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LEVY'S PHENOMENON FOR ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. For entire functions $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $z \in \mathbb{C}$, P. Lévy (1929) established that in the classical Wiman's inequality $M_f(r) \leqslant \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$, $\varepsilon > 0$, which holds outside a set of finite logarithmic measure, the constant $1/2$ can be replaced almost surely in some sense by $1/4$; here $M_f(r) = \max\{|f(z)|\colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n\colon n \geqslant 0\}$, $r > 0$. In this paper we prove that the phenomenon discovered by P. Lévy holds also in the case of Wiman's inequality for entire functions of several variables, which gives an affirmative answer to the question of A. A. Goldberg and M. M. Sheremeta (1996) on the possibility of this phenomenon.

Keywords: Levy's phenomenon, random entire functions of several variables, Wiman's inequality

Mathematics Subject Classification: 30B20, 30D20

1. Introduction

For an entire function of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

we denote $M_f(r) = \max\{|f(z)|\colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n\colon n \geqslant 0\}$, $r > 0$. It is well known ([1], [2]) that for each nonconstant entire function f and each $\varepsilon > 0$ the following inequality

$$M_f(r) \leqslant \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon} \tag{1}$$

holds for $r > 1$ outside an exceptional set $E_f(\varepsilon)$ of finite logarithmic measure ($\int_{E_f(\varepsilon)} \frac{dr}{r} < +\infty$).

In this paper we consider entire functions of p complex variables

$$f(z) = f(z_1, \dots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n, \tag{2}$$

where $z^n = z_1^{n_1} \dots z_p^{n_p}$, $p \in \mathbb{N}$, $n = (n_1, \dots, n_p) \in \mathbb{Z}_+^p$, $\|n\| = \sum_{j=1}^p n_j$. For $r = (r_1, \dots, r_p) \in \mathbb{R}_+^p$ we denote

$$B(R) = \{t \in \mathbb{R}_+^p \colon t_j \geqslant R_j,\ j \in \{1, \dots, p\}\},\ R = (R_1, \dots, R_p),\ \ln_2 x = \ln\ln x,$$
$$r^\wedge = \min_{1 \leqslant i \leqslant p} r_i,\ M_f(r) = \max\{|f(z)|\colon |z_1| = r_1, \dots, |z_p| = r_p\},$$
$$\mu_f(r) = \max\{|a_n| r_1^{n_1} \dots r_p^{n_p} \colon n \in \mathbb{Z}_+^p\},\ \mathfrak{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n| r^n.$$

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By Λ^p we denote the class of entire functions such that $\frac{\partial}{\partial z_j}f(z)\not\equiv 0$ in $\mathbb{C}^p$ for each $j\in\{1,\dots,p\}$. We say that a subset E of $\mathbb{R}^p_+$ is a set of asymptotically finite logarithmic measure [9] if E is Lebesgue measurable in $\mathbb{R}^p_+$ and there exists an $R\in\mathbb{R}^p_+$ such that $E\cap B(R)$ is a set of finite logarithmic measure, i.e.

$$\int\limits_{E\cap B(R)}\cdots\int\prod_{j=1}^{p}\frac{dr_j}{r_j}<+\infty.$$

For entire functions (2) analogues of inequality (1) were proved in [3, 5, 6, 9]. Also analogues of inequality (1) without exceptional sets for entire functions of several complex variables can be found in [10].

In particular, the following statement was proved in [9].

Theorem 1. *Let $f\in\Lambda^p$ and $\delta>0$.*

a) *Then there exist $R\in\mathbb{R}^p_+$ and a subset E of $B(R)$ of finite logarithmic measure such that for each $r\in B(R)\backslash E$ we have*

$$\mathfrak{M}_f(r)\leqslant\mu_f(r)\Big(\prod_{i=1}^{p}\ln^{p-1}r_i\cdot\ln^p\mu_f(r)\Big)^{1/2+\delta}. \tag{3}$$

b) *If for some $\alpha\in\mathbb{R}^p_+$ we have $\mathfrak{M}(r)\geqslant\exp(r^\alpha)=\exp(r_1^{\alpha_1}\dots r_p^{\alpha_p})$, as $r^\wedge\to+\infty$ or more generally, for each $\beta>0$*

$$\int\limits_{B(S)}\cdots\int\frac{\prod\limits_{i=1}^{p}dr_i}{r_1r_2\dots r_p\ln^\beta\mathfrak{M}_f(r)}<+\infty,\ \text{as } S^\wedge\to+\infty, \tag{4}$$

then there exist $R\in\mathbb{R}^p_+$ and a subset E of $B(R)$ of finite logarithmic measure such that for each $r\in B(R)\backslash E$ we have

$$\mathfrak{M}_f(r)\leqslant\mu_f(r)\ln^{p/2+\delta}\mu_f(r).$$

2. Wiman's type inequality for random entire functions of several variables

Let $\Omega=[0,1]$ and P be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega,\mathcal{A},P)$, where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω. Let $X=(X_n(t))$ be some sequence of random variables defined in this space. For an entire function of the form $f(z)=\sum_{n=0}^{+\infty}a_nz^n$ by $K(f,X)$ we denote the class of random entire functions

$$f(z,t)=\sum_{n=0}^{+\infty}a_nX_n(t)z^n. \tag{5}$$

In what follows, the notion "almost surely" will be employed in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure P on $\Omega=[0,1]$. We say that some relation holds almost surely in the class $K(f,X)$ if it holds for each entire function $f(z,t)$ of the form (5) almost surely in t.

In the case $\mathcal{R}=(X_n(t))$ is the Rademacher sequence, i.e. $(X_n(t))$ is a sequence of independent uniformly distributed random variables on $[0,1]$ such that $P\{t\colon X_n(t)=\pm1\}=1/2$, P. Levy [7] proved that for each entire function we can replace the constant $1/2$ by $1/4$ in the inequality (1) almost surely in the class $K(f,\mathcal{R})$. Later P. Erdős and A. Rényi [8] proved the same result for the class $K(f,H)$, where $H=(e^{2\pi i\omega_n(t)})$ is the Steinhaus sequence, i.e. $(\omega_n(t))$ is a sequence of independent uniformly distributed random variables on $[0,1]$. This statement is true also for each class $K(f, X)$, where $X = (X_n(t))$ is multiplicative system (MS) uniformly bounded by the number 1. That is for each $n \in \mathbb{N}$ and $t \in [0, 1]$ we have $|X_n(t)| \leqslant 1$ and

$$(\forall 1 \leqslant i_1 < i_2 < \dots < i_k)\colon\ \mathbf{M}(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0,$$

where $\mathbf{M}\xi$ is the expected value of a random variable ξ ([15]–[16]).

In the spring of 1996 during the report of P. V. Filevych at the Lviv seminar of the theory of analytic functions, professors A. A. Goldberg and M. M. Sheremeta posed the following question (see [12]): *Does Levy's effect take place for analogues of Wiman's inequality for entire functions of several complex variables?*

In the papers [12]–[14] we have found an affirmative answer to this question for Fenton's inequality [4] for entire functions of two complex variables.

In this paper we will give answer to this question for Wiman's type inequality in [9] for entire functions of several complex variables.

The exceptional set in our statements is "smaller" than the exceptional set in the corresponding theorems from [4], [12]–[14]. The method of proof in this paper differs from the method of the papers [4], [12]–[14].

Let $Z = (Z_n(t))$ be a complex sequence of random variables $Z_n(t) = X_n(t) + iY_n(t)$ such that both $X = (X_n(t))$ and $Y = (Y_n(t))$ are real MS and $K(f, Z)$ the class of random entire functions of the form

$$f(z, t) = \sum_{\|n\|=0}^{+\infty} a_n Z_n(t) z_1^{n_1} \dots z_p^{n_p}.$$

Theorem 2. *Let $Z = (Z_n(t))$ be a MS uniformly bounded by the number 1, $\delta > 0$, $f \in \Lambda^p$.*

a) *Then almost surely in $K(f, Z)$ there exist $R \in \mathbb{R}_+^p$ and a subset E^* of $B(R)$ of finite logarithmic measure such that for each $r \in B(R)\backslash E^*$ we have*

$$M_f(r, t) = \max_{|z|=r} |f(z, t)| \leqslant \mu_f(r) \Big(\ln^p \mu_f(r) \cdot \prod_{i=1}^{p} \ln^{p-1} r_i\Big)^{1/4+\delta}. \tag{6}$$

b) *If for some $\alpha \in \mathbb{R}_+^p$ we have*

$$\mathfrak{M}(r) \geqslant \exp(r^\alpha) = \exp(r_1^{\alpha_1} \dots r_p^{\alpha_p}) \text{ as } r^\wedge \to +\infty$$

or more generally, for each $\beta > 0$ inequality (4) holds, then almost surely in $K(f, Z)$ there exist $R \in \mathbb{R}_+^p$ and a subset E of $B(R)$ of finite logarithmic measure such that for each $r \in B(R)\backslash E$ we get

$$M_f(r, t) \leqslant \mu_f(r) \ln^{p/4+\delta} \mu_f(r). \tag{7}$$

Lemma 1 ([10])**.** *Let $X = (X_n(t))$ be a MS uniformly bounded by the number 1. Then for each $\beta > 0$ there exists a constant $A_{\beta p} > 0$, which depends on p and β only such that for each $N \geqslant N_1(p) = \max\{p, 4\pi\}$ and $\{c_n\colon \|n\| \leqslant N\} \subset \mathbb{C}$ we have*

$$P\left\{t\colon \max\left\{\left|\sum_{\|n\|=0}^{N} c_n X_n(t) e^{in_1\psi_1} \dots e^{in_p\psi_p}\right| \colon \psi \in [0, 2\pi]^p\right\} \geqslant A_{\beta p} S_N \ln^{\frac{1}{2}} N\right\} \leqslant \frac{1}{N^\beta}, \tag{8}$$

where $S_N^2 = \sum_{\|n\|=0}^{N} |c_n|^2$.

By H we denote the class of function $h\colon \mathbb{R}_+^p \to \mathbb{R}_+$ such that

$$\int_1^{+\infty} \dots \int_1^{+\infty} \frac{du_1 \dots du_p}{h(u)} < +\infty.$$

For each $i \in \{1, \ldots, p\}$ we also define

$$\partial_i \ln \mathfrak{M}_f(r) = r_i \frac{\partial}{\partial r_i}(\ln \mathfrak{M}_f(r)) = \frac{1}{\mathfrak{M}_f(r)} \sum_{\|n\|=0}^{+\infty} n_i |a_n| r^n$$

Lemma 2 ([9])**.** *Let $h \in H$. Then there exist $R \in \mathbb{R}^p_+$ and a subset E' of $B(R)$ of finite logarithmic measure such that for each $r \in B(R)\backslash E'$ and $s \in \{1, \ldots, p\}$ we have*

$$\partial_s \ln \mathfrak{M}_f(r) \leqslant h(\ln r_1, \ldots, \ln r_{s-1}, \ln \mathfrak{M}_f(r), \ln r_{s+1}, \ldots, \ln r_p). \tag{9}$$

Proof of Theorem 2. Without loss of generality we may suppose that $Z = X = (X_n(t))$ is a MS. Indeed, if $Z_n(t) = X_n(t) + iY_n(t)$, we obtain

$$f(z,t) = \sum_{\|n\|=0}^{+\infty} a_n X_n(t) z^n + \sum_{\|n\|=0}^{+\infty} i a_n Y_n(t) z^n = f_1(z,t) + f_2(z,t),$$

where $f_1, f_2 \in K(f, X)$, and

$$\max\{\mu(r, f_1(\cdot, t)), \mu(r, f_2(\cdot, t))\} \leqslant \mu(r, f) = \max\{|a_n| r_1^{n_1} \ldots r_p^{n_p} \colon n \in \mathbb{Z}^p_+\}$$

for each $r \in \mathbb{R}^p_+$ and $t \in [0,1]$. Hence, by inequality (6) we obtain that there exists a set E_0 of asymptotically finite logarithmic measure such that for each $r \in B(R)\backslash E_0$ almost surely in $K(f, Z)$

$$M_{f_j}(r,t) \leqslant \mu_f(r) \Big(\prod_{i=1}^p \ln^{p-1} r_i \cdot \ln^p \mu_f(r) \Big)^{1/4+\delta_0}, \quad j \in \{1,2\},\ \delta_0 > 0.$$

Thus, for $R^\wedge$ great enough and for each $r \in B(R)\backslash E_0$ almost surely in $K(f, Z)$ we get

$$\begin{aligned} M_f(r,t) \leqslant & M_{f_1}(r,t) + M_{f_2}(r,t) \\ \leqslant & 2\mu_f(r) \Big(\prod_{i=1}^p \ln^{p-1} r_i \cdot \ln^p \mu_f(r) \Big)^{1/4+\delta_0} < \mu_f(r) \Big(\prod_{i=1}^p \ln^{p-1} r_i \cdot \ln^p \mu_f(r) \Big)^{1/4+2\delta_0}. \end{aligned}$$

For each $j \in \{1, \ldots, p\}$ we have

$$\lim_{r_j \to +\infty} \mu_f(r_1^0, \ldots, r_{j-1}^0, r_j, r_{j+1}^0, \ldots, r_p^0) = +\infty \tag{10}$$

for fixed $r_i^0 > 0$, $i \in \{1, \ldots, p\}\backslash\{j\}$. Indeed, if (10) does not hold, there exists a constant $C > 0$ such that for each $r_j > r_j^*$ we have $\mu_f(r_1^0, \ldots, r_{j-1}^0, r_j, r_{j+1}^0, \ldots, r_p^0) < C < +\infty$. Hence, $\#\{n_j \geqslant 1 \colon a_n \neq 0\} = 0$ and $\frac{\partial}{\partial z_j} f(z) \equiv 0$ in $\mathbb{C}^p$. So, $f \notin \Lambda^p$, which gives a contradiction.

For $k \in \mathbb{N} \cup \{0\}$ we denote $G_k = \{r = (r_1, \ldots, r_p) \in \mathbb{R}^p_+ \colon k \leqslant \ln \mu_f(r) < k+1\} \cap [1; +\infty)^p$. Then $G_k \neq \varnothing$ for $k \geqslant k_0$ and by (10) we obtain that for each k the set G_k is a bounded set. Let $G_k^+ = \bigcup_{j=k}^{+\infty} G_j$ and

$$h(r) = \prod_{i=1}^p r_i \ln^{1+\delta_1} r_i \in H,\ \delta_1 > 0.$$

By Lemma 2 there exist $R_j \in \mathbb{R}^p_+$ and a subset E_j of $B(R_j)$ of finite logarithmic measure such that for each $r \in B(R_j)\backslash E_j$ and $j \in \{1, \ldots, p\}$ we have

$$\begin{aligned} \sum_{\|n\|=0}^{+\infty} n_i |a_n| r^n \leqslant & \mathfrak{M}_f(r) h(\ln r_1, \ldots, \ln r_{s-1}, \ln \mathfrak{M}_f(r), \ln r_{s+1}, \ldots, \ln r_n) \\ \leqslant & \mathfrak{M}_f(r) \ln \mathfrak{M}_f(r) \ln_2^{1+\delta_1} \mathfrak{M}_f(r) \prod_{i=1,\ i\neq j}^p \ln r_i \ln_2^{1+\delta_1} r_i. \end{aligned}$$

We can choose $R \in \mathbb{R}^p_+$ so that $B(R) \subset \Big(\bigcap_{j=1}^p B(R_j) \Big) \cap [e^{e^2}, +\infty)^p$.

Thus, for $R^{\wedge}$ great enough and for each $r\in B(R)\backslash(\cup_{i=1}^{p}E_i)$ we obtain

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n \leqslant \mathfrak{M}_f(r)\ln\mathfrak{M}_f(r)\ln_2^{1+\delta_1}\mathfrak{M}_f(r)\sum_{j=1}^{p}\Biggl(\prod_{i=1,\ i\neq j}^{p}\ln r_i\ln_2^{1+\delta_1}r_i\Biggr)$$
$$\leqslant p\cdot\mathfrak{M}_f(r)\ln^{1+\delta_1/2}\mathfrak{M}_f(r)\prod_{i=1}^{p}\ln r_i\ln_2^{1+\delta_1}r_i,$$

By Theorem 1, we get that for $R^{\wedge}$ great enough and for each $r\in B(R)\backslash(\bigcup_{i=1}^{p}E_i)$

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n \leqslant p\mu_f(r)\Bigl(\prod_{i=1}^{p}\ln^{p-1}r_i\cdot\ln^{p}\mu_f(r)\Bigr)^{1/2+\delta_1}\times$$
$$\times\Bigl(\ln\mu_f(r)+\Bigl(\frac{1}{2}+\delta_1\Bigr)\Bigl((p-1)\sum_{i=1}^{p}\ln_2 r_i+p\ln_2\mu_f(r)\Bigr)\Bigr)^{1+\delta_1/2}\prod_{i=1}^{p}\ln r_i\ln_2^{1+\delta_1}r_i$$
$$\leqslant\mu_f(r)(\ln\mu_f(r))^{p/2+(p+1)\delta_1+1}\Bigl(\prod_{i=1}^{p}\ln r_i\Bigr)^{(p-1)(1/2+\delta_1)+1}\Bigl(\prod_{i=1}^{p}\ln_2 r_i\Bigr)^{2+3\delta_1/2},$$

since $a_1x_1+\dots+a_kx_k<x_1\cdot\ldots\cdot x_k$ for $x^{\wedge}\geqslant 1$ great enough, $x=(x_1,\dots,x_k)$. Therefore, as $\delta_2=(p+1)\delta_1$, for $R^{\wedge}$ great enough and for each $r\in B(R)\backslash(\bigcup_{i=1}^{p}E_i)$ we obtain

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\leqslant\mu_f(r)\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^{p}\Bigl(\ln^p r_i\ln_2^2 r_i\Bigr)^{1+\delta_2}.$$

Hence,

$$\sum_{\|n\|\geqslant d}|a_n|r^n\leqslant\sum_{\|n\|\geqslant d}\frac{\|n\|}{d}|a_n|r^n=\frac{1}{d}\sum_{\|n\|\geqslant d}\|n\||a_n|r^n$$
$$\leqslant\frac{1}{d}\mu_f(r)\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^{p}\Bigl(\ln^p r_i\ln_2^2 r_i\Bigr)^{1+\delta_2}=\mu_f(r),\tag{11}$$

where

$$d=d(r)=\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^{p}\Bigl(\ln^p r_i\ln_2^2 r_i\Bigr)^{1+\delta_2}.$$

Let $G_k^*=G_k\setminus E_{p+1}$, $E_{p+1}=\bigcup_{i=1}^{p}(E_i\cup E^*)\cup\Bigl(\bigcup_{i=1}^{k_0-1}G_i\Bigr)$. By I we denote the set of integers $k\geqslant k_0$ such that $G_k^*\neq\varnothing$. Then $\#I=+\infty$. For $k\in I$ we choose a sequence $r^{(k)}\in G_k^*$. Then for each $r\in G_k^*$ we get

$$\mu_f(r^{(k)})<e^{k+1}\leqslant e\mu_f(r),\quad \mu_f(r)<e^{k+1}<e\mu_f(r^{(k)}),\tag{12}$$

and also

$$\bigcup_{k\in I}G_k^*=\bigcup_{k\in I}G_k\setminus E_{p+1}=\bigcup_{k=1}^{+\infty}G_k\setminus E_{p+1}=[1;+\infty)^p\setminus E_{p+1}.$$

For $k\in I$ we denote $N_k=[2d_1(r^{(k)})]$, where

$$d_1(r)=\ln^{p/2+1+\delta_2}(e\mu_f(r))\prod_{i=1}^{p}\Bigl(\ln^p r_i\ln_2^2 r_i\Bigr)^{1+\delta_2},$$

and for $r \in G_k^*$

$$W_{N_k}(r,t) = \max\left\{\left|\sum_{\|n\|\leqslant N_k} a_n r_1^{n_1}\dots r_p^{n_p} e^{in_1\psi_1+\ldots+in_p\psi_p} X_n(t)\right| : \psi \in [0,2\pi]^p\right\}.$$

For a Lebesgue measurable set $G \subset G_k^*$ and for $k \in I$ we denote

$$\nu_k(G) = \frac{\text{meas}_p(G)}{\text{meas}_p(G_k^*)},$$

where meas_p denotes the Lebesgue measure on $\mathbb{R}^p$. Note that ν_k is a probability measure defined on the family of Lebesgue measurable subsets of G_k^*.

Let $\Omega = \bigcup_{k\in I} G_k^*$ and $I = \{k_j : j \geqslant 0\}$, where $k_j < k_{j+1}$, $j \geqslant 0$. Without loss of generality we may assume that $k_0 = 0$. Then $E_{p+1} = \bigcup_{i=1}^p (E_i \cup E^*)$. For Lebesgue measurable subsets G of Ω we denote

$$\nu(G) = \sum_{j=0}^{+\infty} \frac{1}{2^{k_j}}\Big(1 - \Big(\frac{1}{2}\Big)^{k_{j+1}-k_j}\Big) \cdot \nu_{k_{j+1}}(G \cap G_{k_{j+1}}^*). \tag{13}$$

We note that $\nu_{k_{j+1}}(G_{k_{j+1}}^*) = 1$, therefore

$$\nu(\Omega) = \sum_{j=0}^{+\infty} \frac{1}{2^{k_j}}\Big(1 - \Big(\frac{1}{2}\Big)^{k_{j+1}-k_j}\Big)\nu_{k_{j+1}}(G_{k_{j+1}}^*) = \sum_{j=0}^{+\infty}\sum_{s=k_j+1}^{k_{j+1}} \frac{1}{2^s} = \sum_{s=1}^{+\infty}\frac{1}{2^s} = 1.$$

Thus, ν is a probability measure defined on measurable subsets of Ω. On $[0,1]\times\Omega$ we define the probability measure $P_0 = P \otimes \nu$, which is a direct product of the probability measures P and ν. Now for $k \in I$ we define

$$F_k = \{(t,r) \in [0,1]\times\Omega \colon W_{N_k}(r,t) > A_1 S_{N_k}(r)\ln^{1/2} N_k\},$$
$$F_k(r) = \{t \in [0,1] \colon W_{N_k}(r,t) > A_1 S_{N_k}(r)\ln^{1/2} N_k\},$$

where $S_{N_k}^2(r) = \sum_{\|n\|=0}^{N_k} |a_n|^2 r^{2n}$ and A_p is the constant from Lemma 1 with $\beta = 1$. Using Fubini's theorem and Lemma 1 with $c_n = a_n r^n$ and $\beta = 1$, for $k \in I$ we get

$$P_0(F_k) = \int\limits_{\Omega}\Bigg(\int\limits_{F_k(r)} dP\Bigg)d\nu = \int\limits_{\Omega} P(F_k(r))d\nu \leqslant \frac{1}{N_k}\nu(\Omega) = \frac{1}{N_k}.$$

Note that $N_k > \ln^{p/2+1}\mu_f(r^{(k)}) \geqslant k^{3/2}$. Therefore $\sum_{k\in I} P_0(F_k) \leqslant \sum_{k=1}^{+\infty} k^{-3/2} < +\infty$. By Borel-Cantelli's lemma the infinite quantity of the events $\{F_k \colon k \in I\}$ may occur with the zero probability. Thus,

$$P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty}\bigcap_{k\geqslant s, k\in I} \overline{F_k} \subset [0,1]\times\Omega.$$

Then for each point $(t,r) \in F$ there exists $k_0 = k_0(t,r)$ such that for each $k \geqslant k_0$, $k \in I$ we have

$$W_{N_k}(r,t) \leqslant A_1 S_{N_k}(r)\ln^{1/2} N_k. \tag{14}$$

Let P_j be a probability measure defined on $(\Omega_j, \mathcal{A}_j)$, where $\mathcal{A}_j$ is a σ-algebra of subsets Ω_j ($j \in \{1,\dots,p\}$) and P_0 is the direct product of probability measures $P_1,\dots,P_p$ defined on $(\Omega_1\times\ldots\times\Omega_p, \mathcal{A}_1\times\ldots\times\mathcal{A}_p)$. Here $\mathcal{A}_1\times\ldots\times\mathcal{A}_p$ is the σ-algebra, which contains all $A_1\times\ldots\times A_p$, where $A_j \in \mathcal{A}_j$. If $F \subset \mathcal{A}_1\times\ldots\times\mathcal{A}_p$ such that $P_0(F) = 1$, then in the case when projection

$$F_1 = \{t_1 \in \Omega_1 \colon (\exists(t_2,\dots,t_p) \in \Omega_2\times\ldots\times\Omega_p)[(t_1,\dots,t_p) \in F]\}$$

of the set F on Ω_1 is P_1-measurable we have $P_1(F_1) = 1$.

By F_Ω we denote the projection of F on Ω, i.e. $F_\Omega = \{r \in \Omega\colon (\exists t)[(t,r) \in F]\}$. Then $\nu(F_\Omega)=1$. Similarly, the projection of F on $[0,1]$, $F_{[0,1]} = \bigcup_{r\in\Omega} F(r)$, we obtain $P(F_{[0,1]})=1$.

Let $F^\wedge(t) = \{r \in \Omega : (t,r) \in F\}$. By Fubini's theorem we have

$$0 = \int\limits_X (1-\chi_F)dP_0 = \int\limits_0^1 \left(\int\limits_\Omega (1-\chi_{F^\wedge(t)})d\nu\right)dP.$$

Hence, P-almost everywhere $0 = \int_\Omega (1-\chi_{F^\wedge(t)})d\nu = 1-\nu(F^\wedge(t))$, i.e. $\exists\, F_1 \subset F_{[0,1]}$, $P(F_1)=1$ such that for each $t \in F_1$ we get $\nu(F^\wedge(t))=1$.

Indeed, if for some $k \in I$, $k=k_{j+1}$ we obtain $\nu_k(F^\wedge(t)\cap G_k^*) = q < 1$, then

$$\nu(F^\wedge(t)) = \sum_{k\in I}\nu_k(F^\wedge(t)\cap G_k^*) \leqslant \sum_{s=0}^{+\infty}\frac{1}{2^{k_s}}\Big(1-\Big(\frac12\Big)^{k_{s+1}-k_s}\Big) -$$
$$-(1-q)\frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big) = 1-(1-q)\frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big) < 1.$$

For each $t \in F_1$ and $k \in I$ we choose a point $r_0^{(k)}(t) \in G_k^*$ such that

$$W_{N_k}(r_0^{(k)}(t),t) \geqslant \frac34 M_k(t),\ \ M_k(t) \overset{\text{def}}{=} \sup\{W_{N_k}(r,t) : r \in G_k^*\}.$$

Then from $\nu_k(F^\wedge(t)\cap G_k^*) = 1$ for each $k \in I$ it follows that there exists a point $r^{(k)}(t) \in G_k^* \cap F^\wedge(t)$ such that

$$|W_{N_k}(r_0^{(k)}(t),t) - W_{N_k}(r^{(k)}(t),t)| < \frac14 M_k(t)$$

or

$$\frac34 M_k(t) \leqslant W_{N_k}(r_0^{(k)}(t),t) \leqslant W_{N_k}(r^{(k)}(t),t) + \frac14 M_k(t).$$

Since $(t, r^{(k)}(t)) \in F$, from inequality (13) we obtain

$$\frac12 M_k(t) \leqslant W_{N_k}(r^{(k)}(t),t) \leqslant A_1 S_{N_k}(r^{(k)}(t))\ln^{1/2} N_k. \tag{15}$$

Now for $r^{(k)} = r^{(k)}(t)$ we get

$$S_N^2(r^{(k)}) \leqslant \mu_f(r^{(k)})\mathfrak{M}_f(r^{(k)}) \leqslant \mu_f^2(r^{(k)})\Big(\prod_{i=1}^p \ln^{p-1} r_i^{(k)}\cdot \ln^p \mu_f(r^{(k)})\Big)^{1/2+\delta}.$$

Thus, for $t \in F_1$ and each $k \geqslant k_0(t)$, $k \in I$ we obtain

$$S_N(r^{(k)}) \leqslant \mu_f(r^{(k)})\Big(\prod_{i=1}^p \ln^{p-1} r_i^{(k)}\cdot \ln^p \mu_f(r^{(k)})\Big)^{1/4+\delta/2}. \tag{16}$$

It follows from (12) that $d_1(r^{(k)}) \geqslant d(r)$ for $r \in G_k^*$. Then for $t \in F_1$, $r \in F^\wedge(t)\cap G_k^*$, $k \in I,\ k \geqslant k_0(t)$ we get

$$M_f(r,t) \leqslant \sum_{\|n\|\geqslant 2d_1(r^{(k)})}|a_n|r^n + W_{N_k}(r,t) \leqslant \sum_{\|n\|\geqslant 2d(r)}|a_n|r^n + M_k(t).$$

Finally, from (11), (15), (16) for $t \in F_1$, $r \in F^\wedge(t) \cap G_k^*$, $k \in I$ and $k \geqslant k_0(t)$ we obtain

$$\begin{aligned} M_f(r^{(k)}, t) \leqslant & \mu_f(r^{(k)}) + 2A_p S_{N_k}(r^{(k)}) \ln^{1/2} N_k \\ \leqslant & \mu_f(r^{(k)}) + 2A_p \mu_f(r^{(k)}) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i^{(k)} \cdot \ln^p \mu_f(r^{(k)})\Big)^{1/4+\delta/2} \times \\ & \times \Big((p/2+1+\delta_2)\ln_2(e\mu_f(r^{(k)})) + (1+\delta_2)\sum_{i=1}^{p}(p\ln_2 r_i^{(k)} + 2\ln_3 r_i^{(k)})\Big)^{1/2}. \end{aligned}$$

Using inequality (12) we get for $t \in F_1$, $r \in F^\wedge(t) \cap G_k^*$, $k \in I$ and $k \geqslant k_0(t)$

$$M_f(r,t) \leqslant C\mu_f(r)\Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/4+3\delta_2/4}. \tag{17}$$

We choose $k_1 > k_0(t)$ such that for each $r \in G_{k_1}^+$ we have

$$C \leqslant \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{\delta_2/4}. \tag{18}$$

Using (17) and (18) we get that inequality (6) holds almost surely ($t \in F_1$, $P(F_1) = 1$) for each

$$r \in \Big(\bigcup_{k\in I}(G_k^* \cap F^\wedge(t)) \cap G_{k_1}^+\Big) \setminus E^* = ([1,+\infty)^p \cap G_{k_1}^+) \setminus (E^* \cup G^* \cup E_{p+1}) = [1,+\infty)^p \setminus E_{p+2},$$

where

$$E_{p+2} = E_{p+1} \cup G^* \cup E^*, \quad G^* = \bigcup_{k\in I}(G_k^* \setminus F^\wedge(t)).$$

It remains to observe that $\nu(G^*)$ defined in (13) satisfies $\nu(G^*) = \sum_{k\in I}(\nu_k(G_k^*) - \nu_k(F^\wedge(t))) = 0$. Then for each $k \in I$ we obtain

$$\nu_k(G_k^* \setminus F^\wedge(t)) = \frac{\mathrm{meas}_p(G_k^* \setminus F^\wedge(t))}{\mathrm{meas}_p(G_k^*)} = 0,$$
$$\mathrm{meas}_p(G_k^* \setminus F^\wedge(t)) = \int\limits_{G_k^* \setminus F^\wedge(t)} \cdots \int \frac{dr_1 \dots dr_p}{r_1 \dots r_p} = 0. \qquad \square$$

3. Some examples

In this section we prove that the exponent $p/4 + \delta$ in the inequality (7) cannot be replaced by a number smaller than $p/4$. It follows from such a statement.

Theorem 3. *For $f(z) = \exp\{\sum_{i=1}^{p} z_i\}$ almost surely in $K(f,H)$ for $r \in E$ we have*

$$M_f(r,t) \geqslant \frac{1}{(8p)^p}\mu_f(r)\ln^{p/4}\mu_f(r),$$

where E is a set of infinite asymptotically logarithmic measure and $H = \{e^{2\pi i\omega_n}\}$, $\{\omega_n\}$ is a sequence of independent random variables uniformly distributed on $[0,1]$.

In order to prove this theorem we need the following result.

Theorem 4 ([17])**.** *For the entire function $g(z) = e^z$ almost surely in $K(g,H)$ we have*

$$\varliminf_{r\to+\infty} \frac{M_g(r,t)}{\mu_g(r)\ln^{1/4}\mu_g(r)} \geqslant \sqrt{\frac{\pi}{8}}. \tag{19}$$

Proof of Theorem 3. For the entire function $f(z)=\exp\{\sum_{i=1}^p z_i\}$ we have $\ln\mathfrak{M}_f(r)=\sum_{i=1}^p r_i$ and for each $\beta>0$ we get

$$\int\limits_{(1,+\infty)^p}\cdots\int \frac{dr_1\dots dr_p}{r_1\dots r_p(r_1+\dots+r_p)^\beta}<+\infty.$$

Therefore, function $f(z)$ satisfies condition (4). From (19) we have for $r\in(r_0,+\infty)^p$

$$M_f(r,t)>\frac{1}{2^p}\mu_f(r)\prod_{i=1}^p \ln^{1/4}\mu_g(r_i).$$

Denote $\psi(r)=\ln\mu_g(r)$. Note that

$$\begin{aligned}A_t=&\{r\colon r_1=t; r_i\in(t_1,t_2)=(\psi^{-1}(\psi(r_1)/2),\psi^{-1}(2\psi(r_1)))\}\\ \subset&\Big\{r\colon \prod_{i=1}^p\psi(r_i)\geqslant\frac{1}{(4p)^p}\Big(\sum_{i=1}^p\psi(r_i)\Big)^p\Big\}.\end{aligned}$$

Indeed, if $r\in A_t$, for fixed r_1 we obtain

$$\begin{aligned}\prod_{i=1}^p\psi(r_i)=&\psi(r_1)\prod_{i=2}^p\psi(r_i)>\psi(r_1)\prod_{i=2}^p\frac{\psi(r_1)}{2}=\frac{\psi^p(r_1)}{2^{p-1}}\\ =&\frac{1}{2^{p-1}(2p-1)^p}(\psi(r_1)+2\psi(r_1)+\dots+2\psi(r_1))^p>\frac{1}{(4p)^p}\Big(\sum_{i=1}^p\psi(r_i)\Big)^p.\end{aligned}$$

For $r\in A=\bigcup_{t=r_0}^{+\infty}A_t$ we get

$$M_f(r,t)>\frac{1}{2^p}\mu_f(r)\prod_{i=1}^p\ln^{1/4}\mu_g(r_i)>\mu_f(r)\frac{1}{(8p)^p}\Big(\sum_{i=1}^p\ln\mu_g(r_i)\Big)^{p/4}>\frac{1}{(8p)^p}\mu_f(r)\ln^{p/4}\mu_f(r).$$

It remains to prove that the set A has infinite asymptotically logarithmic measure. It is known [11] that $t<\psi^{-1}(t)<3t/2,\ t\to+\infty$. Therefore,

$$\begin{aligned}\mathrm{meas}_p(A)&=\int\limits_{r_0}^{+\infty}\int\limits_{t_1}^{t_2}\dots\int\limits_{t_1}^{t_2}\frac{dr_1\dots dr_p}{r_1\dots r_p}=\int\limits_{r_0}^{+\infty}\Big(\int\limits_{t_1}^{t_2}\frac{dr_2}{r_2}\Big)^{p-1}\frac{dr_1}{r_1}\\ &=\int\limits_{r_0}^{+\infty}\Big(\ln\psi^{-1}(2\psi(r_1))-\ln\psi^{-1}\Big(\frac{\psi(r_1)}{2}\Big)\Big)^{p-1}\frac{dr_1}{r_1}\\ &>\int\limits_{r_0}^{+\infty}\Big(\ln(2\psi(r_1))-\ln\Big(\frac{3\psi(r_1)}{4}\Big)\Big)^{p-1}\frac{dr_1}{r_1}=\ln^{p-1}\frac{8}{3}\cdot\int\limits_{r_0}^{+\infty}\frac{dr_1}{r_1}=+\infty.\end{aligned}$$

□

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