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FICTITIOUS ASYMPTOTIC SOLUTIONS

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Abstract. We provide some examples of the problems with small parameter which have formal asymptotic solutions associated with no exact solutions.
Keywords: small parameter, asymptotics.

1. Introduction

In this note we provide examples of the problems with a small parameter, for which an approximate (asymptotic) solution gives approximation to none of exact solutions. By asymptotic solution we mean a function satisfying all the equations in the problem with a high degree of accuracy. As such functions, in the asymptotic constructions one usually takes the partial sums of the series being the formal asymptotic solution w.r.t. the powers of the small parameter. Sometimes the series is called the asymptotic solution. The coefficients of the series appears from the series of simpler (approximate) equations being more treatable than the original problems. Such simplification, whose realization looks rather like an art [1], is the main advantage of the asymptotic approach. However, it turns out to be an awkward situation if one does not control the error term of the asymptotics. An example of such situation for a particular problem was analyzed in detail in work [2]. In wider aspects this issue was discussed in the talk by D.V. Anosov in the conference in May 2013, [3], as well as in his work [4].

Meanwhile, many specialists dealing with asymptotics are quite sceptical on the necessity of controlling the error term. A widely held feeling is that once one has succeeded to construct an approximate solution giving small errors in the equations, there exists an exact solution close to this approximate solution. Such a faith is based on the experience in studying the problem for the which one can obtain the estimate for the deviation between an exact and approximate solution. For those educated in the traditions of rigorous mathematics the obtaining such estimates, i.e., the justification of the asymptotics, is a necessary milestone in the study. However, in many situations, especially related with applications, the original mathematical models looks so complicated that it is troublesome not only obtaining the estimates, but even proving the existence of an exact solution. This the reason why quite often one does not remember on the estimates for the error terms and employ the notion of "asymptotic solution". The followers of the rigorous mathematics consider of course such approach as an attempt to "bury one's head in the sand" veiling the problem of justification by the terminology. However, one sometimes neglects the justification of an asymptotics even in simpler precisely formulated mathematical problems treating this work as unattractive, often difficult and not deserving the recognition of the colleagues. As the justification of such position one sometimes adduce the arguments on physical interpretation of the simplified model generating the leading terms of the asymptotics. It is clear that it moves the discussion of this issue out the frameworks of the mathematics. In this direction we mention an interesting book [5], in which issues of "asymptotology" are discussed from various points of view but the justification issues remains almost untouched.

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For various problems posed for differential equations both partial and general results on justification of the asymptotic are known. Some of them, especially for nonlinear equations, look very sophisticated [6]–[24]. Such results inspire the hope on the justification in similar problems.

On the other hand, there are known examples of asymptotic solutions associated with none of exact solutions. In the reference literature such constructions are provided for ordinary differential equations in the complex plane, see, for instance, [25]. At that, there can be an impression that such examples are apart from the problems treated in the perturbation theory. However, the example adduced in [2] for the needle crystal model shows that the issue on justification of asymptotics can be topical in the most unexpected situations and its solution can be quite a nontrivial matter.

Similar examples were found recently in the spectral theory. As it is known, the asymptotics in small parameter are widely used in studying spectra of linear operators. For instance, in quantum mechanics, the method of quasiclassical approximation is popular [12]. This approach is an effective tool for approximate solving problems with a self-adjoint operator. However, in the case of a non-self-adjoint operator the quasiclassical approach sometimes fails. To analyze such problems, from 90s of the previous century the theory of pseudospectrum is actively developed. This direction was initiated by examples of operators with a small parameter, for which an asymptotic solution does not approximate the exact solution [27, 26, 28, 29]. In particular, the numbers calculated in the asymptotics and pretending to approximate the spectrum happen to be far from the spectrum [30, 31]. In relation with this fact, the notion of "pseudospectrum" appeared. We note that these studies were initiated by an earlier work on perturbation of non-self-adjoint operators [32]. Problems with interpretation of asymptotic solution under the presence of a spectral parameters appeared also before [33, 34].

One more interesting problem related with the discussed subject is the justification of asymptotic passages to integrable equations. In the mathematical physics of recent decades, there is an actively developed direction on finding equations distinguished by the presence of some integrability properties. Impressive progress was made by employing algebraic methods. Symmetric approach allowed one to form complete lists of integrable difference equations [35, 36]. However, many results in this direction looking quite general are still rather remote from applications in contrast to earlier approaches focused on finding the ways for integrating equations appearing in applications. This is why it is not a surprise that the issue on relation of classified equations with the problems in physics and mechanics attract permanent attention. As a rule, such connection is made by means of asymptotics and requires the justification. There are some rigorous results in this direction, see the survey [37]. The classical example is the justification of passing from the system of surface waves equations to the Korteweg de Vries equation. At the same time, among a huge number of works where asymptotic passages are performed there occur doubtful statements. Some of such situations were analyzed in works [38, 39, 40, 41]. These issues concern a general wide subject and require an independent discussion; here we do not touch them.

In the present note we discuss several examples for ordinary differential equations on the real axis and for partial differential equations. The most part of these examples belongs to the class of well-known problem on a boundary layer. The matter is a power asymptotics, when the formal solutions are constructed as series in powers of a small parameter. The discussed problems split into two series:

1) Problems with exact solutions. Here we construct asymptotic solutions associated with none of the exact solutions. These examples seem to be new, and in the case of nonlinear equations they are not covered by the theory of pseudospectrum.

2) Problems with no exact solutions. Nevertheless, we succeed to construct an asymptotic solution. In the present text we reproduce known examples [2, 3].

Of course, the considered problems were discussed and specialists are familiar with them. However, there are just few publications on this subject not even being well-known, at least, outside rather narrow direction of theory of pseudospectrum. The aim of the present work is to warn the researches carried away by formal constructions against too careless relation to the justification of asymptotics.

The analyzed phenomena are similar to ones occurring in ill-conditioned or unstable problems when small errors in the initial data leads one either to great errors in the solution or to absence of solution. This similarity is especially well seen in the theory of pseudospectrum treating in fact spectrally unstable problems with exponentially increasing (w,r.t. a small parameter) resolvent [31]. However, the formulation of problems on asymptotics w.r.t. the small parameter has almost nothing to do with ill-conditioned problems. The presented asymptotic constructions are not related at all with approximative methods of solving unstable problems [42, 43, 44]. Nevertheless, we should stress that exactly the presence of an exponential instability generates the appearance of fictitious asymptotic expansions. Quite distinctively it is seen in the following rather elementary example.

2. Nefedov's example

The example adduced below was pointed out to the author by professor N.N. Nefedov in a slightly different form. In fact, the matter is the behavior of a function of two variables $\exp(x/\varepsilon)$ in the vicinity of the singular point $x = 0$, $\varepsilon = 0$. The discussed phenomena are related with different asymptotics as $\varepsilon \to 0$ in various domains of x. This example is more suitable for demonstrating an unstable equilibrium. Here we adduce it in order to show for a simplest situation that the appearance of fictitious asymptotic expansions is due to an instability.

We consider the differential equations with a small parameter

$$\varepsilon\frac{du}{dt} = u, \quad t > 0, \quad 0 < \varepsilon \ll 1 \tag{2.1}$$

subject to the homogeneous initial condition

$$u|_{t=0} = 0.$$

The identical zero $u(t;\varepsilon) \equiv 0$ is the only exact solution to the problem.

At the same time, each non-zero function

$$U(t;\varepsilon) = \exp\big((t-1)/\varepsilon\big)$$

solves exactly the differential equations and asymptotically satisfies (up to arbitrarily small error) the initial condition since

$$\exp(-1/\varepsilon) = \mathcal{O}(\varepsilon^n), \ \varepsilon \to 0, \ \forall\, n.$$

This asymptotic solution to the problem happens to be essentially different to zero as $t \geqslant 1$:

$$U(t;\varepsilon) = \exp\big((t-1)/\varepsilon\big) \geqslant 1, \quad \forall\, t \geqslant 1, \ \varepsilon > 0.$$

Therefore, for *a long time* $t \geqslant 1$ the asymptotic solution has nothing to do with the exact solution. Of course, in the vicinity of the initial moment (locally) the asymptotic solution $U(t;\varepsilon)$ corresponds to the exact solution $u(t;\varepsilon)$ and the trivial asymptotic identity

$$\forall\, T < 1, \, n > 0: \ u(t;\varepsilon) = U(t;\varepsilon) + \mathcal{O}(\varepsilon^n), \ \varepsilon \to 0, \forall\, t \in [0,T],$$

holds true.

The uselessness of the asymptotic solution for long time is not a news for Cauchy problems. For instance, in the oscillation theory, while using the simplest version of the asymptotic construction (as the straightforward series of perturbation theory), the restrictions for the time occurs due to so-called secular terms [7]. The non-uniformity in time of such (simplest) asymptotics serves as a basis for introducing more complicated multi-scale expansions. However, the above example is not this case. Here in the asymptotic expansion there are no secular terms. The restriction for the time, $t < 1$, appears in comparing with the exact solution and does not appear in the formal constructions. Similar situation occurs in various problems. Theorem on justification of the asymptotics w.r.t. the small parameter provided in [11] guarantees an appropriate estimate for the error term on a finite, not very large time interval. In the appropriate slow scale this interval is independent of the small parameter [1]. Sometimes for a Cauchy problem one succeeds to construct on the subsequent time scale choosing an appropriate independent variable. However an essential enlarging of the interval of asymptotics applicability is not always possible, cf. [13, 24]. The exponential instability seems to be one of the reasons of breaking the connection between the asymptotic and exact solutions in the general case as well.

If one recall the theorem on continuous dependence of solution on the parameter, one can understand that near the initial moment (locally, in the appropriate time scale), an asymptotic solution to a Cauchy problem is always associated with the exact solution. Long time in such problem is an additional restriction which can make the asymptotics fictitious. Another situation occurs for boundary value problems. Here a part of the domain being far from the boundary is involved in the main formulation of the problem, not in the additional condition. In such problems this far domain can not be excluded by the attempt of localizing the solution at the boundary.

3. Problem on pseudospectrum

In the case of variable coefficients the instability can be local. For instance, the function $U(t;\varepsilon) = \exp(-(t-1)^2/2\varepsilon)$ is exponentially small as $\varepsilon \to 0$ at the initial moment $U(t;\varepsilon)|_{t=0} = \exp(-1/\varepsilon)$ and solves the equation

$$\varepsilon \frac{du}{dt} + (t-1)\, u = 0, \quad t > 0, \quad 0 < \varepsilon \ll 1.$$

This is why for the Cauchy problem with the initial condition $u|_{t=0} = 0$ this function is an asymptotic solution for each $t \geqslant 0$. However, in the vicinity of $t = 1$ it does not provide the asymptotic of exact (zero) solution $u(t;\varepsilon) \equiv 0$ of the problem since $U(t;\varepsilon)|_{t=1} = 1$. At the same time, for long time $t \gg 1 + \varepsilon$ this function is again close to the zero solution. This property is demonstrated by following rather unusual example borrowed from work [28].

We consider first order differential equation on a segment

$$\varepsilon \frac{du}{dx} + x\, u = \lambda\, u, \quad -1 < x < 1, \quad 0 < \varepsilon \ll 1 \tag{3.1}$$

with two boundary conditions

$$u(-1) = 0, \ u(1) = 0. \tag{3.2}$$

It is clear for each $\lambda = \text{const}$ the identical zero $u(x) \equiv 0$ is the only solution. At the same time, the function $U(x;\varepsilon,\lambda) = \exp(-(x-\lambda)^2/2\varepsilon)$ solves the equation and as $\lambda \in \mathbb{R}, \ \lambda \neq \pm 1$,

[1] We note that the form of equation (2.1) corresponds to the writing in slow time scale; here the fast variable is t/ε.

it satisfies asymptotically the boundary conditions

$$\exp(-(x-\lambda)^2/2\varepsilon)|_{x=\pm 1} = \exp(-(\lambda \mp 1)^2/2\varepsilon) = \mathcal{O}(\varepsilon^n), \ \varepsilon \to 0, \ \forall n.$$

If $-1 < \lambda < 1$, the considered function is an asymptotic solution to problem (3.1),(3.2) but it is not small in the interior points of the segment since it has a burst of order one

$$U(x;\varepsilon,\lambda)|_{x=\lambda} = 1.$$

Thus, asymptotic solution $U(x;\varepsilon,\lambda)$ is similar to the asymptotics of an eigenfunction. But there is no true eigenfunction at all. The numbers $\lambda \in \mathbb{C}$ for which there exists an asymptotic solution of this kind are called pseudoeigenvalues and they belong to the set called "pseudospectrum". In the considered problem the pseudospectrum consists of the strip $-1 < \operatorname{Re}\lambda < 1$.

The given example looks artificial and makes an impression of cheating because of two boundary conditions for a first order equation. Nevertheless, it demonstrate very well the idea of the pseudospectrum.

There is a series of other more natural formulations, in which there appears a pseudospectrum[1]. For instance, for the advection-diffusion operator on the axis

$$A_\varepsilon u \equiv \varepsilon^2 \frac{d^2u}{dx^2} + \varepsilon \frac{du}{dx} + \big(\frac{1}{4} - x^2\big)u, \quad x \in \mathbb{R}, \quad 0 < \varepsilon \ll 1$$

the spectrum consists of the eigenvalues $\lambda_n = -(2n+1)\varepsilon$. The pseudospectrum occupies a much wider domain bounded by the parabola $\operatorname{Re}\lambda < 1/4 - (\operatorname{Im}\lambda)^2$, [28]. The appearance of such wide domain of the pseudospectrum is because of the exponential growth in the small parameter ε of the norm for the resolvent at the points far from the spectrum. The latter property is due to the non-self-adjointness of the operator.

This and other examples [28] show that fictitious asymptotics are not uncommon in spectral problems.

4. Problems on stochastic perturbations

The example we consider in what follows motivated the present note.

On a finite interval we consider a linear homogeneous equation

$$\varepsilon \frac{d^2u}{dx^2} - x\frac{du}{dx} = 0, \quad -1 < x < 1, \quad 0 < \varepsilon \ll 1 \tag{4.1}$$

with boundary conditions

$$u(-1) = 0, \ u(1) = 0. \tag{4.2}$$

The pair of functions

$$u_1(x) \equiv 1, \quad u_2(x) = \int_{-1}^{x} \exp\left(\eta^2/2\varepsilon\right) d\eta$$

forms a fundamental system of solutions to equation (4.1). In view of the general solution to the equation

$$u(x) = c_1 + c_2 \int_{-1}^{x} \exp\left(\eta^2/2\varepsilon\right) d\eta, \quad \forall\, c_1, c_2 = \mathrm{const}$$

one can see easily that the homogeneous boundary value problem (4.1),(4.2) has no solutions except the zero one $u(x) \equiv 0$.

[1] Its precise definition is given in terms of resolvent, see [27, 28].

On the other hand, by the boundary layer method [6] we can construct a non-zero asymptotic solution:

$$U(x;\varepsilon) = 1 - \big[\exp(-\xi_-) + \exp(-\xi_+)\big] + \sum_{k=1}^{\infty} \varepsilon^k \big[V_k^-(\xi_-) + V_k^+(\xi_+)\big], \quad \varepsilon \to 0. \tag{4.3}$$

Functions $V_k^{\pm}(\xi_\pm)$ depend on the rescaled variables $\xi_\pm = (1 \mp x)/\varepsilon$. They are determined uniquely by the recurrent system of equations

$$\begin{gathered} V_0'' + V_0' = 0,\ \xi > 0;\quad V_0(0) = -1; \\ V_k'' + V_k' = \xi V_{k-1}',\ \xi > 0;\quad V_k(0) = 0,\ k \geqslant 1 \end{gathered} \tag{4.4}$$

with the conditions $V_k(\xi) \to 0$ as $\xi \to \infty,\ (k \geqslant 0)$. Each of these functions is the product of a polynomial by a decaying exponent

$$V_k^{\pm}(\xi) = P_{2k}(\xi)\,\exp(-\xi),\ \xi \geqslant 0.$$

A partial sum of length n of the above series solves the equation and boundary conditions (4.1) up to an error $\mathcal{O}(\varepsilon^n)$ for each $\forall\, n$. In the internal points of the segment $\forall\, x \in (-1,1)$ this series coincides asymptotically with one, $U(x;\varepsilon) = 1 + \mathcal{O}(\varepsilon^N)$, $\varepsilon \to 0$, $\forall\, N$ and thus *it has no relation with the unique exact (zero) solution.* Moreover, since problem (4.1) is homogeneous, the asymptotic solution is determined non-uniquely, up to a multiplicative constant C. It can be called a pseudoeigenfunction, if we follow the terminology of [26, 28].

A similar problem with nonhomogeneous boundary condition

$$u(-1) = \alpha, \quad u(1) = \beta$$

was analyized in [45]. A formal asymptotic solution has the same structure (4.3) with an indefinite constant C as the leading term. This is why there exists many fictitious asymptotic solutions parameterized by this constant. Meanwhile, there exists the only exact solution. Its asymptotics appears under an appropriate choice of constant C, which can be found by the formula for the exact solution: $C = (\alpha+\beta)/2 + \mathcal{O}(\varepsilon)$. It should be noted that in [45] there was given another way of finding this constant by an integral identity but in this case this result remains conditional since it is formulated as follows: *if the asymptotics of solution reads as* (4.3), *then* $C = (\alpha+\beta)/2 + \mathcal{O}(\varepsilon)$.

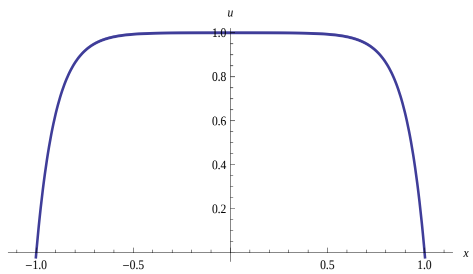

FIGURE 1. Asymptotic solution to problem (4.1),(4.2) not associated with the exact (zero) solution.

A similar construction is made in a slightly more difficult problem

$$\begin{gathered} \varepsilon\Big[\frac{d^2u}{dx^2} + b(x)\frac{du}{dx}\Big] + a(x)\frac{du}{dx} = 0, \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1, \\ u(0) = 0,\ u(1) = 0. \end{gathered} \tag{4.5}$$

Under the smoothness condition for the coefficients $a(x), b(x) \in C^\infty[0,1]$ and the sign-definiteness at the end-points $a(0) > 0,\ a(1) < 0$ we construct a non-zero asymptotic solution

$$U(x;\varepsilon) = 1 + \sum_{k=0}^{\infty} \varepsilon^k \big[V_k^-(\xi_-) + V_k^+(\xi_+)\big], \quad \varepsilon \to 0. \tag{4.6}$$

Here $\xi_- = x/\varepsilon$, $\xi_+ = (1-x)/\varepsilon$. At that, the exact zero solution $u(x) \equiv 0$ is unique as it follows from the explicit formula for the general solution

$$u(x) = c_1 + c_2 \int_0^x \exp\Big(-\varepsilon^{-1} \int_0^\eta [a(\zeta) + \varepsilon b(\zeta)]\, d\zeta \Big)\, d\eta, \quad \forall\, c_1, c_2 = \text{const}.$$

A similar asymptotic construction is possible in a multi-dimensional problem by the way indicated, for instance, in [6]:

$$\begin{gathered} \varepsilon \mathcal{L}_2 u + \mathbf{a}(\mathbf{x}) \partial_{\mathbf{x}} u = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \quad 0 < \varepsilon \ll 1, \\ u(\mathbf{x}) = 0, \ \mathbf{x} \in \partial\Omega. \end{gathered} \tag{4.7}$$

Here $\mathcal{L}_2$ is an elliptic second order differential operator in a bounded domain $\Omega \subset \mathbb{R}^n$ involving the derivatives only. Apart from the smoothness of the coefficients and the boundary, the main assumption is made for vector-function $\mathbf{a}(\mathbf{x})$. We postulate that the boundary layer exists along whole the boundary. Formal condition says that on boundary $\partial\Omega$, vector $\mathbf{a}(\mathbf{x})$ is direct inside the domain so that its scalar product with the internal normal $\mathbf{n}(\mathbf{x})$ is positive: $(\mathbf{a}, \mathbf{n}) > 0$, $\forall\, \mathbf{x} \in \partial\Omega$. This condition ensures the exponential vanishing at infinity w.r.t. the fast variable (in the direction of the internal normal) for the boundary layer functions. A non-zero asymptotic solution has the structure like in (4.6) with x replaced by $\mathbf{x}$. The fast variable is introduced by rescaling along the internal normal. The boundary layer functions involves an additional dependence on a "slow" variable $\mathbf{y} \in \mathbb{R}^{n-1}$ parameterizing the points on the boundary of the domain. At that, the exact zero solution $u(\mathbf{x}) \equiv 0$ remains unique once the problem is not at the spectrum. Thus, we again can construct an *asymptotic solution not related with the exact solution.*

The described phenomenon, appearance of a fictitious asymptotic solution, can be explained in various ways. On the level of asymptotic constructions one usually refers to the structure of the limiting (unperturbed, as $\varepsilon = 0$) first order operator, all the characteristics of which goes inside the domain that guarantees the existence of the boundary layer along whole the domain. A deformation of domain Ω can change the situation drastically. For instance, for equation (4.1) on the segment $1/2 < x < 1$ the above construction of fictitious asymptotic solution becomes impossible. In this case the boundary layer is absent at the left end-point, the external expansion must satisfy the boundary condition and it happens to be zero. Such situation corresponds to a regular degeneration [6].

Another explanation is that problem (4.1) is unstable in certain sense w.r.t. the perturbation of the equation. For example, the solution to the non-homogeneous equation with zero condition at the end-points

$$\varepsilon \frac{d^2u}{dx^2} - x\frac{du}{dx} = 1, \quad -1 < x < 1; \quad u(-1) = 0,\ u(1) = 0$$

grows exponentially as $\varepsilon \to 0$. In the present example such property can be seen by the explicit formula for the solution which thanks to the symmetry $u(x) = u(-x)$ should be written only for $\geqslant 0$:

$$u(x;\varepsilon) = \varepsilon^{-1} \int_x^1 \int_x^y \exp\big((z^2 - y^2)/2\varepsilon\big)\, dz\, dy - C(\varepsilon) \int_x^1 \exp\big(z^2/2\varepsilon\big)\, dz, \quad x \geqslant 0.$$

Here the exponential growth is contained in the second term only, which the solution to the homogeneous equation with a constant:

$$C(\varepsilon)=\varepsilon^{-1}\int_0^1 \exp\big(-y^2/2\varepsilon\big)\,dy=\mathcal{O}(1/\sqrt{\varepsilon}),\ \varepsilon\to 0.$$

For a more general multi-dimensional problem, such fact is well-known in the theory of stochastic perturbations [46], see also [45]. Such problems were discussed many times [47, 48, 49], the precise results on determining the leading term in the asymptotics are presented in [50].

Under the presence of an exponential growth in the exact solution, the construction of asymptotics (at the formal level) allows one to extract and justify exponential asymptotics, but not the power ones[1]. It means that the power asymptotic remains uncontrolled. Such uncontrollability can serve as the explanation for the appearance of a fictitious asymptotic solution in the homogeneous problem.

We observe that an exponential growth (in the small parameter) of the resolvent for a non-self-adjoint operator is related with the appearance of a pseudospectrum [29] and with what we call fictitious asymptotics. Many linear problem of such types are known [28]. In the considered example the resolvent has not been treated and the presence of the pseudospectrum (except the point $\lambda=0$) has not been analyzed. Here we just point out the exponential growth of the inverse operator for a particular problem.

However, all these explanations are just an attempt to select a certain class of problems where one observes the phenomenon of appearance of fictitious asymptotic solution. The following examples select a class of nonlinear problems with the same phenomenon.

5. Bursts

We consider the boundary value problem for the simplest nonlinear equation with the small parameter at the derivative

$$\begin{gathered}\varepsilon^2\frac{d^2u}{dx^2}=4u-6u^2,\quad 0<x<1,\quad 0<\varepsilon^2\ll 1,\\ u(0)=\alpha,\ u(1)=\beta.\end{gathered}\tag{5.1}$$

The construction and justification of the asymptotics for solution to such problems are well-known [9]. We shall show that in these problems it is possible to construct asymptotic solutions associated with none of exact solutions.

Since the equation is very simple, it can be integrated by quadratures. The exact solution is written in terms of elliptic functions whose parameters are related with boundary values α,β. However, in more complicated situations, say, for the equations with variable coefficients, the explicit representations are absent. At the same time, the asymptotics for solution can be also written in terms of primitive functions. In our simple example we make use of the exact partial solutions to the equation

$$u_s(x)=-\frac{1}{\sinh^2(x/\varepsilon)},\quad u_c(x)=\frac{1}{\cosh^2(x/\varepsilon)}$$

associated with two type of separatrix trajectories on the phase plane.

We note that in terms of the fast variable $\xi=x/\varepsilon$ the equation is independent of the parameter:

$$u''_{\xi\xi}=4u-6u^2.$$

[1]Such situation occurs while constructing power asymptotics when exponentially small terms remain uncontrolled; a typical example is provided in [2].

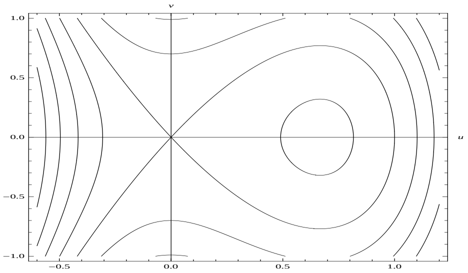

FIGURE 2. Phase portrait for an equation with one point of unstable equilibrium

On the phase plane (Fig. 2), the problem is to find a trajectory connecting the points on the lines $u = \alpha$ and $u = \beta$ for a "large time" $0 \leqslant \xi \leqslant \varepsilon^{-1}$. As it is known, the time of moving along the trajectory increases unboundedly while the trajectory approaches an unstable equilibrium. This is why for $0 < \varepsilon \ll 1$ the desired trajectory can pass near the equilibrium. For boundary values $\alpha, \beta < 0$ on the phase plane one can find two such trajectories. One of them remains to left of the equilibrium: $x < 0$, while the other rounds the separatrix loop. These trajectories are associated with two exact solutions of the boundary value problem.

However, in the formal constructions one can manage without this interpretation. The asymptotics for one of solutions to problem (5.1) is the sum of boundary layers functions

$$u_0(x) = -\frac{1}{\sinh^2(x/\varepsilon + \xi_0)} - \frac{1}{\sinh^2((1-x)/\varepsilon + \xi_1)} + \mathcal{O}(\varepsilon^n),\ \varepsilon \to 0,\ \forall\, n.$$

The phase shifts ξ_0, ξ_1 are determined in terms of initial conditions by the relations

$$-\frac{1}{\sinh^2(\xi_0)} = \alpha, \quad -\frac{1}{\sinh^2(\xi_1)} = \beta$$

under the additional condition $\alpha, \beta < 0$. Such simple structure of the asymptotics (without series) is thanks to the simplicity of the original equation. Under the presence of variable coefficients, for instance, $\varepsilon^2 u'' = a(x)u[b(x) - u]$ the structure of the asymptotics becomes more complicated and the appearance of series in powers of ε can not be avoided, see [9].

We note that the sum of exact partial solutions

$$U_0(x) = -\frac{1}{\sinh^2(x/\varepsilon + \xi_0)} - \frac{1}{\sinh^2((1-x)/\varepsilon + \xi_1)}$$

satisfies the equation and boundary condition asymptotically as $\varepsilon \to 0$ due to the exponential decay of the function $\sinh^{-2}(\xi) = \mathcal{O}(\exp(-2\xi))$, $\xi \to \infty$.

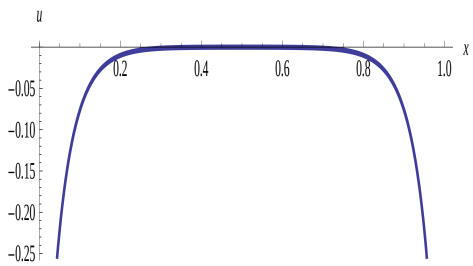

FIGURE 3. Asymptotic solution with a boundary layer associated with the exact solution to problem (5.1)

If we employ the other partial solution to the equation, we can form the sum

$$U_1(x; x_1) = -\frac{1}{\sinh^2(x/\varepsilon + \xi_0)} - \frac{1}{\sinh^2((1-x)/\varepsilon) + \xi_1} + \frac{1}{\cosh^2((x - x_1)/\varepsilon)}.$$

For each fixed value $x_1 \in (0,1)$ this function also satisfies equations (5.1) asymptotically up to an exponentially small term. This function provides the asymptotics for the exact solution

$$u_1(x) = -\frac{1}{\sinh^2(x/\varepsilon+\xi_0)} - \frac{1}{\sinh^2((1-x)/\varepsilon)+\xi_1} + \frac{1}{\cosh^2((x-x_1)/\varepsilon)} + \mathcal{O}(\varepsilon^n),\ \forall n,$$

for the only value $x_1 = x^*$; for instance, $x^* = 1/2$ if $\alpha = \beta < 0$. Sometimes the solution of such kind is called solution (or function) with "burst". It corresponds to a trajectory rounding the separatrix loop.

Thus, under the assumption $\alpha, \beta < 0$, boundary value problem (5.1) has just two exact solutions $u_0(x)$, $u_1(x)$, as one can see by the phase portrait. Meanwhile, it follows from the above construction that for various x_1 we can construct a *family of asymptotic solutions with burst* $U_1(x; x_1)$ *related with none of exact solutions to the boundary value problem.*

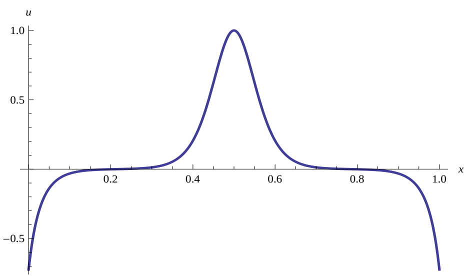

FIGURE 4. Only one of such asymptotic solutions with the boundary layer and burst is associated with the exact solution (5.1) as $\alpha, \beta < 0$

Moreover, there exists a multi-parametric family of asymptotic solutions

$$U_m(x) = -\frac{1}{\sinh^2(x/\varepsilon+\xi_0)} - \frac{1}{\sinh^2((1-x)/\varepsilon)+\xi_1} + \sum_{k=1}^{m} \frac{1}{\cosh^2((x-x_k)/\varepsilon)}$$

each being a function with m bursts; here x_k are different points outside the end-points $0 < x_1 < \ldots < x_m < 1$.

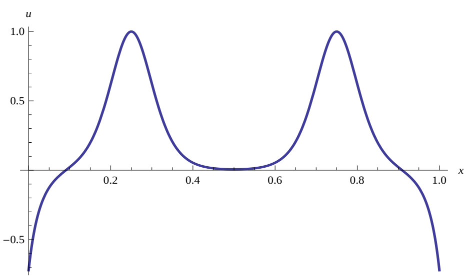

FIGURE 5. Asymptotic solution with a boundary layer and bursts associated with none of exact solutions to problem (5.1) as $\alpha, \beta < 0$

These functions asymptotically satisfy both the equation and boundary conditions (5.1) as $\varepsilon \to 0$. They have m maxima at different points x_k. However, these functions *can not be the asymptotics for a solution to differential equation* (5.1) *with boundary conditions* $\alpha, \beta < 0$. This fact can be easily seen by the phase portrait which implies that this solution can have the only maximum.

We observe that the exact solutions with multiple bursts are possible for $0 < \alpha, \beta < 1$. Their trajectories pass along closed curves of the phase portrait [51].

While interpreting results for the considered problem, there is an temptation for relate the presence of fictitious asymptotic expansions with the non-uniqueness of exact solutions.

In particular, for $\alpha, \beta = 0$ there exist two exact solutions; one corresponds a stable point $u_0(x) \equiv 0$, the other is a burst and corresponds a trajectory near the separatrix loop $u_1(x) = \cosh^{-2}((x-1/2)/\varepsilon) + \mathcal{O}(\varepsilon^n)$, $\forall n$. Nevertheless, the next example shows that fictitious asymptotics remains also in the case of the unique exact solution. The critical condition here the presence of the cell formed by the separatrices in a bounded part of the phase plane.

6. Contrast structures

Consider the boundary value problem for the equation with two unstable equilibria

$$\varepsilon^2 \frac{d^2u}{dx^2} + 2u[1-u^2] = 0, \quad 0 < x < 1, \quad 0 < \varepsilon^2 \ll 1,$$
$$u(0) = \alpha, \ u(1) = \beta. \tag{6.1}$$

Two exact solutions to the equation

$$u_{cot}(x) = \frac{\cosh(x/\varepsilon)}{\sinh(x/\varepsilon)} = \coth(x/\varepsilon), \quad u_{tan}(x) = \frac{\sinh(x/\varepsilon)}{\cosh(x/\varepsilon)} = \tanh(x/\varepsilon)$$

form a basis for asymptotic constructions. On the phase portrait these solutions correspond to the separatrices which either pass to infinity or stay in a bounded part of the plane, respectively.

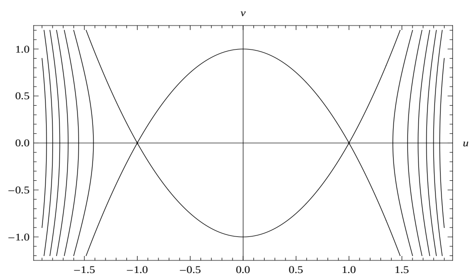

Figure 6. Phase portrait of equation with unstable equilibria

In view of the phase portrait, the boundary value problem has the unique solution if at least one of the boundary values α, β is outside the interval $|u| < 1$. For small ε, the solution is associated with a trajectory passing near unstable equilibria. In particular, as $\alpha, \beta < -1$, the trajectory stay to left of the equilibrium $u = -1$. The asymptotics for this solution is described by the boundary layer functions:

$$u_0(x) = 1 - \coth(x/\varepsilon + \xi_0) - \coth((1-x)/\varepsilon + \xi_1) + O(\varepsilon^n), \ \varepsilon \to 0, \ \forall\, n.$$

Constants ξ_0, ξ_1 are determined by the boundary values:

$$\coth(\xi_0) = -\alpha, \quad \coth(\xi_1) = -\beta.$$

The leading term of this asymptotics

$$U_0(x) = 1 - \coth(x/\varepsilon + \xi_0) - \coth((1-x)/\varepsilon + \xi_1),$$

formed by partial solution of the nonlinear equation is not an exact solution to the problem. However, due to the fast stabilization property $\coth(\xi) = 1 + \mathcal{O}(\exp(-\xi))$, $\xi \to \infty$, this sum satisfies the equation and boundary conditions up to an exponentially small error as $\varepsilon \to 0$.

Apart from such asymptotic solution there exists another function describing the motion along the separatrix via the other unstable equilibrium. Here we use the second exact solution to the equation:

$$U_1(x) = 1 - \coth(x/\varepsilon + \xi_0) - \coth((1-x)/\varepsilon + \xi_1) + \tanh((x-x_0)/\varepsilon) - \tanh((x-y_0)/\varepsilon)$$

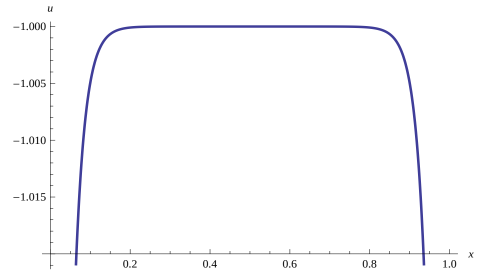

FIGURE 7. Asymptotic solution with boundary layer associated with the only exact solution

with fixed values $0 < x_0 < y_0 < 1$. Due to the stabilization properties, this function asymptotically satisfies the equation and boundary conditions as $\varepsilon \to 0$. But in view of the phase portrait we see that the differential equation has no solutions with such asymptotics.

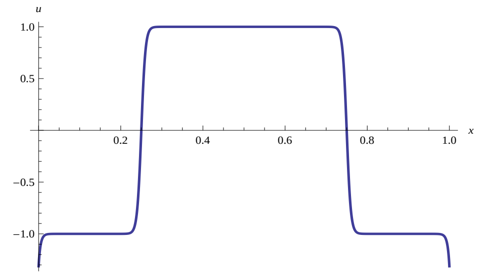

FIGURE 8. Asymptotic solution with the contrast structure associated with none of exact solutions

It is clear that on the base of series of pairs $x_k < y_k$ one can construct fictitious asymptotic solutions with a finite number of such kind bursts. Functions with bursts localized on a finite interval are usually called contrast structures.

All the above described asymptotic solutions to boundary value problems for nonlinear equations are described by the functions which on the phase plane correspond to equilibria and fast motion along the separatrices. The exponential proximity of exact solutions to the separatrix trajectories allows one to construct fictitious asymptotic solutions passing from one trajectory to another with an exponentially small error.

It is seen very clearly when the boundary values correspond to equilibria $\alpha = -1$, $\beta = 1$. In this case there exists the unique exact solution to problem (6.1). Its asymptotics is described by the trajectory connecting equilibria along the upper separatrix:

$$u_1(x) = \tanh((x - 1/2)/\varepsilon) + \mathcal{O}(\varepsilon^n), \quad \varepsilon \to 0, \quad \forall\, n.$$

The graph of this function as a contrast structure with one internal intermediate layer.

At the same time there exists a fictitious asymptotic solution appearing by adding the motion along the lower separatrix and also one more motion along the upper separatrix:

$$U_1(x) = \tanh(x/\varepsilon) - \tanh((x - x_1)/\varepsilon) + \tanh((x - x_2)/\varepsilon), \quad 0 < x_1 < x_2 < 1.$$

The graph of this function contains three internal intermediate layers.

In view of the phase portrait we see that there exist *no exact solution with such asymptotics.*

The class of considered examples can be extended in various directions, say, for equations with a general nonlinearity $\varepsilon^2 u'' = F(u)$. The essential condition is the presence of equilibria which are the solutions to the limiting equation $F(u) = 0$ and cells formed by the separatrices. Exactly these cells give a possibility to construct fictitious contrast structures.

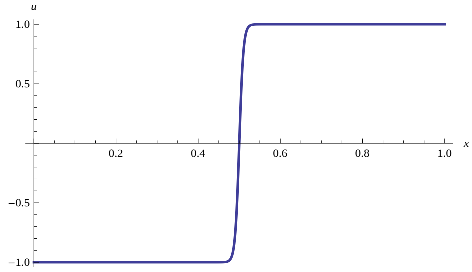

FIGURE 9. Asymptotic solution with a contrast structure associated with the unique exact solution

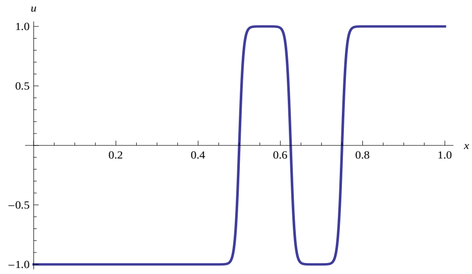

FIGURE 10. Asymptotic solution with a contrast structure associated with none of exact solutions

We note that contrast structures can be constructed also in the case when the roots to the limiting equation $F(u,x)=0$ depend on x, i.e., they are not equilibria [10]. For such problems we can not construct fictitious asymptotic solutions. It is remarkable that the dependence on x happens to be essential for justification of asymptotics and the results of [10] are not applicable for the justification in the above example with equilibria.

7. Multi-dimensional contrast structures

The above constructions of fictitious asymptotics can be easily extended for equations with the first derivative in the perturbation

$$\varepsilon^2[u''+a(u',x)]=F(u).$$

In this way there appears a chance to construct fictitious contrast structures for partial differential equations. One of the simplest examples is for the equation with two independent variables

$$\varepsilon^2\Delta u+2u[1-u^2]=0,\quad x,y\in\mathbb{R}^2.$$

If we consider the problem in a ring with appropriate boundary conditions, the situation happens to be close to the example for ordinary differential equation (6.1). In terms of polar coordinates r, φ the equation and boundary condition read as

$$\begin{gathered}\varepsilon^2\Big(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial\varphi^2}\Big)+2u[1-u^2]=0,\quad 1<r<2,\quad 0<\varepsilon^2\ll 1,\\ u(r,\varphi)|_{r=1}=\alpha(\varphi),\ u(r,\varphi)|_{r=2}=\beta(\varphi),\ 0\leqslant\varphi<2\pi.\end{gathered}\tag{7.1}$$

For the boundary functions we require that at least one of them does not take the values in the interval $(-1,1)$, for instance, $|\alpha(\varphi)|,|\beta(\varphi)|>1$, $\forall\,\varphi$. In the model independent of φ it allows us to avoid the motion along the trajectories located in vicinity of to closed ones as well as to avoid the problems with the uniqueness of exact solution.

In the case of the ring, for small ε the influence of boundary conditions, and thus of angular variable φ is essential only in a narrow strip along the boundary. The asymptotics outside the boundary layers is independent of φ. In fact, the problem is reduced to constructing an asymptotic solution to the ordinary differential equation

$$\varepsilon^2\Big(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\Big)+2u[1-u^2]=0,\quad 1<r<2,\quad 0<\varepsilon^2\ll 1. \tag{7.2}$$

The presence of the first derivative multiplied by a small factor plays no key role although the construction becomes more complicated.

The contrast structure is determined by the asymptotic solution in a narrow intermediate layer in the vicinity of the point $r_j\in(1,2)$. The solution should fast stabilize w.r.t. the rescaled variable $\rho=(r-r_j)/\varepsilon$. In terms of this variable, the equation $u=v(\rho;\varepsilon)$ casts into the form

$$\frac{d^2v}{d\rho^2}+\varepsilon\frac{1}{r_j+\varepsilon\rho}\frac{dv}{d\rho}+2v[1-v^2]=0,\quad \rho\in\mathbb{R},\quad 0<\varepsilon^2\ll 1.$$

The asymptotic solution is constructed as the series

$$V(\rho;r_j)=v_0(\rho)+\sum_{n=1}^{\infty}\varepsilon^n v_n(\rho).$$

The leading term $v_0(\rho)=\tanh\rho$ solves the nonlinear equation as $\varepsilon=0$. The next terms $v_n(\rho)$ are determined by the linearized equation

$$\frac{d^2v_n}{d\rho^2}+2v_n[1-3v_0^2]=f_n(\rho),\quad \rho\in\mathbb{R}$$

with the additional condition $v_n(\rho)\to 0$, $\rho\to\pm\infty$. The right hand sides are calculated in terms of previous $v_k(\rho)$, $k<n$.

At the first step we have $f_1=-v_o'/r_j$. Since the fast decaying function

$$v_o'(\rho)=\frac{1}{\cosh^2(\rho)}$$

solves the homogeneous linearized equation, the solution to the next-to-leading term is written as

$$v_1(\rho)=-\rho\, v_o'/2r_j+c\,v_o'$$

and involves an arbitrary constant c. This freedom is eliminated at the next step, when in the solution for $v_2(\rho)$ we exclude the growing at infinity terms. Such procedure is repeated at each step. As a result, all the coefficients are determined uniquely and they are fast decaying functions $v_n(\rho)=\mathcal{O}(\rho^n\exp(\mp 2\rho)$, $\rho\to\pm\infty$, $n\geqslant 1$.

In this way we construct series $V(\rho;r_j)$ being an analogue of the separatrix solution[1] describing the motion from the equilibrium -1 to $+1$. Due to the symmetry of the equation w.r.t. the change of the sign, the series $-V(\rho;r_j)$ is the asymptotic solution, too. By such series we can combine contrast structures with intermediate layers at various points $r_j\in(1,2)$. For example, the combination

$$-1+V(\rho_-;r_-)-V(\rho_+;r_+),\quad \rho_\pm=(r-r_\pm)/\varepsilon,\ r_-<r_+$$

describes the motion from the point -1 to the point $+1$ and back.

[1]We should understand that equation (7.2) has no exact solutions whose trajectories connect the saddle points. The presence of the first derivative splits the separatrices.

This construction can be easily adapted for solution to partial differential equation (7.1) by adding the series of boundary layer functions

$$U^-(\xi_-,\varphi) = \sum_{n=0}^{\infty} \varepsilon^{2n} u_n^-(\xi_-,\varphi), \quad U^+(\xi_+,\varphi) = \sum_{n=0}^{\infty} \varepsilon^{2n} + u_n^+(\xi_+,\varphi).$$

The variables for the boundary layers are introduced by the formulae $\xi_- = (r-1)/\varepsilon$, $\xi_+ = (2-r)/\varepsilon$, while the coefficients fast decay w.r.t. these variables $u_n^{\pm}(\xi) = \mathcal{O}(\xi^{2n}\exp(-\xi))$, $\xi \to \infty$.

Finally we obtain the asymptotic solution as a contrast structure with the boundary layers; this solution can have an arbitrary number of intermediate layers at the circles of arbitrary radius $r_j \in (1,2)$.

The described construction of contrast structures as concentric rings with arbitrary centers and radii can be performed in an arbitrary domain, not necessarily in the ring. In order to avoid bulky constructions, one can restrict himself by the case when the rays passing from the center of the chosen circles do not touch the domain of the boundary [6].

Thus, under the present of equilibria, for boundary value problem with the small parameter (7.1) it is possible to construct infinitely many asymptotic solutions the most part of which are associated with the exact solution.

The fact that the constructed contrast structure is not associated with the exact solution is to be proven. The most simplest proof is for the problem in the ring when the boundary condition is independent of the angle and the problem is reduced to the ordinary differential equation.

Proposition 7.1. *Boundary value problem for equation* (7.2) *possesses asymptotic solution if* $\alpha,\beta \notin [-1,1]$.

Proof. Since equation (7.2) is not autonomous, we can not draw the phase portrait on the plane and one has to find out the information on the exact solution by other means, namely, by apriori estimates. One can observe that the non-autonomous term with the first derivative plays the role of dissipation (resistance). It allows us to estimate the location of a phase trajectory by writing an analogue of conservation law for equation (7.2):

$$\left[(\varepsilon\, u')^2 + 2u^2 - u^4\right]_{r_0}^{r} = -\varepsilon^2 \int_{r_0}^{r} \frac{1}{r}(u'(\varrho))^2\, d\varrho < 0.$$

It follows from this inequality that the quantity $I(u,u') = (\varepsilon\, u')^2 + 2u^2 - u^4$, whose level set is given in the phase portrait, Fig. 6, decays along the trajectories of equation (7.2). This is why the trajectory being inside the separatrix cell (Fig. 6) as $r = \hat{r}$ should stay there for each $r > \hat{r}$. Therefore, for the solution to equation (7.2) with the boundary values outside the boundary of the cell $\alpha,\beta \notin [-1,1]$, the trajectory is not inside the cell. In particular, the extrema of such solution are outside the segment $[-1,1]$ since as $u'(r) = 0$, we necessarily have $|u(r)| > 1$.

On the other hand, it follows from equation (7.2) that $u(r)[1-u^2(r)] \geqslant 0$ at a maximum. This is why the value of a positive maximum can be only in the segment $[0,1]$. However, in the problem with boundary values $\alpha,\beta < -1$ such maximum is impossible and hence, the solution stays to left of the equilibrium $u(r) < -1$. In the same way one can argue in the general case when $\alpha,\beta \notin [-1,1]$. If the values α, β are of different signs, then the solution to boundary value problem is monotonous $u'(r) \neq 0$.

The constructed above asymptotic solution as a contrast structure with boundary layers

$$-1 + V(\rho_-; r_-) - V(\rho_+; r_+) + U_-(\xi_-) + U_+(\xi_+), \quad (r_- < r_+)$$

is not monotone and takes values close both to $+1$ and -1. This is why it is associated with none of exact solutions to the boundary value problem with $\alpha, \beta \notin [-1, 1]$, i.e., it is fictitious. The proof is complete. $\square$

8. Needle crystal model

We consider third order differential equation with a small parameter at the derivatives

$$\varepsilon^5 \frac{d^3u}{dx^3} + \varepsilon \frac{du}{dx} = \cos u, \quad -1 < x < 1, \quad 0 < \varepsilon \ll 1. \tag{8.1}$$

and with the boundary conditions corresponding to the equilibria:

$$u(-1) = -\pi/2, \ u(1) = \pi/2. \tag{8.2}$$

In such formulation the problem is similar to those considered in the previous sections. Generally speaking, the amount of boundary conditions is insufficient for the well-defined formulation. The additional condition

$$u(0) = 0, \tag{8.3}$$

likely makes the problem well-defined. This condition fixes the moment of passing a given point (here $u = 0$) by a trajectory of the autonomous system.

If we pass to the fast variable $s = x/\varepsilon$, the equation casts into the standard form used in the geometric theory of needle crystal

$$\varepsilon^2 \frac{d^3u}{ds^3} + \frac{du}{ds} = \cos u, \quad 0 < \varepsilon \ll 1. \tag{8.4}$$

At that, instead of large interval $-1/\varepsilon < s < 1/\varepsilon$, in work [2] there was considered the unbounded interval $-\infty < s < \infty$ and the conditions at infinity

$$u(s) \to \mp\pi/2, \ s \to \mp\infty. \tag{8.5}$$

Since the asymptotic solution is constructed in the class of exponentially stabilizing function, such replacement of the domain is not essential.

In work [2] the matter is the attempt to single out a solution of separatrix type whose trajectory goes monotonically from one equilibrium to another. The main result is that such solution with conditions (8.5), (8.3) does not exist.

The instructiveness of this example is that an asymptotic solution to problem (8.4), (8.5), (8.3) is constructed up to arbitrarily small error:

$$U(s, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(s), \ \varepsilon \to 0, \quad s \in \mathbb{R}.$$

The leading term is determined by the limiting nonlinear equation $U_0' = \cos U_0$ and it is a smooth step $U_0(s) = \arcsin \tanh(s)$, see Fig. 11. The solution to the linearized equation $v' + \sin U_0 \cdot v = 0$ fast decays at infinity $v = U_0'(s) = 1/\cosh s$. All the terms in the asymptotics are uniquely defined by inhomogeneous linearized equations with the condition $U_n(0) = 0$ and are smooth odd functions. At infinity they fast tend to zero: $U_n(s) = \mathcal{O}\big(s^n \exp(\mp s)\big)$, $s \to \pm\infty$.

Moreover, at each half-line (either $s \geqslant 0$ or $s \leqslant 0$) formal solution $U(s, \varepsilon)$ is the asymptotics for the unique exact solution [2]. However, these two exact solutions (at different half-lines) do not provide the solution on the whole line because of discrepancy of second derivatives at the point $s = 0$. This discrepancy is exponentially small as $\varepsilon \to 0$ and does not appear in formal constructions.

We note that in numerical experiments there appears an approximate solution whose graph almost coincides with the graph of the asymptotic solution already for $\varepsilon = 0.1$, Fig. 11. This

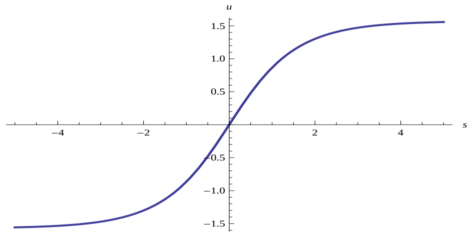

FIGURE 11. Graphs of asymptotic and numerical solutions to equation (8.4) almost coincide as $\varepsilon = 0.1$

fact should alert the fans of justifying the asymptotics by the numerics since in the considered problem there exists no exact solution.

9. Anosov's example

Here we reproduce in the simplest form the example in the report by D.V. Anosov [3].

For the linear equation with a periodic right hand side and a small parameter

$$\varepsilon \frac{d^2u}{dt^2} + u = f(t), \ t > 0, \ 0 < \varepsilon \ll 1 \tag{9.1}$$

we consider a problem on finding a periodic solution. Let the right hand side be a typical 2π-periodic function which can be expanded into the Fourier series

$$f(t) = \sum_{k=0}^{\infty} f_k \exp(ikt)$$

and is not a trigonometric polynomial; thus, there exists a sequence of numbers $k_j \to \infty$ obeying $f_{k_j} \neq 0$. We assume that this function is infinitely differentiable; for instance, all the series

$$\forall\, n > 0, \quad \sum_{k=0}^{\infty} k^n |f_k| < \infty$$

converge. Under these assumptions one can easily construct an *asymptotic solution with 2π-periodic coefficients:*

$$U(t;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n u_n(t), \quad \varepsilon \to 0; \quad u_n(t) = (-1)^n f^{(2n)}(t). \tag{9.2}$$

However, the conjecture that as $\varepsilon \to 0$ this series provides an asymptotics for a periodic solution fails. For an exact solution the following statement holds true.

Proposition 9.1. *If $f(t)$ is a typical periodic function, there exists a sequence of values of parameter $\varepsilon_j \to 0$ for which equation* (9.1) *has no periodic solutions.*

To prove this proposition, one should take the Fourier series for a periodic solution (if it exists)

$$u(t;\varepsilon) = \sum_{k=0}^{\infty} a_k(\varepsilon) \exp(ikt),$$

and by equation write the relations for the coefficients $(1 - \varepsilon k^2) a_k = f_k$. If $f_{k_j} \neq 0$, as $\varepsilon_j = 1/k_j^2$ we obtain the contradiction.

Thus, the existence of asymptotic solution as $\varepsilon \to 0$ with periodic coefficients does not imply the existence of a periodic solution for all sufficiently small ε. In view of this fact the issue on estimating the error makes no sense.

However, it does not imply that the asymptotic solution has no relation with any exact solution. A non-periodic function can have a very small phase shift so that it can be well approximated by a periodic function, as one can see by the example

$$\sin\big(t+\exp(-(t^2+1)/\varepsilon)\big)=\sin t+\mathcal{O}(\exp(-1/\varepsilon)),\ \varepsilon\to 0,\ \forall\, t\in\mathbb{R}.$$

For the considered problem we have a similar situation: there exists an exact solution to the equation and the constructed formal series is its asymptotics as $\varepsilon\to 0$ uniformly in t on a large time interval. Moreover, there are many of such solutions.

Let us consider the Cauchy problem for equation (9.1) with the initial conditions

$$u|_{t=0}=\varphi(\varepsilon),\quad u'|_{t=0}=\psi(\varepsilon). \tag{9.3}$$

The solution can be written as

$$u(t;\varepsilon)=\varepsilon^{-1/2}\int_0^t \sin((t-\eta)/\sqrt{\varepsilon})f(\eta)\,d\eta+\varphi(\varepsilon)\cos(t/\sqrt{\varepsilon})+\sqrt{\varepsilon}\psi(\varepsilon)\sin(t/\sqrt{\varepsilon}). \tag{9.4}$$

Proposition 9.2. *There exists a set of pairs of smooth functions $\varphi(\varepsilon),\psi(\varepsilon)$ with the same power asymptotics as $\varepsilon\to 0$, for which the solution to problem* (9.1), (9.3) *has the asymptotics provided by the above constructed series* (9.2) *with periodic coefficients. This asymptotics is uniform w.r.t. t on the interval $|t|\leqslant\varepsilon^{-p}$ for each fixed $p<\infty$.*

The key ingredient of the proof is an appropriate choice of the coefficients in the asymptotic series

$$\varphi(\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\varphi_n,\quad \psi(\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\psi_n,\quad \varepsilon\to 0.$$

In order to do it, we calculate the asymptotics for the exact solution by integrating by parts in formula (9.4). It is easy to see that after multiple repetition of such operation all the extra-integral terms at the upper limit $\eta=t$ give the coefficients of series (9.2), while at the lower limit $\eta=0$ we obtain the asymptotics series at $\cos(t/\sqrt{\varepsilon})$ and $\sin(t/\sqrt{\varepsilon})$. The asymptotic sums (see [52, 53]) of these series give the desired functions[1] with the opposite sign: $-\varphi(\varepsilon),-\psi(\varepsilon)$.

In this way we obtain the expansion of the exact solution as series (9.2). The estimate for the error term can be proven as follows. If we stop the integration by parts at $2N$th step, we obtain the representation of the solution as a partial sum of the series and the remaining integral

$$u(t;\varepsilon)=\sum_{n=0}^{N-1}\varepsilon^n u_n(t)+(-1)^N\varepsilon^{N-1/2}\int_0^t \sin\big((t-\eta)/\sqrt{\varepsilon}\big)f^{(2N)}(\eta)\,d\eta+\Phi(\varepsilon).$$

Here $\Phi(\varepsilon)=\mathcal{O}(\varepsilon^N)$ are the remainders of the initial functions; this term and the integral with the small factor is the error term of the asymptotic expansion. The integral can be roughly estimated by the length of the interval,

$$\Big|u(t;\varepsilon)-\sum_{n=0}^{N-1}\varepsilon^n u_n(t)\Big|<M(N)\varepsilon^{N-p-1/2},\ M(N)=\mathrm{const}<\infty,\ \forall\,|t|<\varepsilon^{-p}.$$

Since number N is arbitrary, this estimate proves the proposition.

We note that on a finite interval $0\leqslant t<2\pi$, formal 2π-periodic solution (9.2) is the asymptotics of exact solution (9.4) up to arbitrarily small error $\mathcal{O}(\varepsilon^n)$, $\varepsilon\to 0$, $\forall\, n$. Another point is that this exact solution is not 2π-periodic.

On a large interval containing values $t\gg 1$, the estimate for the error term of the asymptotic expansion becomes worse once the length of the interval increases[2]. This is a usual situation

[1] As it is known, there are many of such functions.

[2] Due to the presence of resonance terms, for typical right hand sides $f(t)$ this estimate is the best impossible in the order of ε.

in the problems of oscillation theory with which known perturbation methods deal. The expansions applicable uniformly on an infinite interval are rather an exception than the rule. In particular, such exceptions are provided by the problems on constructing periodic solutions near the equilibrium [7, 54]. The above example (9.1) is not such an exception.

We note that this and other examples adduced by [4] are outside the framework of the problems treated usually in the oscillation theory because of periodicity of solution in the slow variable. More often one considers the problems no conserving the periodicity of solutions in fast variable under a weak perturbation of the equation, i.e., without increasing the order of the derivatives [54]. In the above example a (strong) perturbation of the second derivative leads us to the appearance of a fast time scale $t/\sqrt{\varepsilon}$ and the original scale t becomes slow. In fact, we deal with the equation

$$\frac{d^2u}{dt^2} + u = f(\varepsilon t), \ 0 < \varepsilon \ll 1$$

with the periodicity condition in the variable $\tau = \varepsilon t$. Such problems on slow periodic perturbations was analyzed in [17].

10. Conclusion

For various classes of problem with a small parameter we have constructed formal asymptotic solutions not providing the asymptotics for an exact solution. All the examples, for which we have constructed fictitious asymptotic solutions (except Nefedov's and Anosov's examples), possess two features:

1) presence of an equilibrium for a differential equation;
2) boundary conditions on the boundary.

Among variety of presented constructions there are such that in particular cases they are associated with exact solutions (bursts, contrast structures). However, on the level of formal constructions they are not distinguished among fictitious solutions. The justification of the asymptotics in such particular cases can happen to be not a simple problem; the results of [10] are apriori not applicable.

Anosov's example is close in spirit to the Kruskal–Segur example. In both cases they consider a problem having no exact solution but one can construct an asymptotic solution. In the general situation the absence of exact solution and the proof of phantomness for a formal solution can happen to be a very nontrivial problem as it was demonstrated in [2].

It seems that in all the examples the solution of the contradiction between the exact and asymptotic solutions lie beyond the power asymptotics. To separate fictitious asymptotics, one has to take into consideration exponentially small terms. For ordinary differential equations there exists a way with going out to the complex plane, cf. [2, 55, 56]. For partial differential equations the issue remains open.

One more remark concerns the usage of numerical methods. Due to the complexity of the issue on errors and absence of the desire to deal with the justification, sometimes one justifies the validity of asymptotic formulae by the results of numerical experiments with the perturbed problem. The closeness of numerical and asymptotical solutions is interpreted as the justification of the asymptotics. However, if one is not careful enough, such "justification" can be performed in all the above examples. That is, the numerical solutions can happen to be fictitious. Here the problems occurs first of all because of finite-dimensional approximation. In connection with the problems on pseudospectrum this issue was studied in [57], where it was indicated that the spectrum of finite-dimensional approximations not necessarily approximates the spectrum of the original operator. Another reasons of inefficiency of numerical methods as well as of asymptotic formulae is hidden in troublesome control of exponentially small terms leading in time to great errors. However, for the problems with a small parameter at higher derivative one needs special numerical methods even in the case when the degeneration is regular in the sense [6] and fictitious asymptotics are absent. This is a well-known problem [58] and there are many publications on this subject. The problems with fictitious asymptotics were not touched in this direction.

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