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ESTIMATES OF DECAY RATE FOR SOLUTION TO PARABOLIC EQUATION WITH NON-POWER NONLINEARITIES

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Abstract. We study the Dirichlet mixed problem for a class of parabolic equation with double non-power nonlinearities in cylindrical domain $D = (t > 0) \times \Omega$. By the Galerkin approximations method suggested by Mukminov F.Kh. for a parabolic equation with double nonlinearities we prove the existence of strong solutions in Sobolev-Orlicz space. The maximum principle as well as upper and lower estimates characterizing powerlike decay of solution as $t \to \infty$ in bounded and unbounded domains $\Omega \subset \mathbb{R}^n$ are established.

Keywords: parabolic equation, N-functions, existence of solution, estimate of decay rate of solution, Sobolev-Orlicz spaces.

Mathematics Subject Classification: 35D05, 35B50, 35B45, 35K55

1. Introduction

Let Ω be an arbitrary domain in the space $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n)\}$, $n \geqslant 2$. In the cylindrical domain $D = \{t > 0\} \times \Omega$ we consider the equation

$$(\beta(x,u))_t = \sum_{i=1}^{n} \left(a_{p_i}(x, \nabla u)\right)_{x_i}, \text{ where } a(x,\nabla u) = a(x,p)\Big|_{p=\nabla u}, \tag{1}$$

with boundary and initial conditions

$$u(t,x)\Big|_S = 0, \quad S = \{t > 0\} \times \partial\Omega; \tag{2}$$

$$u(0,x) = u_0(x). \tag{3}$$

Hereinafter the subscripts t, x_i, p_i denote the derivatives w.r.t. the indicated variables.

Suppose that function $a(x,p)$ is convex w.r.t. $p = (p_1, p_2, \ldots, p_n)$ and satisfies Caratheodory condition for $p \in \mathbb{R}^n$ and $x \in \Omega$. Function $\beta(x,u)$, $\beta(x,0) = 0$,

$$|\beta(x,u)| \leqslant c_\beta |u\beta'_u(x,u)|, \tag{4}$$

is absolutely continuous and increases w.r.t. u, as well as it is measurable w.r.t. $x \in \Omega$ as $u \in \mathbb{R}$.

The existence and uniqueness of the solutions to nonlinear parabolic equations were considered in works [1]–[4], [7], [19]–[25] and others. The problem were mainly considered for a bounded domain Ω and on a bounded time interval $[0,T]$ for an arbitrary $T > 0$. In work [1] there was proven the existence of weak solutions to quasilinear second order parabolic equations with a double-nonlinearity in a bounded domain. The existence of weak solution to a parabolic equation with two variable nonlinearities in appropriate Sobolev-Orlicz space for a bounded

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domain Ω was proven in [2]. In [3] there were proven the existence and uniqueness theorems for the generalized solution to the Dirichlet problem for degenerating parabolic equations linear w.r.t. ∇u and having a variable nonlinearity index w.r.t. u. The existence of W- and H-solutions for second order parabolic equations with a variable nonlinearity index was proven in work [4].

Dealing with a weak solution is troublesome in studying, say, the decay of solution as $t \to \infty$. In the present work for constructing a strong solution to problem (1) – (3) on the whole time interval $[0, \infty)$ we employ the Galerkin approximations method (domain Ω can be unbounded). By this method the solution to a parabolic equation was constructed in work [5] on the bounded time interval $[0, T]$ for each $T > 0$ and in work [6] on an unbounded time interval.

The Galerkin approximations are smooth functions that simplifies the proving necessary estimates which then are extended by passage to the limit for the solution to problem (1) – (3). In the present work we obtain both upper and lower estimates characterizing the power decay of the solution as $t \to \infty$ in the case of both bounded and unbounded domains $\Omega \subset \mathbb{R}^n$.

Work [6] was devoted to the study of the behavior as $t \to \infty$ of solution to a mixed problem for an isotropic parabolic equations with a double nonlinearity, while for anisotropic equations with a double nonlinearity the same was done in works [7]–[9]. In work [10] there was studied the degeneration property for the solution to a nonlinear parabolic equation with a non-standard anisotropic growth conditions in a finite time interval. The same authors in [11] established the sufficient conditions for the blow-up of the solution to the homogeneous Dirichlet problem for an anisotropic parabolic equations with a variable nonlinearity in a finite time interval. In [12] there were established the estimates of the higher integrability for a weak solution to a parabolic system with a variable index of nonlinearity. The exact two-sided estimates for the decay rate of the norm of solution to a linear and quasilinear parabolic equation in an unbounded domain there were established in works [13, 14], while in [15] it was for an anisotropic parabolic equation. The study of the behavior to linear and quasilinear parabolic equations was done in works [16] – [18].

2. Functional spaces

Here we introduce functional spaces employed in the work and we also provide some known facts in the theory of Sobolev-Orlicz spaces [26].

We shall say that N-function $B(s)$ satisfies $\triangle_2$-condition for great values of s, if there exist numbers $k > 0$, $s_0 > 0$, such that $B(2s) \leqslant kB(s)$ $\forall s \geqslant s_0$. $\triangle_2$-condition is equivalent to the inequality

$$B(ls) \leqslant kl^m B(s), \tag{5}$$

for great values of s, where l can be an arbitrary number greater than one, m is positive. Usually one considers bounded domains only and then condition (5) as $s \geqslant s_0 > 1$ is sufficient. If the domain is unbounded, then (see, for instance, the proof of Lemma 1 below) one has to let $s_0 = 0$. In what follows we assume that all the considered N-functions satisfy $\triangle_2$-condition for all values of $s > 0$ (i.e., $s_0 = 0$). We shall indicate all N-functions by capital Latin letters.

All the constants appearing in the work are positive.

The N-function

$$\overline{B}(z) = \sup_{t \geqslant 0}(t|z| - B(t))$$

is called additional. The following property of additional functions is known (cf. [26]):

$$|zs| \leqslant B(z) + \overline{B}(s). \tag{6}$$

For N-function we shall write $B_1(s) \prec B_2(s)$, if there exist constants s_0, k such that

$$B_1(s) \leqslant B_2(ks), \quad \text{for } s \geqslant s_0. \tag{7}$$

Suppose that for a.e. $x \in \Omega$ a function $\beta_1(x, u)$ is absolutely continuous w.r.t. $u \in \mathbb{R}$ and is defined by the identity

$$\beta'_{1u}(x, u) = u\beta'_u(x, u), \quad \beta_1(x, 0) = 0. \tag{8}$$

At that we assume that $\beta'_u(x,u) \geqslant 0$ is even w.r.t. u, bounded in each bounded domains of (x,u), not vanishing a.e. in each interval w.r.t. u.

Let for each $u \in \mathbb{R}$, $\in \mathbb{R}^n$, and $x \in \Omega$ the conditions

$$\sum_{i=1}^{n} a_{p_i}(x,p)p_i \geqslant \sum_{i=1}^{n} B_i(p_i); \tag{9}$$

$$\Gamma \sum_{i=1}^{n} B_i(p_i) \geqslant a(x,p) \geqslant \delta \sum_{i=1}^{n} a_{p_i}(x,p)p_i; \tag{10}$$

$$\sum_{i=1}^{n} \overline{B}_i(a_{p_i}(x,p)) \leqslant c \sum_{i=1}^{n} B_i(p_i); \tag{11}$$

$$u\beta'_{1u}(x,u) \leqslant \alpha\beta_1(x,u),\ \alpha > 0,\ \forall u \in \mathbb{R}. \tag{12}$$

hold true. Here $B_1(z), B_2(z), \dots, B_n(z)$ are N-functions.

We also suppose the existence of N-function $G(s)$ (then $G(s^2)$ is a N-function as well) such that

$$G(u^2) \leqslant \beta_1(x,u) \leqslant c_1 G(u^2); \tag{13}$$

$$\overline{G}(\beta'_u(x,u)) \leqslant c_2 G(u^2). \tag{14}$$

Hereinafter by c_1, c_2, ... we denote constants which generally saying do not coincide even for the same subscripts.

By $L_B(Q)$ we denote the Orlicz space corresponding to N-function $B(s)$ with the Luxembourg norm

$$\|u\|_{L_B(Q)} = \|u\|_{B,Q} = \inf\left\{k \geqslant 0 : \int\limits_Q B\left(\frac{u(x)}{k}\right) dx \leqslant 1\right\}.$$

In what follows as Q we can choose domains Ω, D^T, and others.

The Orlicz spaces corresponding to the N-function $G(s^2)$ is indicated by $L_{G_2}(Q)$ and the symbol $L_{\overline{G}_2}(Q)$ stands for its dual space.

We also define Sobolev-Orlicz space $\overset{\circ}{W}{}^1_{G,B}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ w.r.t. the norm

$$\|u\|_{W^1_{G,B}(\Omega)} = \sum_{i=1}^{n} \|u_{x_i}\|_{B_i,\Omega} + \|u\|_{G_2,\Omega}.$$

By $V(D^T)$ we shall denote the completion of $C_0^\infty(D^T)$ w.r.t. the norm

$$\|u\|_{V(D^T)} = \sum_{i=1}^{n} \|u_{x_i}\|_{B_i,D^T} + \|u\|_{G_2,D^T}.$$

The Luxembourg norm satisfies the inequality (cf. [26])

$$\|u(x)\|_{L_B(Q)} \leqslant 1 + \int\limits_Q B(u(x))dx. \tag{15}$$

The following simple statement holds true.

Lemma 1. *If $u_j \to u$ in $L_B(\Omega)$ and B satisfies $\triangle_2$-condition, then there exists C such that*

$$\int\limits_\Omega B(u_j)dx \leqslant C.$$

Proof. Since sequence u_j converges, we have $\|u_j\|_{L_B(\Omega)} \leqslant c$. Then by employing $\triangle_2$-condition we obtain

$$\int\limits_\Omega B(u_j)dx = \int\limits_\Omega B\left(\|u_j\|_{L_B(\Omega)}\frac{u_j}{\|u_j\|_{L_B(\Omega)}}\right)dx \leqslant \int\limits_\Omega B\left(c\frac{u_j}{\|u_j\|_{L_B(\Omega)}}\right)dx$$
$$\leqslant kc^m \int\limits_\Omega B\left(\frac{u_j}{\|u_j\|_{L_B(\Omega)}}\right)dx \leqslant kc^m.$$

The proof is complete. □

We define function $h(s)$ as

$$h(s) = s^{-\frac{1}{n}}\left(\prod_{i=1}^n \widetilde{B}_i^{-1}(s)\right)^{\frac{1}{n}}, \tag{16}$$

as

$$\widetilde{B}_i(s) = \begin{cases} B_i(s), & \text{as } |s| \geqslant 1, \\ s^\kappa B_i(1), & \text{as } |s| \leqslant 1. \end{cases}$$

We note that since function B_i are convex, then the inequality $B_i'(1+) > B_i(1)$ holds true. We choose $\kappa \in (1, n)$ to satisfy the inequalities

$$B_i'(1) > \kappa B_i(1), \ i = 1, 2, \ldots, n. \tag{17}$$

We also define a N-function $B^*(z)$ by the formula

$$(B^*)^{-1}(z) = \int\limits_0^{|z|}\frac{h(s)}{s}ds, \tag{18}$$

if the integral

$$\int\limits_0^\infty \frac{h(s)}{s}ds \tag{19}$$

diverges to the infinity and

$$\|u\|_{B^*,Q} = \sup_Q |u|,$$

if integral (19) is bounded. The convergence of the latter integral at zero is ensured by the inequality $\kappa < n$. There is a known embedding theorem of A. G. Korolev [27] implied by the inequality

$$\|u\|_{B^*,Q} \leqslant C\sum_{i=1}^n \|u_{x_i}\|_{\widetilde{B}_i,Q}, \tag{20}$$

which is valid for functions $u \in C_0^\infty(Q)$ in the case of convergence of the integral $\int\limits_0^1 \frac{h(s)}{s}ds$ at zero. We also note that inequality (20) proven in [27] for bounded domains is also true for unbounded domains Q having finite measure.

3. Formulation of main results

Theorem. *Let $u_0 \in \overset{\circ}{W}{}^1_{G,B}(\Omega)$ and suppose that conditions (9)–(14) hold true. Then there exists a generalized solution to problem (1)–(3) satisfying the relations*

$$u \in L_\infty([0,\infty); \overset{\circ}{W}{}^1_{G,B}(\Omega)),$$
$$\beta(x,u) \in C([0,\infty); L_{\overline{G}_2}(\Omega)),$$
$$(\beta(x,u))_t \in L_{\overline{G}_2}(D^T),$$
$$(\beta'_u(x,u))^{\frac{1}{2}} u_t \in L_2(D^T), \ \forall T > 0.$$

The uniqueness of solution to problem (1)–(3) with the properties established in Theorem 1 will proven in another work. Formally we can assume in the following statements we discuss arbitrary solution with the properties established in Theorem 1.

Lemma 2. *Let Ω be a bounded domain. If the initial function is bounded ($u_0(x) \leqslant b$), then the generalized solution to problem* (1)–(3) *is bounded, i.e.,*

$$\operatorname*{vraisup}_D u(t,x) \leqslant b.$$

Remark. If the initial function satisfies the inequality $u_0(x) \geqslant -b$, then the function $-u$ is also a solution to some other equation belonging to the same class (due to the evenness of N-function). This is why applying lemma to the function $-u$, we obtain $-u \leqslant b$, or $u \geqslant -b$.

Lemma 3. *Suppose that domain Ω is located in the half-space $x_1 > 0$ and $\int\limits_1^\infty \frac{h(s)}{s} ds = \infty$, and the condition*

$$B_1 \prec B^* \tag{21}$$

holds true. If the initial function is bounded ($u_0(x) \leqslant b$), then the generalized solution to problem (1)–(3) is bounded, i.e.,

$$\operatorname*{vraisup}_D u(t,x) \leqslant b. \tag{22}$$

Lemma 4. *Let domain Ω be arbitrary and $\int\limits_1^\infty \frac{h(s)}{s} ds < \infty$. Then for each function $u \in V(D^T)$ the inequality*

$$\operatorname*{vraisup}_{D^T} |u(t,x)| \leqslant c \tag{23}$$

holds true, where c is an increasing function of $\|u\|_{V(D^T)}$.

While estimating from below the norm of the solution to problem (1)–(3), we need the following condition: there exist numbers $q > 1$, $c > 0$ such that the inequality

$$\beta_1^q(x,u) \leqslant c(B^*(u) + 1), \ x \in \Omega, \ u \in \mathbb{R}, \tag{24}$$

is valid.

Theorem. *Let Ω be bounded and conditions (9)–(14) be satisfied. Suppose also that condition (24) holds true, if integral (19) diverges. Then there exists a positive number C ($C = C(u_0)$) such that solution $u(t,x)$ to problem* (1)–(3) *satisfies inequalities*

$$\int\limits_\Omega G(u^2(t,x))dx \geqslant \int\limits_\Omega G(u_0^2(x))dx(1+Ct)^{\frac{1}{1-\gamma}}, \ \text{as } \gamma > 1, \ t \geqslant 0, \tag{25}$$

$$\int\limits_\Omega G(u^2(t,x))dx \geqslant \int\limits_\Omega G(u_0^2(x))dx(1-Ct)^{\frac{1}{1-\gamma}}, \ \text{as } \gamma < 1, \ t \leqslant 1/C, \tag{26}$$

where $\gamma = \frac{1}{\Gamma(\max(2,\alpha))}$.

Remark. We roughen the case $\gamma = 1$ to $\gamma < 1$ by increasing Γ.

In the next statement we consider a domain Ω located along the axis Ox_1. In what follows we shall make use of the notation $\Omega_a^b = \{x \in \Omega | a < x_1 < b\}$, the values $a = 0$, $b = \infty$ are omitted. We let $S(r) = \{x \in \Omega | x_1 = r\}$. We assume that $\Omega_{-\infty}^0 = \oslash$ and there exists number $d > 0$ such that

$$\mathrm{mes}(\Omega^r) \leqslant r^d, \ r \geqslant r_0. \tag{27}$$

To study the decay of the solution to problem (1)–(3) as $x_1 \to \infty$, we define the function

$$\nu(r) = \inf_{u \in C_0^\infty(\Omega)} \sup\{z : \int\limits_{S(r)} B_2(zu)dx' \leqslant \int\limits_{S(r)} B_2(u_{x_2})dx'\}, \tag{28}$$

where $x' = \{x_2, x_3, \ldots, x_n\}$. We shall assume that domain Ω satisfies the condition

$$\int\limits_1^\infty \nu(r)dr = \infty. \tag{29}$$

We suppose that the initial function has a compact support

$$\mathrm{supp} u_0 \subset \Omega^{r_0}, \ r_0 > 0. \tag{30}$$

Let

$$B_1(s) \leqslant gB_2(s), \ s < 1, \ g \geqslant 1. \tag{31}$$

Theorem. *Suppose that conditions (9)–(14), (21), (29)-(31) hold true, domain Ω is located along axis Ox_1, and the inequalities*

$$\nu(r) \leqslant \nu_0, \ \text{as } r \geqslant r_0; \ |u_0(x)| \leqslant v_0, \ x \in \Omega, \tag{32}$$

hold true. Then the solution $u(t,x)$ to problem (1)–(3) obeys the estimate

$$\int\limits_{\Omega_r} \beta_1(x, u(t,x))dx \leqslant M \exp\left(-\lambda \int\limits_{2R_0}^r \nu(\rho)d\rho\right) \tag{33}$$

for each $t \geqslant 0$, $r \geqslant 2r_0$ with some numbers $M, \lambda > 0$.

In the next theorem we assume that there exists a number $q > 1$ such that function $\beta_1(x,u)$ satisfies the relation

$$(\beta_1(x,u))^q \leqslant c_3 B_1(u), \forall u \in \mathbb{R}. \tag{34}$$

We take μ so that

$$\mu > m + d(q-1), \tag{35}$$

where m is the number in $\triangle_2$-condition (5) for function B_1 and d is from (27). Let $r(t)$ be an arbitrary positive function satisfying inequality

$$M \exp\left(-\lambda \int\limits_{2r_0}^{r(t)} \nu(\rho)d\rho\right) \leqslant \left(\frac{r^\mu(t)}{(q-1)t}\right)^{\frac{1}{q-1}} \tag{36}$$

for t great enough to obey $r(t) \geqslant 2r_0$.

Theorem. *Suppose that conditions (9)–(14), (21), (29)–(32), (34) hold true and domain Ω is located along axis Ox_1. Then solution $u(t,x)$ to problem (1)–(3) satisfies the estimate*

$$\int\limits_\Omega \beta_1(x, u(t,x))dx \leqslant C(r^\mu(t)t^{-1})^{\frac{1}{q-1}}, \quad C = 2(q-1)^{\frac{1}{1-q}} \tag{37}$$

for t such that $r(t) \geqslant 2r_0$.

If instead of (29) a stronger requirement

$$\lim_{r\to\infty}\frac{1}{\ln r}\int\limits_1^r \nu(t)dt=\infty \tag{38}$$

is fulfilled, we can choose $r(t)=t^{\frac{1}{2\mu}}$, and then estimate (37) casts into the form

$$\int\limits_\Omega \beta_1(x,u(t,x))dx\leqslant Ct^{\frac{-1}{2(q-1)}}. \tag{39}$$

4. Proof of existence theorem

A generalized solution to problem (1)–(3) is a function $u(t,x)$ belonging to space $V(D^T)$ for each $T>0$ and satisfying the identity

$$\int\limits_{D^T}\left(-\beta(x,u)\varphi_t(t,x)+\sum_{i=1}^n a_{p_i}(x,\nabla u)\varphi_{x_i}(t,x)\right)dxdt=\int\limits_\Omega \beta(x,u_0)\varphi(0,x)dx \tag{40}$$

for each $\varphi\in C_0^\infty(D^T_{-1})$.

We choose a sequence $\omega_k\in C_0^\infty(\Omega)$ of linearly independent functions whose linear span is dense in $\overset{\circ}{W}{}^1_{G,B}(\Omega)$. We let $I_m=\cup_{k=1}^m \operatorname{supp}\omega_k$. We seek the Galerkin approximations for the solution as follows,

$$u_m(t,x)=\sum_{k=1}^m c_{mk}(t)\omega_k(x),$$

where functions $c_{mk}(t)$ are determined by the equations

$$\int\limits_\Omega\left(\omega_j\frac{\partial}{\partial t}\left(\frac{u_m}{b_m}+\beta(x,u_m)\right)+\sum_{i=1}^n a_{p_i}(x,\nabla u_m)(\omega_j)_{x_i}\right)dx=0,\qquad j=1,2,\ldots,m. \tag{41}$$

We shall determine numbers $b_m>0$ later. Let us make sure that equations (41) are solvable w.r.t. the derivatives c'_{mk}. It is obvious that they read as

$$\sum_{k=1}^m A_{jk}(t)c'_{mk}=F_j(c_{m1},c_{m2},\ldots,c_{mm}).$$

For each t, the matrix of coefficients

$$A_{jk}(t)=\int\limits_\Omega\left(\frac{1}{b_m}+\beta'_u(x,u_m)\right)\omega_j\omega_k dx$$

is the Gram matrix of linearly independent vectors ω_k, $k=1,2,\ldots,m$, and thus it is invertible. By equations (41) and the initial conditions $c_{mk}(0)$ chosen so that $u_m(0,x)\to u_0(x)$ in $\overset{\circ}{W}{}^1_{G,B}(\Omega)$ we find functions $c_{mk}(t)$. First we find these functions on a small time interval, but the boundedness of Galerkin approximations allows us to define them on an infinite time interval. We choose numbers b_m so that $\|u_m(0)\|_2^2/b_m\to 0$ as $m\to\infty$.

Let us establish the estimates for the Galerkin approximations. We multiply equations (41) by $c_{mj}(t)$, sum up and use formula (8). Then

$$\int\limits_\Omega\left(\left(\frac{u_m^2}{2b_m}\right)_t+\beta'_{1u}(x,u_m)(u_m)_t+\sum_{i=1}^n a_{p_i}(x,\nabla u_m)u_{mx_i}\right)dx=0. \tag{42}$$

Employing inequality (9), we get

$$\int\limits_{\Omega}\left(\left(\frac{u_m^2}{2b_m}\right)_t+(\beta_1(x,u_m))_t+\sum_{i=1}^{n}B_i(u_{mx_i})\right)dx\leqslant 0.$$

Integrating w.r.t. t, due to (13) we have

$$\int\limits_{\Omega}\left(\frac{u_m^2(t,x)}{2b_m}+G(u_m^2(t,x))\right)dx+\int\limits_{D_0^t}\sum_{i=1}^{n}B_i(u_{mx_i})dxdt\leqslant$$
$$\leqslant\int\limits_{\Omega}\left(\frac{u_m^2(0,x)}{2b_m}+c_1G(u_m^2(0,x))\right)dx.$$

The latter integral in the right hand side is bounded due to the chosen convergences. Hence, we obtain the estimate

$$\int\limits_{\Omega}G(u_m^2(t,x))dx+\int\limits_{D_0^t}\sum_{i=1}^{n}B_i(u_{mx_i})dxdt\leqslant\overline{c}. \tag{43}$$

Now (43) implies the boundedness of sequence u_m in the space $L_\infty([0,T];L_{G_2}(\Omega))$ and in the space $V(D^T)$ for each $T>0$.

We multiply equations (41) by $c'_{mj}(t)$ and sum up to obtain

$$\int\limits_{\Omega}\left(\left(\frac{1}{b_m}+\beta'_u(x,u_m)\right)(u_m)_t^2+\sum_{i=1}^{n}a_{p_i}(x,\nabla u_m)(u_{mx_i})_t\right)dx=0,$$

or

$$\int\limits_{\Omega}\left(\left(\frac{1}{b_m}+\beta'_u(x,u_m)\right)(u_m)_t^2+a(x,\nabla u_m)_t\right)dx=0. \tag{44}$$

We integrate the latter identity w.r.t. t:

$$\begin{aligned}\int\limits_{D^T}\left(\frac{1}{b_m}+\beta'_u(x,u_m)\right)(u_m)_t^2dxdt+\int\limits_{\Omega}a(x,\nabla u_m(T,x))dx&\\=\int\limits_{\Omega}a(x,\nabla u_m(0,x))dx=I_\Omega.&\end{aligned} \tag{45}$$

To estimate integral I_Ω, we employ inequality (10) and Lemma 1:

$$I_\Omega\leqslant\Gamma\int\limits_{\Omega}\sum_{i=1}^{n}B_i(u_{mx_i}(0,x))dx\leqslant C.$$

Combining the obtained estimate and identity (45) and applying (10) and (9), we get

$$\int\limits_{D^T}(\beta'_u(x,u_m))(u_{mt})^2dxdt+\delta\int\limits_{\Omega}\sum_{i=1}^{n}B_i(u_{mx_i}(T,x))dx\leqslant C.$$

Hence, by the latter inequality and (43) we prove the boundedness of the sequence $(\beta'_u)^{\frac{1}{2}}(u_m)_t$ in $L_2(D^T)$ for each $T>0$ and of sequence u_m in the space $L_\infty([0,\infty];\overset{\circ}{W}{}^1_{G,B}(\Omega))$.

The established facts, by the diagonal process, allow us to choose a sequence u_{m_k} weakly converging in the given below spaces. For the sake of simplifying the notations, we omit the subscript k in the subsequences:

$$u_m \to u \quad \text{weakly in} \quad V(D^T),$$

$$(\beta_u'(x,u_m))^{\frac{1}{2}}\,(u_m)_t \to \tilde{u} \quad \text{weakly in} \quad L_2(D^T), \quad \forall T>0.$$

Let us show that by (11) the functional

$$\widetilde{a}(u_m) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_{p_i}(x, \nabla u_m)$$

is bounded on the unit ball in $V(D^T)$:

$$(\widetilde{a}(u_m), v) = \sum_{i=1}^{n} \int\limits_{D^T} a_{p_i}(x,\nabla u_m) v_{x_i} dxdt \leqslant \sum_{i=1}^{n} \int\limits_{D^T} (B_i(v_{x_i}) + \overline{B}_i(a_{p_i}(x,\nabla u_m)))dxdt$$

$$\leqslant \sum_{i=1}^{n} \int\limits_{D^T} (B_i(v_{x_i}) + cB_i(u_{x_i}))dxdt \leqslant c_1, \qquad \|v\|_{V(D^T)} \leqslant 1.$$

Therefore, $\widetilde{a}(u_m)$ is a bounded sequence in space $(V(D^T))'$ and we can choose a weakly converging subsequence

$$\tilde{a}(u_m) \to \chi \qquad \text{weakly in} \qquad (V(D^T))'.$$

The convergence holds true for each $T = 1, 2, \ldots$, at that, the limiting functions coincide on the joint domain. Then, in fact, the convergence holds true for each $T > 0$.

In what follows we shall show that $\tilde{u} = (\beta_u'(x,u))^{\frac{1}{2}}\, u_t$, $\chi = \tilde{a}(u)$, and function u is a generalized solution to problem (1)-(3). We split the appropriate arguments into three steps.

Step 1. Sequence $u_m(t)$ is bounded in the space $\overset{\circ}{W}{}^1_{G,B}(\Omega)$ for each $t > 0$:

$$\|u_m(t)\|_{W^1_{G,B}(\Omega)} \leqslant C, \ m = 1, 2, \ldots$$

We fix a countable dense set $\{t_s\} \subset [0,\infty]$. We can assume that $t_0 = 0$. For each bounded domain $\Omega^r \subset \Omega$ with a smooth boundary the compact embedding $W_1^1(\Omega^r) \subset L_1(\Omega^r)$ is known. Since $\overset{\circ}{W}{}^1_{G,B}(\Omega) \subset W_1^1(\Omega^r)$, by the diagonal process we choose a subsequence $u_{m_k}(t_s) \to h_s$ strongly in $L_1(\Omega^r)$ for each natural s. Choosing a subsequence once again and omitting the subscripts, we can suppose that $u_m(t_s, x) \to h_s(x)$ a.e. in Ω^r for each t_s. In particular, as $t_0 = 0$, we have $u_m(0, x) \to u_0(x)$ a.e. in Ω.

At the next step we make use of the lemma proven in [6].

Lemma 5. *Suppose a sequence $v_m(t) \in C([0,T]; L_2(\Omega))$ possesses the properties:*
1) $v_m(t_s, x)$ converges a.e. in Ω^r for each t_s and some $r > 0$,
2) sequence v_{mt} is bounded in $L_2(D^T)$.

Then there exists a subsequence v_{m_k} converging to a function v in the space $C([0,T]; L_1(\Omega^r))$ and $v_{m_k} \to v$ a.e. in $(0,T) \times \Omega^r$.

Step 2. We apply Lemma 5 to the sequence $v_m = f(x, u_m) = \int\limits_0^{u_m} (\beta_u'(x,\tau))^{\frac{1}{2}} d\tau$. Then $(v_m)_t = (\beta_u')^{\frac{1}{2}} (u_m)_t$. The belonging of $v_m(t)$ to $L_2(\Omega)$ for each $t > 0$ follows from the boundedness of the support of function $u_m(t)$, its smoothness, and the boundedness of $\beta_u'(x,u)$ on a bounded set of the arguments. Thanks to the arbitrary choice of $r > 0$ and $T = 1, 2, \ldots$, by the diagonal process one can choose a subsequence v_{m_k} converging a.e. in D. Since β_u' is not identically zero on intervals, then the function $f(x, u_m)$ increasing in u_m has the inverse function: $u_m = f^{-1}(x, v_m)$. Then the convergence of sequence v_{m_k} implies the convergence of sequence u_{m_k} a.e. in D to u. The limiting function is exactly u that is implied by the following statement (cf. [19, Ch. 1, Sect. 1.4, Lm. 1.3]):

Lemma 6. *Suppose a sequence g_m converges to g a.e. in Q and is bounded in $L_q(Q)$. Then $g_m \to g$ weakly in $L_q(Q)$.*

The inequality $\int\limits_{\Omega^r} u^2 dx \leqslant \int\limits_{\Omega^r} (G(u^2) + \overline{G}(1))dx$ implies the continuous embedding $V(D^T) \subset L_2([0,T] \times \Omega^r)$. This is why the weak convergence $u_m \to u$ in $V(D^T)$ implies the weak convergence $u_m \to u$ in $L_2([0,T] \times \Omega^r)$. It also follows from Lemma 6 that $v_{m_k} \to v = \int\limits_0^u (\beta'_u(x,\tau))^{\frac{1}{2}} d\tau$ weakly in $L_2(D^T)$ for each $T > 0$.

By Lemma 5 we know that $v_{m_k}(T) \to v(T)$ in $L_1(\Omega^r)$. Then we can select a subsequence converging a.e. in Ω^r: $v_{m_k}(T,x) \to v(T,x) \Rightarrow u_{m_k}(T,x) \to u(T,x)$ a.e. in Ω^r (and hence in Ω). Since the sequence $u_m(T)$ is bounded in the space $\mathring{W}^1_{G,B}(\Omega)$, we can choose a subsequence such that

$$u_{m_k}(T) \to u(T) \ \text{ weakly in } \ \mathring{W}^1_{G,B}(\Omega), \text{ for a fixed } T. \tag{46}$$

It follows that $u \in L_\infty([0,\infty); \mathring{W}^1_{G,B}(\Omega))$.

Then, $((v_m)_t, \varphi)_{D^T} = -(v_m, \varphi_t)_{D^T}$ $\varphi \in C_0^\infty(D^T)$. Passing to the limit as $m \to \infty$, we obtain

$$(\tilde{u}, \varphi)_{D^T} = -(v, \varphi_t)_{D^T}.$$

It follows that $\tilde{u} = v_t = (\beta'_u(x,u))^{\frac{1}{2}} u_t$.

Let us show that sequence $\beta'_u(x,u_m)(u_m)_t$ is bounded in $L_{\overline{G}_2}(D^T)$. Indeed,

$$\begin{aligned} |(\beta'_u(x,u_m)(u_m)_t, \varphi)_{D^T}| =& |((\beta'_u))^{\frac{1}{2}}(u_m)_t, \varphi(\beta'_u)^{\frac{1}{2}})_{D^T}| \leqslant c\|\varphi(\beta'_u)^{\frac{1}{2}}\|_{L_2(D^T)} \leqslant \\ &\leqslant c_1 \left(\int\limits_{D^T} G(\varphi^2)dxdt + \int\limits_{D^T} \overline{G}(\beta'_u(u_m))dxdt \right)^{\frac{1}{2}} < c_2, \ \|\varphi\|_{G_2,D^T} \leqslant 1, \end{aligned}$$

since the latter integral is estimated by formula (14). Then we can assume that $\beta'_u(x,u_m)(u_m)_t \to \overline{u}$ weakly in $L_{\overline{G}_2}(D^T)$.

Let us show that $\beta(x,u(t,x))$ belongs to the space $C([0,\infty]; L_{\overline{G}_2}(\Omega))$. We introduce the functional

$$l(\varphi) = \int\limits_\Omega \varphi(x)\beta(x,u(t,x))dx.$$

Employing (4), let us show that it is bounded on the unit ball in the space $L_{G_2}(\Omega)$

$$|l(\varphi)| \leqslant c_\beta \int\limits_\Omega |\varphi u \beta'_u(x,u)|dx \leqslant c_\beta \int\limits_\Omega \left(\overline{G}(\beta'_u(x,u)) + G(|\varphi u|)\right) dx. \tag{47}$$

The first integral in the right hand side of this inequality is bounded due to condition (14). We estimate the second integral

$$G(|\varphi u|) \leqslant G(\frac{\varphi^2 + u^2}{2}) \leqslant G(\varphi^2) + G(u^2).$$

Since $\int\limits_\Omega G(\varphi^2)dx \leqslant 1$, the second integral in the right hand side of (47) is bounded. The upper bound by a constant independent of $t \in [0,\infty)$ is ensured for the functional by the belonging of $\beta(x,u)$ to the space $L_\infty([0,\infty); L_{\overline{G}_2}(\Omega))$. Passing to limit in the identity

$$(\beta(x,u_m), \varphi_t)_{D^T} = -(\beta'_u(x,u_m)(u_m)_t, \varphi)_{D^T},$$

we obtain that

$$(\beta(x,u), \varphi_t)_{D^T} = -(\overline{u}, \varphi)_{D^T},$$

i.e., $(\beta(x,u))_t = \overline{u} \in L_{\overline{G}_2}(D^T)$. Therefore, since $T > 0$ is arbitrary, $\beta(x,u) \in C([0,\infty); L_{\overline{G}_2}(\Omega))$. At that,

$$\beta(x, u(0)) = \beta(x, u_0). \tag{48}$$

Indeed, at the Step 1 we mentioned the convergence $u_m(0,x) \to u_0(x)$ a.e. in Ω. Then, by Lemma 5, the convergence $v_m(0) \to v(0)$ in $L_1(\Omega^r)$, $r > 0$, implies the convergence $u_m(0,x) = f^{-1}(x, v_m(0,x)) \to u(0,x)$ a.e. in Ω by a suitable subsequence. It guarantees the validity of identity (48).

Step 3. We proceed to proving the identity $\chi = \tilde{a}(u)$. We multiply equation (41) by a smooth function $d_j(t)$, integrate w.r.t. t and pass to the limit as $m \to \infty$, denoting $d_j(t)\omega_j(x)$ by φ in the final expression:

$$(\beta_t(x,u), \varphi)_{D^T} + (\chi, \varphi)_{D^T} = 0. \tag{49}$$

We note that

$$\Big(\frac{(u_m)_t}{b_m}, \varphi\Big)_{D^T} = \frac{1}{b_m}(-(u_m, \varphi_t)_{D^T} + (u_m(T), \varphi(T))_\Omega - (u_m(0), \varphi(0))_\Omega) \to 0$$

by the boundedness of u_m in $L_\infty([0,T]; L_{G_2}(\Omega))$ and since $b_m \to \infty$. It is also easy to see that each function in $V(D^T)$ can be approximated by linear combinations of the form

$$\sum_{i=1}^{N} d_j(t)\omega_j(x).$$

Thus, (49) is valid for functions φ in space $V(D^T)$ as well. Hence, u is the generalized solution to problem (1)–(3) once we show that $\chi = \tilde{a}(u)$.

Let $w_m = (\beta_1(x,u_m))^{\frac{1}{2}}$, $w_m \to w = (\beta_1(x,u))^{\frac{1}{2}}$ a.e. in D. If we show that $w \in L_2(D^T)$ has the generalized derivative $w_t \in L_2(D^T)$, it will imply the identity

$$\int_0^T \frac{\partial}{\partial t}\|w\|_2^2 dt = \|w(T)\|_{L_2(\Omega)} - \|w(0)\|_{L_2(\Omega)}. \tag{50}$$

We employ (13):

$$\int_\Omega \beta_1(x, u_m(T,x))dx \leqslant c_1 \int_\Omega G(u_m^2(T,x))dx < c_2. \tag{51}$$

Hence, the sequence $w_m(T)$ is bounded in $L_2(\Omega)$ and by Lemma 6 there exists a subsequence converging to $w(T)$ weakly in $L_2(\Omega)$. We note that then $\|w\|_2^2 = \lim(w, w_m) \leqslant \liminf \|w\|_2 \|w_m\|_2$. It yields the inequality

$$\liminf \|\beta_1(x, u_m(T))\|_{L_1(\Omega)} \geqslant \|\beta_1(x, u(T))\|_{L_1(\Omega)}. \tag{52}$$

Integrating inequality (51) w.r.t. T, we obtain that sequence w_m is bounded in $L_2(D^T)$ and by Lemma 6 we can choose a subsequence weakly converging to w in $L_2(D^T)$.

To prove that $w_t \in L_2(D^T)$, we apply condition (12), then

$$\int_{D^T} \Big((\beta_1^{\frac{1}{2}}(x,u_m))_t\Big)^2 dxdt = \int_{D^T} \left(\frac{\beta'_{1u}(x,u_m)(u_m)_t}{2\beta_1^{\frac{1}{2}}(x,u_m)}\right)^2 dxdt \leqslant$$
$$\leqslant \int_{D^T} \frac{\alpha(\beta'_{1u}(x,u_m))^2(u_m)_t^2}{4u\beta'_{1u}(x,u_m)} dxdt = \frac{\alpha}{4}\int_{D^T} \beta'_u(x,u_m)(u_m)_t^2 dxdt < c.$$

The latter inequality follows from (45). Therefore, $(w_m)_t$ weakly converges to $\overline{w}$ in $L_2(D^T)$. Then, $(w_m, \varphi_t)_{D^T} = -((w_m)_t, \varphi)_{D^T}$, $\varphi \in C_0^\infty(D^T)$. Passing to the limit, we obtain $(w, \varphi_t)_{D^T} = -(\overline{w}, \varphi)_{D^T}$. Hence, $\overline{w} = w_t$.

We substitute $\varphi = u$ into (49) and apply (50) to obtain

$$(-\chi, u)_{D^T} = ((\beta(x,u))_t, u)_{D^T} = \int\limits_{D^T} \beta'_{1u}(x,u) u_t dxdt$$
$$= \int\limits_0^T \frac{\partial}{\partial t} \|\beta_1^{\frac{1}{2}}(x,u)\|_2^2 dt = \|\beta_1(u(T))\|_{L_1(\Omega)} - \|\beta_1(u(0))\|_{L_1(\Omega)}. \tag{53}$$

Then we employ the monotonicity of operator $\widetilde{a}$. It is easy to check (see [19, Ch. 2, Sect. 1, Prop. 1.1]) that

$$X_m = \int\limits_0^T (\tilde{a}(u_m(t)) - \tilde{a}(h(t)), u_m(t) - h(t))_\Omega dt \geqslant 0, \qquad \forall h \in V(D^T).$$

Equations (41) imply easily the relations

$$(\tilde{a}(u_m), u_m)_{D^T} = \|\beta_1(u_m(0))\|_{L_1(\Omega)} - \|\beta_1(u_m(T))\|_{L_1(\Omega)} + \frac{1}{2b_m}\left(\|u_m(0)\|_2^2 - \|u_m(T)\|_2^2\right). \tag{54}$$

Hence,

$$X_m = \|\beta_1(u_m(0))\|_{L_1(\Omega)} - \|\beta_1(u_m(T))\|_{L_1(\Omega)} + \frac{1}{2b_m}(\|u_m(0)\|_2^2 - \|u_m(T)\|_2^2) - (\tilde{a}(u_m), h)_{D^T} - (\tilde{a}(h), u_m - h)_{D^T}.$$

Employing (52), we get

$$0 \leqslant \limsup X_m \leqslant \|\beta_1(u(0))\|_{L_1(\Omega)} - \|\beta_1(u(T))\|_{L_1(\Omega)} - (\chi, h)_{D^T} - (\tilde{a}(h), u-h)_{D^T}.$$

Applying 53), we obtain

$$(\chi - \tilde{a}(h), u - h)_{D^T} \geqslant 0.$$

We let $h = u - \lambda\omega$, $\lambda > 0$, $\omega \in V(D^T)$, then

$$\lambda(\chi - \tilde{a}(u - \lambda\omega), \omega)_{D^T} \geqslant 0.$$

Letting $\lambda \to 0$, we have $(\chi - \tilde{a}(u), \omega) \geqslant 0$, $\forall \omega$. Hence, $\chi = \tilde{a}(u)$.

For further using we write (53) as

$$\|\beta_1(u(T))\|_{L_1(\Omega)} + \sum_{i=1}^n (a_{p_i}(x, \nabla u), u_{x_i})_{D^T} = \|\beta_1(u(0))\|_{L_1(\Omega)}. \tag{55}$$

5. Proof of Lemmata 2–4

We denote by $\{u > b\}$ the set $\{(t,x) \in D^T | u(t,x) > b\}$. We note that

$$\text{mes}\{u > b\} < \frac{c}{G(b^2)}, \ u \in L_{G_2}(D^T). \tag{56}$$

Indeed,

$$c > \int\limits_{D^T} G(u^2) dxdt > \int\limits_{D^T \cap \{u > b\}} G(b^2) dxdt = G(b^2) \text{mes}\{u > b\}$$

that yields inequality (56).

For a domain located along the axis Ox_1 let us prove the following inequality

$$\int\limits_{\Omega^r} B_1(v(x)) dx \leqslant \int\limits_{\Omega^r} B_1(r v_{x_1}(x)) dx, \ v \in C_0^\infty(\Omega). \tag{57}$$

Let $f(x_1) \in C[0,r]$, $f(0)=0$. We employ Newton-Leibniz formula to get

$$|f(x_1)| = |\int_0^{x_1} f'(x_1)dx_1| \leqslant \int_0^r |f'(x_1)|dx_1, \quad x_1 \in [0,r].$$

Now we apply Jensen integral inequality (see [26, Ch. 2, Sect. 8.2, Ineq. (8.2)]), then

$$B_1\left(\frac{f(x_1)}{r}\right) \leqslant B_1\left(\frac{\int_0^r |f'(x_1)|dx_1}{r}\right) \leqslant \frac{1}{r}\int_0^r B_1(f'(x_1))dx_1.$$

We integrate the latter inequality w.r.t. x_1

$$\int_0^r B_1\left(\frac{f(x_1)}{r}\right)dx_1 \leqslant \int_0^r B_1(f'(x_1))dx_1.$$

Then, substituting $f(x_1) = rv(x)$ and integrating w.r.t. $x' = \{x_2, \dots, x_n\}$, we obtain (57).

Proof of Lemmata2, 3. Let us show that if $u_0(x) \leqslant b$ for a.e. $x \in \Omega$, then (22) holds true. We let $u^{(b)}(t,x) = \max(u(t,x)-b, 0)$ and employ identity (49) for $\varphi = u^{(b)}(t,x)\xi(x)$, where $\xi(x)$ is a Lipschitz compactly supported function

$$\big((\beta(x,u))_t, u^{(b)}(t,x)\xi(x)\big)_{D^T} - \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} a_{p_i}(x,\nabla u), u^{(b)}(t,x)\xi(x)\right)_{D^T} = 0,$$

$$\begin{aligned}\big((\beta(x,u))_t, u^{(b)}(t,x)\xi(x)\big)_{D^T} &+ \sum_{i=1}^n \big(a_{p_i}(x,\nabla u), u^{(b)}_{x_i}(t,x)\xi(x)\big)_{D^T} \\ &+ \sum_{i=1}^n \big(a_{p_i}(x,\nabla u), u^{(b)}(t,x)\xi_{x_i}(x)\big)_{D^T} = 0.\end{aligned} \tag{58}$$

We choose $\xi = \eta(x_1)$, where

$$\eta(\rho) = \begin{cases} 1, & \text{as} \quad \rho < r, \\ 0, & \text{as} \quad \rho > R, \\ \dfrac{R-\rho}{R-r}, & \text{as} \quad \rho \in [r,R]. \end{cases}$$

Then $|\xi_{x_1}| \leqslant \frac{1}{R-r}$. We note that $u^{(b)}(0,x) = u_0^{(b)}(x) = 0$ for a.e. $x \in \Omega$. We then estimate the integrals involved in identity (58) by condition (9):

$$\sum_{i=1}^n \int_{D^T} a_{p_i}(x,\nabla u)u^{(b)}_{x_i}(t,x)\xi(x)dxdt \geqslant \sum_{i=1}^n \int_{D^T\cap\{u>b\}} B_i(u_{x_i})\xi(x)dxdt. \tag{59}$$

Now we transform the first integral in (58):

$$\begin{aligned} I_1 &= \int_{D^T} \beta_t(x,u)u^{(b)}(t,x)\xi(x)dxdt = \int_{D^T\cap\{u>b\}} \beta'_u(x,u)u_t(t,x)u^{(b)}(t,x)\xi(x)dxdt \\ &= \int_{D^T\cap\{u>b\}} \beta'_u(x,b+u^{(b)})(u^{(b)})_t u^{(b)}\xi(x)dxdt. \end{aligned}$$

We let $h(x,y)=\int\limits_0^y \beta'_u(x,b+v)vdv$, then $h'_y(x,u^{(b)})\geqslant 0$, since $\beta'_u\geqslant 0$. Hence, integral I_1 casts into the form

$$I_1=\int\limits_\Omega \xi(x)h(x,u^{(b)}(t,x))|_0^T dx=\int\limits_\Omega \xi(x)u^{(b)}h'_y(x,\theta(x)u^{(b)})|_{t=T}dx\geqslant 0, \qquad (60)$$

where $0<\theta(x)<1$.

In the case of a bounded domain Ω we choose r so that it is contained in the ball of radius r. Then $\xi_{x_i}=0$ in D^T and by (58), (59) we obtain the inequality

$$\sum_{i=1}^n \int\limits_{D^T\cap\{u>b\}} B_i(u_{x_i})\xi(x)dxdt\leqslant 0.$$

It yields

$$B_i(u_{x_i})=0,\ i=1,2,\ldots,n, \qquad (61)$$

for a.e. $t,x\in D^T\cap\{u>b\}\cap\{x_1<r\}$. Applying inequality (57) to function $u^{(b)}(t,x)$, we find

$$\int\limits_{\Omega^r} B_1(u^{(b)}(t,x))dx\leqslant\int\limits_{\Omega^r} B_1(ru^{(b)}_{x_1}(t,x))dx=0,\qquad \text{for a.e.}\qquad t\in(0,T).$$

Therefore, $u^{(b)}(t,x)=0$ for a.e. $x\in\Omega^r$, $t\in(0,T)$.

To estimate the latter integral in (58) in the case of an unbounded domain Ω, we employ sequentially inequalities (6) and (11):

$$\begin{aligned}\int\limits_{D^T} a_{p_1}(x,\nabla u)u^{(b)}(t,x)\xi_{x_1}(x)dxdt &\leqslant \frac{1}{R-r}\int\limits_{D^T}|a_{p_1}(x,\nabla u)u^{(b)}(t,x)|dxdt\\ &\leqslant\frac{1}{R-r}\int\limits_{D^T}\left(\overline{B}_1(a_{p_1}(x,\nabla u))+B_1(u^{(b)}(t,x))\right)dxdt\\ &\leqslant\frac{1}{R-r}\int\limits_{D^T}\left(c\sum_{i=1}^n B_i(u_{x_i})+B_1(u^{(b)}(t,x))\right)dxdt.\end{aligned} \qquad (62)$$

Taking into consideration (58)–(60), (62), we get

$$\sum_{i=1}^n\int\limits_{D^T\cap\{u>b\}} B_i(u_{x_i}(t,x))\xi(x)dxdt\leqslant\frac{c}{R-r}\int\limits_{D^T}\left(\sum_{i=1}^n B_i(u_{x_i})+B_1(u^{(b)})\right)dxdt. \qquad (63)$$

Let us show that the integral $\int\limits_{D^T} B_1(u^{(b)}(t,x))dxdt$ is bounded. We employ conditions (21),(5), and (20)

$$\begin{aligned}\int\limits_{D^T} B_1(u^{(b)})dxdt &\leqslant \int\limits_{D^T\cap\{u^{(b)}>s_0\}} B^*(ku^{(b)})dxdt+\int\limits_{D^T\cap\{u>b\}} B_1(s_0)dxdt\\ &\leqslant\int\limits_{D^T} B^*\left(k\|u^{(b)}\|_{B^*,D^T}\frac{u^{(b)}}{\|u^{(b)}\|_{B^*,D^T}}\right)dxdt+B_1(s_0)\mathrm{mes}\{u>b\}\\ &\leqslant k^*(k\|u^{(b)}\|_{B^*,D^T})^m+c_3\leqslant k^*\left(Ck\sum_{i=1}^n\|u^{(b)}_{x_i}\|_{\widetilde{B}_i,D^T}\right)^m+c_3,\end{aligned} \qquad (64)$$

where k, s_0 is taken from definition (7), and k^*, m come from definition (5).

Let us show that

$$\sum_{i=1}^{n} \|u^{(b)}_{x_i}\|_{\widetilde{B}_i,D^T} \leqslant c_4. \tag{65}$$

Applying (56), as well as inequality (15), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \|u^{(b)}_{x_i}\|_{\widetilde{B}_i,D^T} &\leqslant n + \int\limits_{D^T\cap\{u>b\}} \sum_{i=1}^{n} \widetilde{B}_i(u_{x_i})dxdt \\
&= n + \int\limits_{D^T\cap\{|u_{x_i}|>1\}\cap\{u>b\}} \sum_{i=1}^{n} \widetilde{B}_i(u_{x_i})dxdt + \int\limits_{D^T\cap\{|u_{x_i}|\leqslant 1\}\cap\{u>b\}} \sum_{i=1}^{n} \widetilde{B}_i(u_{x_i})dxdt \\
&\leqslant n + \int\limits_{D^T\cap\{|u_{x_i}|>1\}} \sum_{i=1}^{n} B_i(u_{x_i})dxdt + \int\limits_{D^T\cap\{u>b\}} \sum_{i=1}^{n} B_i(1)dxdt \leqslant c_4.
\end{aligned}$$

The boundedness of the integral $\int\limits_{D^T} B_1(u^{(b)}(t,x))dxdt$ is proven.

Hence, the right hand side of (63) tends to zero as $R \to \infty$. Thus, (61) is valid for an unbounded domain Ω as well. Then $u^{(b)}(t,x) = 0$ a.e. $(0,T)\times\Omega^r$. Since $r > 0$, $T > 0$ are arbitrary, it implies that $u(t,x) \leqslant b$ for a.e. $(t,x) \in D$. □

Proof of Lemma 4. We take an arbitrary $b > 0$ and employ (20) and (65) to obtain

$$\|u^{(b)}\|_{\infty,D^T} \leqslant C\sum_{i=1}^{n} \|u^{(b)}_{x_i}\|_{\widetilde{B}_i,D^T} \leqslant c_4.$$

Since $u(t,x) \leqslant b + u^{(b)}(t,x)$, inequality (23) is valid. □

6. Proof of Theorem 2

Suppose that domain Ω is bounded. Let us establish the lower estimates for the decay rate of solution to problem (1)–(3) as $t \to \infty$.

We introduce the notations

$$e_m(t) = e(t) = \int\limits_{\Omega} \left(\beta_1(x,u_m(t,x)) + \frac{u_m^2(t,x)}{2b_m}\right) dx,$$

$$h(t) = \int\limits_{\Omega} a(x,\nabla u_m)dx,$$

omitting the subscript m if it is possible. It follows from (42) that

$$e'(t) = -\int\limits_{\Omega} \sum_{i=1}^{n} a_{p_i}(x,\nabla u_m)u_{mx_i}dx. \tag{66}$$

By (44) we have

$$-h'(t) = \int\limits_{\Omega} \left(\frac{1}{b_m} + \beta'_u(x,u_m)\right) u_{mt}^2 dx.$$

Thus,

$$(e'(t))^2 = \left(\int\limits_\Omega \left(\beta'_{1u}(x,u_m)(u_m)_t + \frac{u_m(t)u_{mt}(t)}{b_m} \right) dx \right)^2$$
$$\leqslant \left(\|(u_m)_t(\beta'_u(x,u_m))^{\frac{1}{2}}\|_2 \|u_m(\beta'_u(x,u_m))^{\frac{1}{2}}\|_2 + \frac{\|u_m(t)\|_2\|u_{mt}(t)\|_2}{b_m} \right)^2 .$$

We apply Cauchy-Schwarz inequality for the scalar product in $\mathbb{R}_2$ and employ condition (12). Then

$$(e'(t))^2 \leqslant \int\limits_\Omega \left(\beta'_u(x,u_m)((u_m)_t)^2 + \frac{u^2_{mt}(t)}{b_m} \right) dx \int\limits_\Omega \left(\beta'_u(x,u_m)u_m^2 + \frac{u^2_m(t)}{b_m} \right) dx$$
$$\leqslant -\overline{\alpha} h'(t)e(t), \quad \overline{\alpha} = \max(\alpha, 2).$$

By means of (66) we rewrite the latter as

$$e'(t) \left(\int\limits_\Omega \sum_{i=1}^n a_{p_i}(x, \nabla u_m) u_{mx_i} dx \right) \geqslant \overline{\alpha} h'(t) e(t).$$

By the left inequality in (10) and by (9) it yields

$$\frac{e'(t)}{e(t)} \geqslant \frac{\overline{\alpha} h'(t)}{h(t)} \frac{h(t)}{\int\limits_\Omega \sum_{i=1}^n a_{p_i}(x,\nabla u_m)u_{mx_i} dx} \geqslant \overline{\alpha}\Gamma \frac{h'(t)}{h(t)},$$

or

$$\gamma \frac{e'(t)}{e(t)} \geqslant \frac{h'(t)}{h(t)}, \quad \text{where} \quad \gamma = \frac{1}{\overline{\alpha}\Gamma}.$$

After the integration we have

$$h(t) \leqslant \frac{h(0)e^\gamma(t)}{e^\gamma(0)}.$$

Then, in view of (66) and condition (10),

$$e'(t) \geqslant -h(t)/\delta \geqslant -\frac{h(0)e^\gamma(t)}{\delta e^\gamma(0)},$$

or

$$\frac{e'}{e^\gamma} \geqslant -\frac{h(0)}{\delta e^\gamma(0)}.$$

It yields

$$e^{1-\gamma}(t) - e^{1-\gamma}(0) \leqslant (\gamma - 1)\frac{h(0)t}{\delta e^\gamma(0)} \qquad \text{for the case} \quad \gamma > 1;$$
$$e^{1-\gamma}(t) - e^{1-\gamma}(0) \geqslant -(1 - \gamma)\frac{h(0)t}{\delta e^\gamma(0)} \qquad \text{for the case} \quad \gamma < 1.$$

Thus, we obtain

$$e(t) \geqslant e(0) \left(1 + (\gamma - 1)\frac{h(0)t}{\delta e(0)} \right)^{\frac{1}{1-\gamma}}, \qquad \text{for} \quad \gamma > 1; \tag{67}$$

$$e(t) \geqslant e(0) \left(1 - (1 - \gamma)\frac{h(0)t}{\delta e(0)} \right)^{\frac{1}{1-\gamma}}, \qquad \text{for } \gamma < 1, \quad t \in \left[0, \frac{\delta e(0)}{(1-\gamma)h(0)} \right). \tag{68}$$

Let us prove the passage to the limit

$$\int\limits_{\Omega} \beta_1(x, u_m)dx \to \int\limits_{\Omega} \beta_1(x, u)dx. \tag{69}$$

If integral (19) converges, by Lemma 4, $|u_m| \leqslant c$. Then Lebesgue's dominated convergence theorem allows us to pass to the limit as in (69). Assume now that integral (19) diverges and condition (24) is obeyed. By Egorov's theorem, the convergence $u_m(t, x) \to u(t, x)$ for a.e. $x \in \Omega$ implies the uniform convergence on the set $\Omega_\delta \subset \Omega$, mes $\Omega/\Omega_\delta < \delta$. If for sufficiently large m the inequality

$$|u_m(t, x) - u(t, x)| < \varepsilon,\ x \in \Omega_\delta$$

holds true, then

$$\beta_1(x, u_m) \leqslant c_1 G(u_m^2(t, x)) \leqslant c_1 G((|u(t, x)| + \varepsilon)^2).$$

This is why the Lebesgue's dominated convergence theorem yields $\int\limits_{\Omega_\delta} \beta_1(x, u_m)dx \to \int\limits_{\Omega_\delta} \beta_1(x, u)dx$. Then

$$\begin{aligned} I_{m,\delta} = \int\limits_{\Omega/\Omega_\delta} |\beta_1(x, u_m)|dx &\leqslant \|\beta_1(x, u_m)\|_{L_q(\Omega)} \|1\|_{L_{\overline{q}}(\Omega/\Omega_\delta)} \\ &\leqslant \delta^{1/\overline{q}} \|\beta_1(x, u_m)\|_{L_q(\Omega)}. \end{aligned}$$

Employing condition (24), as well as the assertions (64) and (65), we obtain

$$I_{m,\delta} \leqslant c\delta^{1/\overline{q}} \left(\int\limits_{\Omega} (B^*(u_m) + 1)dx \right)^{1/q} \leqslant \overline{c}\delta^{1/\overline{q}}.$$

Now it is easy to complete the proof of (69).

The functions

$$u_m(t) = \sum_{k=1}^{n} c_{mk}(t)\omega_k$$

belong to the linear space of functions $\omega_1, \omega_2, \ldots, \omega_m$. In a finite-dimensional space all the norms are equivalent and hence

$$\int\limits_{\Omega} u_m^2(t)dx \leqslant c_m \|u_m(t)\|^2_{L_{G_2}(\Omega)} \leqslant \widetilde{c_m}, \forall t > 0.$$

We choose numbers b_m so that $\widetilde{c_m} \leqslant b_m/m$. Then employing (69), by means of formula (13) we obtain

$$e_m(t) \to \int\limits_{\Omega} \beta_1(x, u)dx \leqslant c_1 \int\limits_{\Omega} G(u^2(t))dx.$$

By the passage to the limit as $m \to \infty$ in (67) and (68), where $e(t) = e_m(t)$, we obtain estimates (25), (26).

6.1. Proof of Theorem 3. Let $\theta(\rho)$, $\rho > 0$, be an absolutely continuous function being one as $\rho \geqslant r$, vanishing as $\rho \leqslant r_0$, being linear as $\rho \in [r_0, 2r_0]$, and satisfying the equation

$$\theta'(\rho) = \lambda\nu(\rho)\theta(\rho), \quad \rho \in (2r_0, r); \tag{70}$$

we shall define constant λ later. Solving this equation, we find

$$\theta(\rho) = \exp\left(-\lambda \int\limits_{\rho}^{r} \nu(t)dt\right), \quad \rho \in (2r_0, r).$$

As $\rho \in (r_0, 2r_0)$, we have

$$\theta'(\rho) = \frac{\theta(2r_0)}{r_0} = \frac{1}{r_0}\exp\left(-\lambda\int\limits_{2r_0}^{r}\nu(t)dt\right), \quad \rho \in (r_0, 2r_0). \tag{71}$$

Let $\xi(x)$ be a Lipschitz non-negative cut-off function. Substituting $\varphi = u\xi$ into (49), we obtain

$$(\beta(x,u)_t, u\xi)_{D^T} + (\chi, u\xi)_{D^T} = 0.$$

We rewrite it as

$$\int\limits_{D^T}\left(\beta'_{1u}(x,u)u_t\xi + \sum_{i=1}^{n} a_{p_i}(x,\nabla u)(u\xi)_{x_i}\right)dxdt = 0.$$

We let $\xi(x) = \theta(x_1)$. Employing (9) and bearing in mind that the supports of ξ and u_0 do not intersect, by the integrating of the first term w.r.t. t and applying (70), (71), we get

$$\begin{aligned}
&\int\limits_{\Omega}\beta_1(x,u(T))\theta(x_1)dx + \int\limits_{D^T}\sum_{i=1}^{n} B_i(u_{x_i})\theta(x_1)dxdt \\
&\leqslant \int\limits_{D^T}|ua_{p_1}(x,\nabla u)\theta'(x_1)|dxdt \leqslant \int\limits_{D^T\cap\{2r_0<x_1<r\}}|ua_{p_1}(x,\nabla u)\lambda\nu(x_1)\theta(x_1)|dxdt \\
&+ \int\limits_{D^T\cap\{r_0<x_1<2r_0\}}\left|ua_{p_1}(x,\nabla u)\frac{\theta(2r_0)}{r_0}\right|dxdt = I_1 + I_2.
\end{aligned} \tag{72}$$

We note that $B(su) \leqslant sB(u)$ as $s \leqslant 1$. Employing then the boundedness of functions ν ($\nu \leqslant \nu_0$) and u ($|u| \leqslant v_0$) (see (32)), by means of (6), (11), (31) we estimate the first integral

$$\begin{aligned}
I_1 &\leqslant \int\limits_{D^T\cap\{2r_0<x_1<r\}}\theta(x_1)\left(\overline{B}_1(\varepsilon a_{p_1}(x,\nabla u)) + B_1(u\nu\frac{\lambda}{\varepsilon})\right)dxdt \\
&\leqslant \int\limits_{D^T\cap\{2r_0<x_1<r\}}\theta(x_1)\left(\varepsilon c\sum_{i=1}^{n} B_i(u_{x_i}) + gB_2(u\nu\frac{\lambda}{\varepsilon})\right)dxdt.
\end{aligned}$$

We choose $\varepsilon = \frac{1}{2c}$ and λ so that $\frac{\lambda}{\varepsilon}\nu_0 v_0 \leqslant 1$ and $\frac{\lambda}{\varepsilon}g \leqslant \frac{1}{2}$. Then employing the definition of function ν, we obtain

$$\begin{aligned}
I_1 \leqslant &\frac{1}{2}\int\limits_{D^T\cap\{2r_0<x_1<r\}}\theta(x_1)\sum_{i=1}^{n} B_i(u_{x_i})dxdt + \frac{1}{2}\int\limits_{0}^{T}dt\int\limits_{2r_0}^{r}dx_1\theta(x_1)\int\limits_{\gamma(x_1)}B_2(u\nu)dx' \\
\leqslant &\frac{1}{2}\int\limits_{D^T\cap\{2r_0<x_1<r\}}\theta(x_1)\sum_{i=1}^{n} B_i(u_{x_i})dxdt + \frac{1}{2}\int\limits_{0}^{T}dt\int\limits_{2r_0}^{r}dx_1\theta(x_1)\int\limits_{\gamma(x_1)}B_2(u_{x_2})dx' \\
\leqslant &\frac{1}{2}\int\limits_{D^T}\theta(x_1)\left(\sum_{i=1}^{n} B_i(u_{x_i}) + B_2(u_{x_2})\right)dxdt.
\end{aligned}$$

For I_2, employing inequality (11), we obtain the estimate

$$I_2 \leqslant \frac{\theta(2r_0)}{r_0} \int\limits_{D^T\cap\{r_0<x_1<2r_0\}} (B_1(u) + \overline{B}_1(a_{p_1}(x,\nabla u)))dxdt$$
$$\leqslant \frac{\theta(2r_0)}{r_0} \int\limits_{D^T\cap\{r_0<x_1<2r_0\}} (B_1(u) + c\sum_{i=1}^{n} B_i(u_{x_i}))dxdt.$$

Then, in view of (57), (5),

$$\int\limits_{D^T\cap\{x_1<2r_0\}} B_1(u)dxdt \leqslant \int\limits_{D^T\cap\{x_1<2r_0\}} B_1(2r_0 u_{x_1})dxdt \leqslant c\int\limits_{D^T} B_1(u_{x_1})dxdt.$$

Employing the estimates for I_1, I_2 in (72), we find

$$\int\limits_{\Omega} \beta_1(x,u(T))\theta(x_1)dx \leqslant \frac{\theta(2r_0)}{r_0}\int\limits_{D^T} c_1\sum_{i=1}^{n} B_i(u_{x_i})dxdt.$$

The boundedness of the latter integral is obtained from (43) by passing to the limit as $m\to\infty$. Since $\theta(x_1) = 1$ as $x_1 \geqslant r$, we arrive at inequality (33).

6.2. Proof of Theorem 4. We choose a positive number $r \geqslant 2r_0$. We introduce the notation

$$\varepsilon(r) = M\exp(-\lambda\int\limits_{2r_0}^{r} \nu(t)dt)$$

and employing (33), we write the relation

$$\Phi(t) \equiv \int\limits_{\Omega} \beta_1(x,u(t,x))dx \leqslant \int\limits_{\Omega^r} \beta_1(x,u(t,x))dx + \varepsilon(r).$$

Let t_r be a point in the interval $(0,\infty)$ such that $\Phi(t_r) = \varepsilon(r)$. If there is no such point, then either $\Phi(t) > \varepsilon(r)$ for each $t>0$ and we let $t_r=\infty$, or $\Phi(t) < \varepsilon(r)$ for each $t\geqslant 0$. In the latter case the desired estimate (74) holds true. It follows from (55) that function $\Phi(t)$ is non-increasing, and thus

$$0 \leqslant \Phi(t) - \varepsilon(r) \leqslant \int\limits_{\Omega^r} \beta_1(x,u(t,x))dx, \quad t\in[0,t_r). \tag{73}$$

Employing condition (34), (27), we write the inequalities

$$\Phi(t)-\varepsilon(r) \leqslant \left(\int\limits_{\Omega^r} \beta_1(x,u(t,x))^q dx\right)^{1/q} (\mathrm{mes}\Omega^r)^{1/\overline{q}} \leqslant \left(c_3\int\limits_{\Omega^r} B_1(u(t,x))dx\right)^{1/q} r^{d/\overline{q}}, \qquad r\geqslant r_0.$$

We employ inequality (57) as well as $\triangle_2$-condition (5), (9), (35), and (55), and obtain

$$\Phi(t)-\varepsilon(r) \leqslant \left(c_3\int\limits_{\Omega^r} B_1(ru_{x_1})dx\right)^{1/q} r^{d/\overline{q}} \leqslant c_4 r^{d/\overline{q}}\left(\int\limits_{\Omega^r} r^m B_1(u_{x_1})dx\right)^{1/q}.$$

We shall assume that numbers μ, r_0 are chosen so that the inequality $c_4 r^{d/\overline{q}+m/q} \leqslant r^{\mu/q}$ holds as $r\geqslant r_0$. Then

$$\Phi(t)-\varepsilon(r) \leqslant r^{\mu/q}\left(\int\limits_{\Omega}\sum_{i=1}^{n} a_{p_i}(x,\nabla u)u_{x_i}dx\right)^{1/q} = r^{\mu/q}\left(-\frac{d}{dt}\int\limits_{\Omega}\beta_1(x,u(t))dx\right)^{1/q}.$$

Solving this differential inequality, we find

$$\Phi(t) \leqslant \varepsilon(r) + \left(\frac{r^{\mu}}{(q-1)t}\right)^{\frac{1}{q-1}}. \tag{74}$$

The latter inequality is valid for each $r \geqslant 2r_0$. Letting $r = r(t)$ (see (36)), we obtain (37). The proof is complete.

7. Examples

We adduce examples of equations satisfying conditions $(4), (9) - (14), (24), (34)$.

7.1. Example 1. We introduce the following notation

$$t^{[a,b]} = \begin{cases} |t|^a, & \text{as } |t| < 1, \\ |t|^b, & \text{as } |t| \geqslant 1. \end{cases}$$

Let $n = 2$. We choose N-functions $B_1(s)$, $B_2(s)$, $G(s)$, and functions $\beta(x,u)$, $a(x,p)$ as follows

$$B_1(s) = s^{[2,5/2]},\ B_2(s) = s^{[5/4,3/2]},\ G(s) = s^{[5/4,11/10]},\ \sum_{i=1}^{2} a_{p_i}(x,p)p_i = B_1(p_1) + B_2(p_2)\frac{2+|x|}{1+|x|}.$$

It is clear that the dependence on x can appear in function $\beta(x,u)$, but in order to avoid bulky formulae, we restrict ourselves by the simplest example of the dependence on x:

$$\beta(u) = \begin{cases} \frac{5}{3}|u|^{\frac{3}{2}}, & \text{as } |u| < 1, \\ \frac{5}{3} + \frac{11}{6}(|u|^{\frac{6}{5}} - 1), & \text{as } |u| \geqslant 1. \end{cases}$$

By formula (8) we find function

$$\beta_1(u) = u^{[5/2,11/5]}.$$

It is easy to check that these functions satisfy conditions (4),(9)–(14), (24), (31), (34). Then by formula (16) for $\kappa = \frac{3}{2}$ we obtain

$$\widetilde{B}_1(s) = s^{[3/2,5/2]}, \quad \widetilde{B}_2(s) = |s|^{3/2}, \quad h(s) = s^{[1/6,1/30]}.$$

Since integral (19) diverges to infinity, by formula (18) we find

$$(B^*)^{-1}(z) = \begin{cases} 6z^{\frac{1}{6}}, & \text{as } |z| < 1, \\ 30z^{\frac{1}{30}} - 24, & \text{as } |z| \geqslant 1, \end{cases}$$

$$B^*(s) = \begin{cases} (\frac{s}{6})^6, & \text{as } |s| < 6, \\ \left(\frac{|s|+24}{30}\right)^{30}, & \text{as } |s| \geqslant 6. \end{cases}$$

In view of conditions (10), (12) one can see easily that it is possible to take $\Gamma = \frac{8}{5}$, $\alpha = \frac{5}{2}$. Thus, by (26) we obtain the estimate in the case of a bounded domain Ω

$$\int\limits_{\Omega} G(u^2(t,x))dx \geqslant \int\limits_{\Omega} G(u_0^2(x))dx(1-Ct)^{\frac{4}{3}},\ \text{as } t \leqslant 1/C.$$

Now as Ω we choose the following domain

$$\Omega(f) = \{x | x_1 > 0,\ -x_1^{\frac{1}{2}} + f(x_1) \leqslant x_2 \leqslant x_1^{\frac{1}{2}} + f(x_1)\},$$

where f is an arbitrary continuous function. Then mes $\Omega^r(f) = \frac{4}{3}r^{\frac{3}{2}} \leqslant r^2$, $r \geqslant 2$. By (28) we find $\nu(r) = \frac{1}{2\sqrt{r}}$. It is easy to make sure that domain $\Omega(f)$ satisfies conditions (29), (38). Then in condition (34) we can choose $q = \frac{25}{22}$. And then by (39) we find the upper estimates

$$\int\limits_{\Omega(f)} \beta_1(u(t,x))dx \leqslant Ct^{\frac{-11}{3}}.$$

7.2. Example 2. Let $n = 2$. We define N-functions $B_1(s)$, $B_2(s)$, $G(s)$ and functions $\beta(x,u)$, $a(x,p)$ as

$$B_1(s) = s^{[9/2,6]}, \quad B_2(s) = s^{[17/4,6]}, \quad G(s) = s^{[3/2,2]}, \quad \sum_{i=1}^{2} a_{p_i}(p)p_i = B_1(p_1) + B_2(p_2);$$

$$\beta(u) = \begin{cases} \dfrac{3}{2}|u|^2, & \text{as } |u| < 1, \\ \dfrac{3}{2} + \dfrac{4}{3}(|u|^3 - 1), & \text{as } |u| \geqslant 1. \end{cases}$$

By formula (8) we get

$$\beta_1(u) = u^{[3,4]}.$$

It is easy to make sure that these functions satisfy conditions (4), (9)–(14), (31), (34). By formula (16) for $\kappa = \frac{3}{2}$ we obtain

$$\widetilde{B}_1(s) = s^{[3/2,6]}, \quad \widetilde{B}_2(s) = s^{[3/2,6]}, \quad h(s) = s^{[1/6,-1/3]}.$$

Hence, integral (19) converges.

Due to conditions (10), (12) one can easily make sure that one can take $\Gamma = \frac{4}{17}$, $\alpha = 4$. Then from (25) we get the estimate in the case of a bounded domain Ω

$$\int\limits_{\Omega} G(u^2(t,x))dx \geqslant \int\limits_{\Omega} G(u_0^2(x))dx(1 + Ct)^{-16}, \text{ as } t \geqslant 0.$$

In condition (34) we can choose $q = \frac{3}{2}$. Then (39) implies the upper estimate

$$\int\limits_{\Omega(f)} \beta_1(u(t,x))dx \leqslant Ct^{-1}.$$

Remark. Since for $|u| < 1$ and $|u| \geqslant 1$ the functions have different growth indices, the power upper and lower estimates have also different exponents.

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