X}} &\leq \|x^1(T; Px_0; u_1) + x^0(T; (I - P)x_0; u_1) - \tilde{x}\|_{\mathcal{X}} \\ &\leq \|x^1(T; Px_0; u) + x^0(T; u) - \tilde{x}\|_{\mathcal{X}} + \left\| \int_0^T X^{T-s} L_1^{-1} Q B(s) (u_1(s) - u(s)) ds \right\|_{\mathcal{X}} \leq 2\varepsilon \end{aligned}$$

that completes the proof. $\square$

Reamrk 4. In what follows we shall make us of Lemma 2 and Corollary 1 implicitly and in the proof of ε -controllability we shall employ only the existence of an appropriate control function in the whole space $W_q^{p+1}(0, T; \mathcal{U})$ but not in $H_{\partial}(x_0, y)$.

The next result states that controlling two systems (10) and (11) by one control function, we nevertheless can move simultaneously the trajectories of both the systems into ε -neighborhoods of desired points.

Theorem 3. *Suppose that operator M is strongly (L, p) -radial and conditions (6), (7) are satisfied. Then system (9) ε -controllable in time T if and only if systems (10) and (11) are ε -controllable in time T .*

Proof. The direct statement of theorem is obvious, since system (9) splits into two subsystems (10) and (11) on mutually complementing subspaces.

We proceed to the proof of the inverse statement of theorem. Let

$$\begin{aligned} & \forall x_0^0 \in \text{dom } M_0 \quad \forall \tilde{x}^0 \in \mathcal{X}^0 \quad \forall \varepsilon > 0 \quad \exists v \in W_q^{p+1}(0, T; \mathcal{U}) \\ & \left\| - \sum_{l=0}^p \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T) v^{(l)}(T) - \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(T) - \tilde{x}^0 \right\|_{\mathcal{X}} \leq \varepsilon/3, \\ & \forall x_0^1 \in \text{dom } M_1 \quad \forall \tilde{x}^1 \in \mathcal{X}^1 \quad \forall \varepsilon > 0 \quad \exists w \in W_q^{p+1}(0, T; \mathcal{U}) \\ & \left\| X^T x_0^1 + \int_0^T X^{T-s} L_1^{-1} Q (B(s) w(s) + y(s)) ds - \tilde{x}^1 \right\|_{\mathcal{X}} \leq \varepsilon/3. \end{aligned}$$

For $x_0 \in \text{dom } M$, $\tilde{x} \in \mathcal{X}$ we take $x_0^0 = (I - P)x_0$, $\tilde{x}^0 = (I - P)\tilde{x}$, $x_0^1 = Px_0$, $\tilde{x}^1 = P\tilde{x}$. By these vectors and $\varepsilon > 0$ we choose the appropriate control functions v, w . We denote $u(t) = w(T - t)$ as $t \in [0, T]$, $u_k = (-1)^k v^{(k)}(T)$, $k = 0, 1, \dots, p$, and, as it was done in the proof of Lemma 2, we construct function $u_1 \in W_q^{p+1}(0, T; \mathcal{U})$ obeying $u_1^{(k)}(0) = (-1)^k v^{(k)}(T)$,

$$\|u - u_1\|_{L_1(0, T; \mathcal{U})} \leq \frac{\varepsilon}{3K e^{|a|T} \|L_1^{-1} Q\|_{\mathcal{L}(\mathcal{Y}; \mathcal{X})} \|B\|_{C([0, \delta]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))}}.$$

We choose the control function $v_1(t) = u_1(T - t)$ for system (9). Then we get $v_1^{(k)}(T) = (-1)^k u_1^{(k)}(0) = v^{(k)}(T)$ and therefore

$$\|x(T; x_0; v_1) - \tilde{x}\|_{\mathcal{X}} \leq 2\varepsilon/3 + \left\| \int_0^T X^{T-s} L_1^{-1} Q B(v_1(s) - w(s)) ds \right\|_{\mathcal{X}} \leq \varepsilon.$$

The proof is complete. $\square$

4. RELATIONS BETWEEN VARIOUS NOTIONS OF ε -CONTROLLABILITY

We introduce extra two definitions of ε -controllability employed actively in considering systems (10) (see, for instance, [8, 10]).

System (9) is called ε -controllable to zero in time $T > 0$, if for each $x_0 \in \text{dom } M$, $\varepsilon > 0$ there exists a control $u \in H_\partial(x_0, y)$ such that $\|x(T; x_0; u)\|_{\mathcal{X}} \leq \varepsilon$.

System (9) is called ε -controllable from zero in time $T > 0$, if for each $\tilde{x} \in \mathcal{X}$, $\varepsilon > 0$ there exists a control $u \in H_\partial(0, y)$ such that inequality $\|x(T; 0; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$ holds true.

Let us count the relations between various notions of ε -controllability for system (9); we provisionally agree to call the notion introduced in the previous section ε -controllability from any point to another.

Proposition 1. *Let operator M be strongly (L, p) -radial, $(I - Q)B \in C^{p+1}([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)y \equiv 0$. Then the notion of ε -controllability to zero in time T makes no sense for system (11).*

Proof. Letting $u \equiv 0 \in W_q^{p+1}(0, T; \mathcal{U})$, we obtain even the exact controllability ($\varepsilon = 0$) to zero in any time $T > 0$ for system (11). $\square$

Proposition 2. *Let operator M be strongly (L, p) -radial and $QB \in C^1([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $Qy \in W_q^1(0, T; \mathcal{U})$. For system (10) the notions of ε -controllability from zero and from any point to another in time T are equivalent.*

Proof. Denote $\tilde{\tilde{x}} = \tilde{x} - X^T x_0$. Then by Remark 1 $x(T; x_0; u) - \tilde{x} = x(T; 0; u) - \tilde{\tilde{x}}$. The arbitrariness of $\tilde{x}$ means the same for $\tilde{\tilde{x}}$ and vice versa. Then the notions of ε -controllability from zero and from any point to another are equivalent. $\square$

Proposition 3. *Let operator M be strongly (L, p) -radial and conditions (6), (7) are satisfied. Then system (9) is ε -controllable from any point to another in time T if and only if it is ε -controllable from zero in time T and condition (12) is obeyed.*

Proof. The direct statement is obvious. To prove the inverse, we observe that, as it was shown in Theorem 3, ε -controllability of system (9) is equivalent to ε -controllability of subsystems (10) and (11). By Proposition 2, for the former ε -controllability from zero is equivalent to ε -controllability from any point to another. The solution to the latter system at time $T > 0$ is independent of the initial condition, as it was said in Remark 1. $\square$

Remark 5. As one can see easily, a similar statement is true for system (11).

Proposition 3 allows us to restrict ourselves by considering the notion of ε -controllability from any point to another in time T . As above, we shall call it simply ε -controllability in time T .

5. CRITERION OF ε -CONTROLLABILITY IN TIME T

Lemma 3. *Let operator M be strongly (L, p) -radial, $QB \in C^1([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, and $Qy \in W_q^1(0, T; \mathcal{Y})$. Then system (10) is ε -controllable in time T if and only if*

$$\overline{\text{span}} \{ \text{im } X^{T-s} L_1^{-1} QB(s), s \in [0, T] \} = \mathcal{X}^1.$$

Proof. We first of all observe that by Proposition 2 we can consider only ε -controllability from zero. Assume that the system is not ε -controllable. Then, making the change $\tilde{x} = \tilde{x} - \int_0^T X^{T-s} L_1^{-1} Qy(s) ds$, we obtain that the set of vectors

$$\int_0^T X^{T-s} L_1^{-1} QB(s) u(s) ds, \quad \text{where } u \in W_q^{p+1}(0, T; \mathcal{U}),$$

is not dense in space $\mathcal{X}^1$. Then by Hahn-Banach theorem there exists a functional $f \in \mathcal{X}^{1*} \setminus \{0\}$ such that

$$0 = f \left(\int_0^T X^{T-s} L_1^{-1} QB(s) u(s) ds \right) = \int_0^T f(X^{T-s} L_1^{-1} QB(s) u(s)) ds \quad (13)$$

for each $u \in W_q^{p+1}(0, T; \mathcal{U})$.

For each function $v \in L_1(0, T; \mathcal{U})$ there exists a sequence $\{u_n\} \subset W_q^{p+1}(0, T; \mathcal{U})$ satisfying $\lim_{n \rightarrow \infty} u_n = v$ in $L_1(0, T; \mathcal{U})$. It implies

$$\begin{aligned} & \left| \int_0^T f(X^{T-s} L_1^{-1} QB(s) u_n(s)) ds - \int_0^T f(X^{T-s} L_1^{-1} QB(s) v(s)) ds \right| \\ & \leq \int_0^T |f(X^{T-s} L_1^{-1} QB(s) (u_n(s) - v(s)))| ds \\ & \leq \|f\|_{\mathcal{X}^{1*}} K e^{|a|T} \|L_1^{-1}\|_{\mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)} \|QB\|_{C([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))} \int_0^T \|u_n(s) - v(s)\|_{\mathcal{U}} ds \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, identity (13) is valid for each function $u \in L_1(0, T; \mathcal{U})$. We take $t_0 \in (0, T)$ and small $\delta > 0$, $u_\delta(t) = w \in \mathcal{U}$ as $t \in [t_0 - \delta, t_0 + \delta]$, $u_\delta(t) = 0$ as $t \in [0, T] \setminus [t_0 - \delta, t_0 + \delta]$.

Then by the continuity of the semi-group $\{X^t \in \mathcal{L}(\mathcal{X}) : t \in \overline{\mathbb{R}}_+\}$ and operator function $B(\cdot)$ the identity

$$0 = \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} f(X^{T-s}L_1^{-1}QB(s)w) ds = f(X^{T-\xi}L_1^{-1}QB(\xi)w)$$

is valid, where $\xi \in (t_0 - \delta, t_0 + \delta)$. Passing to the limit as $\delta \rightarrow 0+$, we arrive at the identity $f(X^{T-t_0}L_1^{-1}QB(t_0)w) = 0$ for each $t_0 \in (0, T)$, $w \in \mathcal{U}$. By the continuity of functional f it follows that $f(X^{T-s}L_1^{-1}QB(s)w) = 0$ for each $s \in [0, T]$. Thus, the set $\text{span}\{\text{im}X^{T-s}L_1^{-1}QB(s), s \in [0, T]\}$ is not dense in space $\mathcal{X}^1$.

The inverse statement is obvious due to the integral form of solution to equation (10) and the definition of integral. $\square$

Let us formulate the criterion of ε -controllability of system (11).

Lemma 4. *Let operator M be strongly (L, p) -radial and $(I - Q)B \in C^{p+1}([0, T]; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$. Then system (11) is ε -controllable in time T if and only if condition (12) and*

$$\overline{\text{span}} \left\{ \text{im} \sum_{k=l}^p \frac{k!G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T), l = 0, 1, \dots, p \right\} = \mathcal{X}^0,$$

are satisfied.

Proof. The direct statement follows from Lemma 1, the form of solution to system (11), and the arbitrariness of the vector $\tilde{x} = \tilde{x} + \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(T)$ in $\mathcal{X}^0$. Let us prove the inverse statement.

By the linearity of the operators used for the vector $\tilde{x} + \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(T) \in \mathcal{X}^0$, for each $\varepsilon > 0$ there exist $u_0, u_1, \dots, u_p \in \mathcal{U}$, such that

$$\left\| - \sum_{l=0}^p \sum_{k=l}^p \frac{k!G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T) u_l - \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(T) - \tilde{x} \right\|_{\mathcal{X}} \leq \varepsilon.$$

Hence, the function $u(t) = \sum_{k=0}^p \frac{(t-T)^k}{k!} u_k$ in $W_q^{p+1}(0, T; \mathcal{U})$ satisfies the inequality $\|x^0(T; x_0; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$. The proof is complete. $\square$

Theorem 4. *Let operator M be strongly (L, p) -radial and conditions (6), (7) are satisfied. Then system (9) is ε -controllable in time T if and only if condition (12) is satisfied and*

$$\overline{\text{span}} \{ \text{im} X^{T-s} L_1^{-1} Q B(s), s \in [0, T] \} = \mathcal{X}^1, \quad (14)$$

$$\overline{\text{span}} \left\{ \text{im} \sum_{k=l}^p \frac{k!G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T), l = 0, 1, \dots, p \right\} = \mathcal{X}^0. \quad (15)$$

Proof. The necessity of conditions (12), (14), and (15) follows from Lemmata 1, 3, and 4. And they are sufficient by the same lemmata and Theorem 3. $\square$

In the case of a constant operator function B , by the obtained criteria one can easily arrive at the following statements.

Corollary 2. *Let M be strongly (L, p) -radial for each $t \in [0, T]$ $(I - Q)B(t) \equiv B_1$, $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$. Then system (11) is ε -controllable in time T if and only if*

$$\text{span} \{ \text{im} G^k M_0^{-1} B_1, k = 0, 1, \dots, p \} = \text{dom} M_0. \quad (16)$$

Proof. By Lemma 1, ε -controllability of system (11) implies condition (12), which in the case of a constant operator function $(I - Q)B$ reads as (16).

Vice versa, condition (16) and the dense definiteness of operator M_0 in $\mathcal{X}^0$ (see Theorem 1, Item (ii)) yield that $\overline{\text{span}} \{\text{im } G^k M_0^{-1} B_1, k = 0, 1, \dots, p\} = \mathcal{X}^0$, and in accordance with Lemma 4 we obtain ε -controllability for system (11). $\square$

Remark 6. As one can see by the latter statement, in the case of a constant operator function $(I - Q)B$, ε -controllability in time T of system (11) implies its ε -controllability in any other time $T_1 > 0$.

Corollary 3. *Let operator M be strongly (L, p) -radial, $B(t) \equiv B_1$ for each $t \in [0, T]$ and conditions (7) hold true. Then system (9) is ε -controllable in time T if and only if*

$$\overline{\text{span}} \{\text{im } X^s L_1^{-1} Q B_1, s \in [0, T]\} = \mathcal{X}^1,$$

$$\text{span} \{\text{im } G^k M_0^{-1} (I - Q) B_1, k = 0, 1, \dots, p\} = \text{dom } M_0.$$

6. NOTION AND CRITERION OF ε -CONTROLLABILITY IN A FREE TIME

We introduce one more notion of controllability.

System (9) is called ε -controllable in a free time if for each $x_0 \in \text{dom } M$, $\tilde{x} \in \mathcal{X}$, and $\varepsilon > 0$ there exists a time $T > 0$ and a control function $u \in H_\partial(x_0, y)$ such that $\|x(T; x_0; u) - \tilde{x}\|_{\mathcal{X}} \leq \varepsilon$.

Remark 7. It is obvious that ε -controllability in time T implies ε -controllability in a free time.

Lemma 5. *Let operator M be strongly (L, p) -radial. If $(I - Q)B \in C^{p+1}([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, for each $T > 0$ $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$, then ε -controllability in a free time of system (11) implies identity (12).*

Proof. Time T has played no role in the proof of similar Lemma 1. $\square$

Theorem 5. *Let operator M be strongly (L, p) -radial, and for each $T > 0$ one have $y \in W_q^1(0, T; \mathcal{Y})$, $(I - Q)y \in W_q^{p+1}(0, T; \mathcal{Y})$, $B \in C^1([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)B \in C^{p+1}([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$. Then system (9) is ε -controllable in a free time if and only if systems (10) and (11) are ε -controllable in a free time.*

Proof. The direct statement of theorem is obvious; let us prove the inverse. We take $x_0 \in \text{dom } M_0$, $\tilde{x} \in \mathcal{X}$. Then there exists a time $T > 0$ and a control $u \in W_q^{p+1}(0, T; \mathcal{Y})$ moving the trajectory of system (10) into the ε -neighborhood of point $P\tilde{x}$. As in the proof of Theorem 3, we change function u in a sufficiently small neighborhood of point T to obtain a control $u_1 \in W_q^{p+1}(0, T; \mathcal{Y})$, which in time T also moves a trajectory of system (11) into the ε -neighborhood of the point $P\tilde{x}$. At that, a trajectory of system comes into the ε -neighborhood of the point $(I - P)\tilde{x}$ in time T . Thus, a trajectory of system (9) comes into the ε -neighborhood of the point $\tilde{x} = P\tilde{x} + (I - P)\tilde{x}$. The proof is complete. $\square$

Theorem 6. *Let operator M be strongly (L, p) -radial, $(I - Q)B(t) \equiv B_1$, $(I - Q)y \equiv 0$ for each $t \geq 0$. Then system (11) is ε -controllable in a free time if and only if it is ε -controllable in time T for each $T > 0$.*

Proof. The inverse statement of the theorem is obvious and we prove only the direct one. Suppose that system (11) is ε -controllable in a free time, i.e., for each $\tilde{x} \in \mathcal{X}$ and $\varepsilon > 0$ there exist $T_{\tilde{x}, \varepsilon} > 0$ and control $u \in W_q^{p+1}(0, T_{\tilde{x}, \varepsilon}; \mathcal{U})$ such that

$$\left\| -\sum_{k=0}^p G^k M_0^{-1} B_1 u^{(k)}(T_{\tilde{x}, \varepsilon}) - \tilde{x} \right\|_{\mathcal{X}} \leq \varepsilon.$$

Let us show that system (11) is ε -controllable in time T . If $T > T_{\tilde{x},\varepsilon}$, we take the control function $v(t) = u(t - T + T_{\tilde{x},\varepsilon})$ as $t \in [T - T_{\tilde{x},\varepsilon}, T]$,

$$v(t) = \sum_{k=0}^p u^{(k)}(0) \frac{(t - T + T_{\tilde{x},\varepsilon})^k}{k!}, \quad t \in [0, T - T_{\tilde{x},\varepsilon}].$$

If $T \leq T_{\tilde{x},\varepsilon}$, it is sufficient to take $v(t) = u(t - T + T_{\tilde{x},\varepsilon})$ as $t \in [0, T]$. $\square$

We prove criteria for ε -controllability in a free time only in the case $y \equiv 0$.

Lemma 6. *Let operator M be strongly (L, p) -radial, and $Qy \equiv 0$, $QB \in C^1([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$. Then system (10) is ε -controllable in a free time if and only if*

$$\overline{\text{span}} \{ \text{im } X^{s_1} L_1^{-1} QB(s_2), s_1, s_2 \geq 0 \} = \mathcal{X}^1.$$

Proof. Arguing by contradiction, as in the proof of Lemma 3, we obtain the identity $f(X^{T-t_0} L_1^{-1} QB(t_0)w) = 0$ for each $T > 0$, $t_0 \in [0, T]$, $w \in \mathcal{U}$. It follows that $f(X^{s_1} L_1^{-1} QB(s_2)w) = 0$ for each $s_1, s_2 \geq 0$ since $T \geq 0$ is arbitrary. Hence, the assumption that system (10) is not ε -controllable in a free time implies that the set $\text{span}\{ \text{im } X^{s_1} L_1^{-1} QB(s_2), s_1, s_2 \geq 0 \}$ is not dense in space $\mathcal{X}^1$.

The direct statement of the lemma follows from the integral form of the solution and the definition of integral. $\square$

Lemma 7. *Let operator M be strongly (L, p) -radial and $(I - Q)B \in C^{p+1}([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)y \equiv 0$. Then system (11) is ε -controllable in a free time if and only if*

$$\text{span} \left\{ \text{im } \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), l = 0, 1, \dots, p \right\} = \text{dom } M_0. \quad (17)$$

Proof. By Lemma 5, ε -controllability of system (11) yields condition (17).

Arguing as in the proof of Lemma 4, one can show easily that ε -controllability in a free time of system (11) is equivalent to the condition

$$\bigcup_{T>0} \overline{\text{span}} \left\{ \text{im } \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T), l = 0, 1, \dots, p \right\} = \mathcal{X}^0. \quad (18)$$

Since subspace $\mathcal{X}^0$ is closed and operator functions $B_0^{(k)}$, $k = 0, 1, \dots, p$, are continuous, set (18) coincides with the set

$$\bigcup_{T \geq 0} \overline{\text{span}} \left\{ \text{im } \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(T), l = 0, 1, \dots, p \right\}$$

and hence, it contains the set

$$\overline{\text{span}} \left\{ \text{im } \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), l = 0, 1, \dots, p \right\}.$$

The strong (L, p) -radiality of operator M implies that operator M_0 is densely defined in $\mathcal{X}^0$. Thus, by the above arguments, identity (17) implies identity (18), and hence, system (11) is ε -controllable in a free time. $\square$

Remark 8. By Lemmata 4 and 7 one can obtain easily another proof of Theorem 6.

Theorem 5 and Lemmata 6, 7 yield the criterion of ε -controllability in a free time for system (9).

Theorem 7. *Let operator M be strongly (L, p) -radial, $y \equiv 0$, $B \in C^1([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$, $(I - Q)B \in C^{p+1}([0, +\infty); \mathcal{L}(\mathcal{U}; \mathcal{Y}))$. Then system (9) is ε -controllable in a free time if and only if*

$$\overline{\text{span}} \{ \text{im } X^{s_1} L_1^{-1} Q B(s_2), s_1, s_2 \geq 0 \} = \mathcal{X}^1, \quad (19)$$

$$\text{span} \left\{ \text{im } \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} B_0^{(k-l)}(0), l = 0, 1, \dots, p \right\} = \text{dom } M_0. \quad (20)$$

As in the case of ε -controllability in time T , for a constant operator B , the latter statement casts into a simpler form.

Corollary 4. *Let operator M be strongly (L, p) -radial, $B(t) \equiv B_1$ for each $t \geq 0$, $y \equiv 0$. System (9) is ε -controllable in a free time if and only if*

$$\begin{aligned} \overline{\text{span}} \{ \text{im } X^s L_1^{-1} Q B_1, s \geq 0 \} &= \mathcal{X}^1, \\ \text{span} \{ \text{im } G^k M_0^{-1} (I - Q) B_1, k = 0, 1, \dots, p \} &= \text{dom } M_0. \end{aligned}$$

7. CONTROLLABILITY OF DZEKTSER EQUATION

Let $\Omega \subset \mathbb{R}^s$ be a bounded domain with a C^∞ boundary $\partial\Omega$. We consider the control system described by Dzektsler equation

$$(\lambda - \Delta)w_t(x, t) = \alpha \Delta w(x, t) - \beta \Delta^2 w(x, t) + c(t) \Delta u(x, t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \quad (21)$$

$\lambda, \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$, $c : \overline{\mathbb{R}}_+ \rightarrow \mathbb{R}$, subject to the boundary conditions

$$\nu \frac{\partial}{\partial n} w + (1 - \nu)w = \nu \frac{\partial}{\partial n} \Delta w + (1 - \nu) \Delta w = 0, \quad (x, t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \quad (22)$$

where $\nu \in \mathbb{R}$. To reduce it to system (9), we choose

$$\begin{aligned} \mathcal{X} &= H_\nu^2(\Omega) = \left\{ v \in W_2^2(\Omega) : \nu \frac{\partial}{\partial n} v(x) + (1 - \nu)v(x) = 0, x \in \partial\Omega \right\}, \\ \mathcal{Y} &= L_2(\Omega), \quad L = \lambda - \Delta \in \mathcal{L}(\mathcal{X}; \mathcal{Y}), \quad M = \alpha \Delta - \beta \Delta^2 \in \mathcal{Cl}(\mathcal{X}; \mathcal{Y}), \\ \text{dom } M &= H_\nu^4(\Omega) = \left\{ v \in W_2^4(\Omega) : \left(\nu \frac{\partial}{\partial n} + 1 - \nu \right) v(x) = \left(\nu \frac{\partial}{\partial n} + 1 - \nu \right) \Delta v(x) = 0, x \in \partial\Omega \right\}. \end{aligned}$$

At that, $B(t) = c(t) \Delta \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ for each $t \geq 0$.

In what follows we employ the symbol A_ν to denote the self-adjoint operator in $\mathcal{Cl}(L_2(\Omega))$ with the domain $\text{dom } A_\nu = H_\nu^2(\Omega)$, $A_\nu z = \Delta z$, $z \in \text{dom } A_\nu$. By $\{\varphi_k : k \in \mathbb{N}\}$ we denote the eigenfunctions of operator A_ν orthonormalized in the sense of the scalar product $\langle \cdot, \cdot \rangle$ in $L_2(\Omega)$; the eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ are taken in the ascending order counting the multiplicities. At that we employ the fact that the spectrum $\sigma(A_\nu)$ of operator A_ν is discrete and can accumulate to $-\infty$ only.

Theorem 8. *Let $\beta > 0$, $\alpha\lambda - \beta\lambda^2 \neq 0$, $c \in C^1(\overline{\mathbb{R}}_+; \mathbb{R})$. Then system (21), (22) is ε -controllable in a free time if and only if $c(0) \neq 0$. If $c(0) \neq 0$, $c(T) \neq 0$, then system (21), (22) is ε -controllable in time $T > 0$.*

Proof. It was proven in [23, Thm. 5] that operator M is strongly $(L, 0)$ -radial, where L, M correspond to a class of boundary value problems involving problem (21), (22). In accordance with this result, if $\beta > 0$, $\alpha\lambda - \beta\lambda^2 \neq 0$, operator M is strongly $(L, 0)$ -radial, where L, M are defined in the present section. At that, $Q = \sum_{\lambda_k \neq \lambda} \langle \cdot, \varphi_k \rangle \varphi_k$, $I - Q = \sum_{\lambda_k = \lambda} \langle \cdot, \varphi_k \rangle \varphi_k$.

It is easy to see that in the considered situation the inequality $c(t) \neq 0$ for some $t \geq 0$ yields $\text{im } B(t) = \mathcal{Y}$, $\text{im } QB(t) = \mathcal{Y}^1$, $\text{im } (I - Q)B(t) = \mathcal{Y}^0$, $\text{im } M_0^{-1}(I - Q)B(t) = \text{dom } M_0$.

Thus, if $c(0) \neq 0$, in view of Remark 3 condition (20) of Theorem 7 is satisfied and since operator $L_1 : \mathcal{X}^1 \rightarrow \mathcal{Y}^1$ is homeomorphic, the identity $\text{im} L_1^{-1}QB(0) = \mathcal{X}^1$ holds true. This is why condition (19) is obeyed and system (21), (22) is ε -controllable in a free time. Vice versa, if $c(0) = 0$, identity (20) fails and by Theorem 7 system (21), (22) is not ε -controllable in a free time.

Suppose $c(0) \neq 0$, $c(T) \neq 0$, then condition (15) holds and $\text{im} L_1^{-1}QB(T) = \mathcal{X}^1$. Therefore, condition (14) of Theorem 4 holds as well and system (21), (22) is ε -controllable in time T . $\square$

Remark 9. We note that condition $c(T) \neq 0$ is not necessary for ε -controllability in time $T > 0$ of system (21), (22).

8. LINEARIZED QUASI-STATIONARY SYSTEM OF PHASE FIELD EQUATIONS

Let us study the controllability of the linearized quasi-stationary system of phase field equations

$$\begin{aligned} \frac{\partial}{\partial t} z(x, t) &= \Delta z(x, t) - \Delta \theta(x, t) + a_{11}(t)u_1(x, t) + a_{12}(t)u_2(x, t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ \Delta \theta(x, t) - \beta \theta(x, t) + z(x, t) + a_{21}(t)u_1(x, t) + a_{22}(t)u_2(x, t) &= 0, \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \end{aligned} \quad (23)$$

subject to the boundary conditions

$$\begin{aligned} \nu \frac{\partial}{\partial n} z(x, t) + (1 - \nu)z(x, t) &= 0, \quad (x, t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \\ \nu \frac{\partial}{\partial n} \theta(x, t) + (1 - \nu)\theta(x, t) &= 0, \quad (x, t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \end{aligned} \quad (24)$$

where $\Omega \subset \mathbb{R}^s$ is a bounded domain with a C^∞ -boundary $\partial\Omega$, $\nu, \beta \in \mathbb{R}$. We let $\mathcal{X} = \mathcal{Y} = \mathcal{U} = (L_2(\Omega))^2$,

$$L = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}((L_2(\Omega))^2), \quad M = \begin{pmatrix} \Delta & -\Delta \\ I & -\beta I + \Delta \end{pmatrix} \in \mathcal{Cl}((L_2(\Omega))^2),$$

$$B(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \in \mathcal{L}((L_2(\Omega))^2), \quad t \geq 0.$$

Theorem 9. Let $\beta \notin \sigma(A_\nu)$, $a_{ij} \in C^1(\overline{\mathbb{R}}_+; \mathbb{R})$, $i, j = 1, 2$. Then under the inequality $a_{11}(0)a_{22}(0) - a_{12}(0)a_{21}(0) \neq 0$ system (23), (24) is ε -controllable in a free time. If inequalities $a_{21}^2(0) + a_{22}^2(0) \neq 0$, $a_{11}(T)a_{22}(T) - a_{12}(T)a_{21}(T) \neq 0$ hold true, then system (23), (24) is ε -controllable in time T .

Proof. It was shown in work [24] that once $\beta \notin \sigma(A_\nu)$, operator M is strongly $(L, 0)$ -radial,

$$P = \begin{pmatrix} I & \mathbb{O} \\ (\beta I - A_\nu)^{-1} & \mathbb{O} \end{pmatrix}, \quad Q = \begin{pmatrix} I & -A_\nu(\beta I - A_\nu)^{-1} \\ \mathbb{O} & \mathbb{O} \end{pmatrix},$$

$$\mathcal{X}^1 = \text{im } P = \{(v, (\beta I - A_\nu)^{-1}v) : v \in L_2(\Omega)\},$$

$$\mathcal{X}^0 = \ker P = \{0\} \times L_2(\Omega), \quad \mathcal{Y}^1 = \text{im } Q = L_2(\Omega) \times \{0\},$$

$$\mathcal{Y}^0 = \ker Q = \{(v, w) \in (L_2(\Omega))^2 : v = A_\nu(\beta I - A_\nu)^{-1}w\}.$$

It is to make sure that operator $B(t)$ in this problem is so that for $t \geq 0$ we have $\text{im } B(t) = \mathcal{Y}$ if and only if $a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t) \neq 0$; moreover, $\text{im } (I - Q)B(t) = \mathcal{Y}^0$ if and only if $a_{21}^2(t) + a_{22}^2(t) \neq 0$.

This is why in the case $a_{11}(0)a_{22}(0) - a_{12}(0)a_{21}(0) \neq 0$ we obtain $\text{im } QB(0) = \mathcal{Y}^1$, $\text{im } (I - Q)B(0) = \mathcal{Y}^0$ and in view of Remark 3, the homeomorphy of operator L_1 , and Theorem 7, system (23), (24) is ε -controllable in a free time.

At that, condition $a_{21}^2(0) + a_{22}^2(0) \neq 0$ means exactly the validity of necessary condition (12) for ε -controllability of system (23), (24) in time T . By the above arguments, the condition $a_{11}(T)a_{22}(T) - a_{12}(T)a_{21}(T) \neq 0$ in the framework of the present problem means that $\text{im } QB(T) = \mathcal{Y}^1$, $\text{im } (I - Q)B(T) = \mathcal{Y}^0$. This is why condition (15) is obeyed thanks to Remark 3. It remains to observe that condition (14) holds true, since we have the chain of embeddings

$$\mathcal{X}^1 \supset \text{span} \{ \text{im } X^{T-s} L_1^{-1} QB(s) : s \in [0, T] \} \supset \text{im } L_1^{-1} QB(T) = \mathcal{X}^1.$$

□

9. CONTROLLABILITY OF ALGEBRAIC-DIFFERENTIAL SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Consider the system of equations

$$\begin{aligned} v_{1t}(x, t) &= \Delta v_1(x, t) + \alpha(t)u_1(x, t), & (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ v_{3t}(x, t) &= \Delta v_2(x, t) + \beta(t)u_2(x, t), & (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ 0 &= \Delta v_3(x, t) + \gamma(t)u_3(x, t), & (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \end{aligned} \quad (25)$$

subject to the boundary conditions

$$v_i(x, t) = 0, \quad (x, t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \quad i = 1, 2, 3. \quad (26)$$

Here $\Omega \subset \mathbb{R}^s$ is a bounded domain with a C^∞ -boundary $\partial\Omega$, and $\alpha, \beta, \gamma : \overline{\mathbb{R}}_+ \rightarrow \mathbb{R}$ are functions.

We let $\mathcal{X} = \mathcal{Y} = \mathcal{U} = (L_2(\Omega))^3$, $\text{dom } M = (H_0^2(\Omega))^3$,

$$L = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}, \quad B(t) = \begin{pmatrix} \alpha(t) & 0 & 0 \\ 0 & \beta(t) & 0 \\ 0 & 0 & \gamma(t) \end{pmatrix}$$

as $t \geq 0$. It was shown in [17] that operator M is strongly $(L, 1)$ -radial, where L, M are the above introduced operators, and there were found the subspaces

$$\mathcal{X}^0 = \mathcal{Y}^0 = \{0\} \times L_2(\Omega) \times L_2(\Omega), \quad \mathcal{X}^1 = \mathcal{Y}^1 = L_2(\Omega) \times \{0\} \times \{0\}$$

and operators $L_1^{-1} = I : L_2(\Omega) \rightarrow L_2(\Omega)$,

$$M_0^{-1} = \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \Delta^{-1} \\ 0 & 0 \end{pmatrix}, \quad GM_0^{-1} = \begin{pmatrix} 0 & \Delta^{-2} \\ 0 & 0 \end{pmatrix}.$$

Here degenerate subsystem (11) reads as

$$\begin{pmatrix} \Delta^{-1}v_{3t}(x, t) \\ 0 \end{pmatrix} = \begin{pmatrix} v_2(x, t) + \beta(t)\Delta^{-1}u_2(x, t) \\ v_3(x, t) + \gamma(t)\Delta^{-1}u_3(x, t) \end{pmatrix}. \quad (27)$$

Lemma 8. *Let $\beta \in C^1(\overline{\mathbb{R}}_+; \mathbb{R})$, $\gamma \in C^2(\overline{\mathbb{R}}_+; \mathbb{R})$. System (26), (27) is ε -controllable in a free time if and only if $\beta(0)\gamma(0) \neq 0$. System (26), (27) is ε -controllable in time T if and only if $\beta(0)\gamma(0)\beta(T)\gamma(T) \neq 0$.*

Proof. According to Lemma 6 (condition (17)), the criterion of ε -controllability in a free time for system (26), (27) reads as

$$\text{span} \left\{ \text{im} \begin{pmatrix} \beta(0)\Delta^{-1} & \gamma'(0)\Delta^{-2} \\ 0 & \gamma(0)\Delta^{-1} \end{pmatrix}, \text{im} \begin{pmatrix} 0 & \gamma(0)\Delta^{-2} \\ 0 & 0 \end{pmatrix} \right\} = (H_0^2(\Omega))^2.$$

It is clear that this identity holds true if and only if $\beta(0) \neq 0$, $\gamma(0) \neq 0$.

At the same time, this condition is necessary for ε -controllability in time T of system (26), (27). Once it is obeyed, by Lemma 4, the criterion of ε -controllability in time T is the identity

$$\text{span} \left\{ \text{im} \begin{pmatrix} \beta(T)\Delta^{-1} & \gamma'(T)\Delta^{-2} \\ 0 & \gamma(T)\Delta^{-1} \end{pmatrix}, \text{im} \begin{pmatrix} 0 & \gamma(T)\Delta^{-2} \\ 0 & 0 \end{pmatrix} \right\} = (H_0^2(\Omega))^2.$$

The proof is complete. $\square$

Theorem 10. *Let $\alpha, \beta \in C^1(\overline{\mathbb{R}}_+; \mathbb{R})$, $\gamma \in C^2(\overline{\mathbb{R}}_+; \mathbb{R})$. If $\alpha(t)\beta(0)\gamma(0) \neq 0$ for some $t \geq 0$, system (25), (26) is ε -controllable in a free time. If $\beta(0)\gamma(0)\alpha(T)\beta(T)\gamma(T) \neq 0$, system (25), (26) is ε -controllable in time T .*

Proof. By Theorem 7, the criterion of ε -controllability in a free time of system (25), (26) is two conditions: $\beta(0)\gamma(0) \neq 0$ (according to Lemma 8) and $\overline{\text{span}}\{\text{im} X^{s_1} QB(s_2), s_1, s_2 \geq 0\} = \mathcal{Y}^1$. In our situation this identity is satisfied if function α is not identically zero; and the action of operator $QB(t)$ as $t \geq 0$ is the multiplication by $\alpha(t)$.

We have the embedding

$$\overline{\text{im}} L_1^{-1} QB(T) \subset \overline{\text{span}}\{\text{im} X^{T-s} L_1^{-1} QB(s), 0 \leq s \leq T\}.$$

If $\alpha(T) \neq 0$, then $\text{im} QB(T) = \mathcal{Y}^1$. Hence, $\text{im} L_1^{-1} QB(T) = \mathcal{X}^1$ and condition (14) in Theorem 4 is satisfied. As $\beta(0)\gamma(0)\beta(T)\gamma(T) \neq 0$, by Lemma 8 we obtain ε -controllability in time T for system (25), (26). $\square$

Remark 10. Thus, if $\beta(0)\gamma(0) = 0$, system (25), (26) is not ε -controllable even in a free time.

Remark 11. If we follow strictly the results of Sections 4 and 5, in Lemma 8 and Theorem 10, we should assume condition $\beta \in C^2(\overline{\mathbb{R}}_+; \mathbb{R})$. However, studying this system, one can observe that the condition $\beta \in C^1(\overline{\mathbb{R}}_+; \mathbb{R})$ is sufficient, while the excessive smoothness is inherited from the abstract formulation of the problem.

10. ON ε -CONTROLLABILITY OF DEGENERATE SYSTEM WITH FINITE-DIMENSIONAL INPUT

Given vector functions $y : \overline{\mathbb{R}}_+ \rightarrow \mathcal{Y}$, $b_i : \overline{\mathbb{R}}_+ \rightarrow \mathcal{Y}$, $i = 1, 2, \dots, m$, we consider the control system

$$L\dot{x}(t) = Mx(t) + \sum_{i=1}^m b_i(t)u_i(t) + y(t). \quad (28)$$

It is a particular case of system (9). To make sure, it is sufficient to take $\mathcal{U} = \mathbb{R}^m$, $B(t)u(t) = \sum_{i=1}^m b_i(t)u_i(t)$ as $t \geq 0$. Such control systems are called systems with a finite-dimensional input.

It is clear that for each $t \geq 0$ we have $B(t) \in \mathcal{L}(\mathbb{R}^m; \mathcal{Y})$.

According to Theorem 1, equation (28) is reduced to the system of two equations

$$\dot{x}^1(t) = S_1 x^1(t) + L_1^{-1} \sum_{i=1}^m b_i^1(t)u_i(t) + L_1^{-1} y^1(t), \quad (29)$$

$$G\dot{x}^0(t) = x^0(t) + M_0^{-1} \sum_{i=1}^m b_i^0(t)u_i(t) + M_0^{-1} y^0(t). \quad (30)$$

Here $b_i^1(t) = Qb_i(t)$, $b_i^0(t) = (I - Q)b_i(t)$, $i = 1, 2, \dots, m$, $x^1(t) = Qx(t)$, $x^0(t) = (I - Q)x(t)$, $y^1(t) = Qy(t)$, $y^0(t) = (I - Q)y(t)$, $t \geq 0$. The solution to the Cauchy problem $x(0) = x_0$ for equation (28) reads as

$$x(t) = X^t x_0 + \int_0^t X^{t-s} L_1^{-1} \left(\sum_{i=1}^m b_i^1(s)u_i(s) + y^1(s) \right) ds - \sum_{k=0}^p G^k M_0^{-1} \left(\sum_{i=1}^m b_i^0(t)u_i(t) + y^0(t) \right)^{(k)}.$$

At that, first two terms solve equation (29), and the latter sum is a solution to equation (30). Control vector functions $u = (u_1, \dots, u_m)$ will be chosen in the space $W_q^{p+1}(0, T; \mathbb{R}^m)$, $q \geq 1$, for some $T > 0$. By $H_\partial(x_0, y)$ we denote the set of vector functions $u = (u_1, u_2, \dots, u_m) \in W_q^{p+1}(0, T; \mathbb{R}^m)$, satisfying the condition

$$\begin{aligned} (I - P)x_0 &= - \sum_{k=0}^p G^k M_0^{-1} \left(\sum_{i=1}^m b_i^0 u_i + y^0 \right)^{(k)} \quad (0) \\ &= - \sum_{i=1}^m \sum_{k=0}^p \sum_{l=0}^k \frac{k!}{l!(k-l)!} G^k M_0^{-1} b_i^{0(k-l)}(0) u_i^{(l)}(0) - \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(0), \end{aligned}$$

which is necessary for the solvability of the Cauchy problem for equation (28) (see Theorem 2).

Lemma 9. *Let operator M be strongly (L, p) -radial, $b_i^0 \in C^{p+1}([0, T]; \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y^0 \in W_q^{p+1}(0, T; \mathcal{Y}^0)$. If system (30) is ε -controllable in time T , then $\mathcal{X}^0$ is at most $(p+1)m$ -dimensional, at that,*

$$\text{span} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(0), \quad l = 0, 1, \dots, p, \quad i = 1, 2, \dots, m \right\} = \text{dom } M_0 = \mathcal{X}^0. \quad (31)$$

Proof. Lemma 1 follows the former identity in (31). The latter follows from the proven finite dimension of domain $\text{dom } M_0$ and its density in $\mathcal{X}^0$. $\square$

Corollary 5. *Let operator M be strongly (L, p) -radial, $b_i^0 \in C^{p+1}([0, T]; \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y^0 \in W_q^{p+1}(0, T; \mathcal{Y}^0)$. Then ε -controllability of system (30) in time T is equivalent to its exact controllability in time T .*

Proof. In a finite-dimensional space the ε -controllability of a system is equivalent to its exact controllability ($\varepsilon = 0$). $\square$

Corollary 6. *Let operator M be strongly (L, p) -radial, $b_i^0 \in C^{p+1}([0, T]; \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y^0 \in W_q^{p+1}(0, T; \mathcal{Y}^0)$. Then ε -controllability in a free time of system (30) implies that $M_0 \in \mathcal{L}(\mathcal{X}^0; \mathcal{Y}^0)$.*

Proof. The statement follows immediately from the second identity in (31) and the closedness of operator M_0 . $\square$

Lemma 10. *Let operator M be strongly (L, p) -radial, $b_i^0 \in C^{p+1}([0, T]; \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y^0 \in W_q^{p+1}(0, T; \mathcal{Y}^0)$. Then system (30) is ε -controllable in time T if and only if (31) is satisfied and*

$$\text{span} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(T), \quad l = 0, 1, \dots, p, \quad i = 1, 2, \dots, m \right\} = \text{dom } M_0 = \mathcal{X}^0.$$

Proof. Lemma 9 implies identity (31), while Lemma 4 follows the identity

$$\begin{aligned} \mathcal{X}^0 &= \overline{\text{span}} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(T), \quad l = 0, 1, \dots, p, \quad i = 1, 2, \dots, m \right\} \\ &= \text{span} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(T), \quad l = 0, 1, \dots, p, \quad i = 1, 2, \dots, m \right\}, \end{aligned}$$

since the system of vectors is finite.

Vice versa, let for each $x^0 \in \text{dom } M_0 = \mathcal{X}^0$ there exist $c_i^l \in \mathbb{R}$, $l = 0, 1, \dots, p$, $i = 1, 2, \dots, m$, such that $x^0 = \sum_{i=1}^m \sum_{l=0}^p c_i^l \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(T)$. We choose constants for

$x^0 = -\tilde{x} - \sum_{k=0}^p G^k M_0^{-1} y^{0(k)}(T)$ and constructing the control vector function by the identities

$$u_i(t) = - \sum_{l=0}^p c_i^l \frac{(t-T)^l}{l!}, \quad i = 1, 2, \dots, m,$$

we obtain $x(T; u) = \tilde{x}$. We have proven the exact controllability of the considered system in time T . $\square$

Lemmata 9, 10 imply a criterion of ε -controllability in time T for system (28).

Theorem 11. *Let operator M be strongly (L, p) -radial, vector functions $b_i \in C^1([0, T]; \mathcal{Y})$, $i = 1, 2, \dots, m$, $y \in W_q^1(0, T; \mathcal{Y})$ be such that $b_i^0 \in C^{p+1}([0, T]; \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y^0 \in W_q^{p+1}(0, T; \mathcal{Y}^0)$. Then system (28) is ε -controllable in time T if and only if condition (31) holds true, and*

$$\overline{\text{span}} \{X^{T-s} L_1^{-1} b_i^1(s), s \in [0, T], i = 1, 2, \dots, m\} = \mathcal{X}^1,$$

$$\text{span} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(T), l = 0, 1, \dots, p, i = 1, 2, \dots, m \right\} = \text{dom } M_0 = \mathcal{X}^0.$$

Remark 12. As $m = 1$ and $m = 2$ and $b_1, b_2 \in \mathcal{Y}$ are constant, Theorem 11 imply the main results of work [21].

Similar results on ε -controllability of systems (28), (30) in free time follow from Lemma 7 and Theorem 7.

A criterion of ε -controllability of system (28) in a free time, as in the case of system with a control operator of general form, has a simpler form than a criterion for ε -controllability in time T .

Theorem 12. *Let operator M be strongly (L, p) -radial, $b_i \in C^1([0, +\infty); \mathcal{Y})$ be such that $b_i^0 \in C^{p+1}([0, +\infty); \mathcal{Y}^0)$, $i = 1, 2, \dots, m$, $y \equiv 0$. Then system (28) ε -controllable in a free time if and only if*

$$\overline{\text{span}} \{X^{s_1} L_1^{-1} b_i^1(s_2), s_1, s_2 \geq 0, i = 1, 2, \dots, m\} = \mathcal{X}^1,$$

and space $\mathcal{X}^0$ is at most $(p+1)m$ -dimensional, at that,

$$\text{span} \left\{ \sum_{k=l}^p \frac{k! G^k M_0^{-1}}{l!(k-l)!} b_i^{0(k-l)}(0), l = 0, 1, \dots, p, i = 1, 2, \dots, m \right\} = \text{dom } M_0 = \mathcal{X}^0.$$

11. EXAMPLES OF SYSTEMS WITH FINITE-DIMENSIONAL INPUT

In a bounded domain $\Omega \subset \mathbb{R}^s$ with C^∞ -boundary $\partial\Omega$ we consider a control system with a finite-dimensional input

$$(\lambda - \Delta)v_t(x, t) = \alpha \Delta v(x, t) - \beta \Delta^2 v(x, t) + \sum_{i=1}^m \hat{b}_i(x, t) u_i(t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \quad (32)$$

subject to boundary conditions (22). Here $\hat{b}_i(\cdot, t) \in L_2(\Omega)$ as $t \geq 0$, $i = 1, 2, \dots, m$.

Proposition 4. *Let $\beta > 0$, $\alpha\lambda - \beta\lambda^2 \neq 0$, and $\hat{b}_i \in C^1([0, +\infty); L_2(\Omega))$, $i = 1, 2, \dots, m$. If system (32), (22) is ε -controllable in a free time, then the system of vectors*

$$\left\{ \sum_{\lambda_k=\lambda} \langle \hat{b}_i(\cdot, 0), \varphi_k \rangle \varphi_k, i = 1, 2, \dots, m \right\} \subset L_2(\Omega)$$

contains a basis of the subspace $\mathcal{Y}^0 = \text{span} \{\varphi_k : \lambda_k = \lambda\}$.

Proof. We choose operators L, M as in Section 7. By Theorem 12 and in view of Remark 3, in the case of ε -controllability in a free time of system (32), (22), the identity $\text{dom } M_0 = \text{span} \{M_0^{-1}b_i^0(0), i = 1, 2, \dots, m\}$ is satisfied. This identity is equivalent to the identity $\mathcal{Y}^0 = \text{span} \{b_i^0(0), i = 1, 2, \dots, m\}$, at that, $b_i^0(0) = \sum_{\lambda_k=\lambda} \langle \hat{b}_i(\cdot, 0), \varphi_k \rangle \varphi_k$. $\square$

Remark 13. In particular, Proposition 4 implies that if system (32), (22) is ε -controllable in a free time, the multiplicity of an eigenvalue $\lambda \in \sigma(A_\nu)$ is at most m . Indeed, by Lemma 12, subspace $\mathcal{X}^0$ is at most m -dimensional and thus the same is true for $\mathcal{Y}^0$.

For the system of equations

$$\begin{aligned} v_{1t}(x, t) &= \Delta v_1(x, t) + \sum_{i=1}^m \hat{b}_i^1(x, t) u_i(t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ v_{3t}(x, t) &= \Delta v_2(x, t) + \sum_{i=1}^m \hat{b}_i^2(x, t) u_i(t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ 0 &= \Delta v_1(x, t) + \sum_{i=1}^m \hat{b}_i^3(x, t) u_i(t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \end{aligned}$$

subject to boundary conditions (26), where $\hat{b}_i^j(\cdot, t) \in L_2(\Omega)$ as $t \geq 0, i = 1, 2, \dots, m, j = 1, 2, 3$, the space $\mathcal{X}^0 = \{0\} \times L_2(\Omega) \times L_2(\Omega)$ is infinite-dimensional, see Section 9. This is why the hypothesis of Theorem 12 is not satisfied and this system is not ε -controllable even in a free time for each $m \in \mathbb{N}$. The same can be stated on the linearized quasi-stationary system of phase field equations (see Section 8)

$$\begin{aligned} \frac{\partial}{\partial t} z(x, t) &= \Delta z(x, t) - \Delta \theta(x, t) + \sum_{i=1}^m \hat{b}_i^1(x, t) u_i(t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \\ \Delta \theta(x, t) - \beta \theta(x, t) + z(x, t) + \sum_{i=1}^m \hat{b}_i^2(x, t) u_i(t) &= 0, \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \end{aligned}$$

subject to boundary conditions (24).

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