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ON STRUCTURE OF INTEGRALS FOR SYSTEMS OF DISCRETE EQUATIONS

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Abstract. In the work we describe the structure of integrals of systems of discrete equations. We consider an example of discrete Toda chain corresponding to Lie algebra of series A_2.

Keywords: system of discrete equations, complete set of integrals.

Mathematics Subject Classification: 35Q53, 37K10

1. Introduction

The present paper is devoted to describing the structure of integrals for the general system of equations on a square lattice

$$u^i_{1,1} = f^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \mathbf{u}_{0,1}), \quad i = 1, 2, \dots, N, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(n, m)$ is a vector-function of two discrete arguments defined on $\mathbb{C}^N$: $\mathbf{u} = (u^1, u^2, \dots, u^N)^T$. The subscripts indicates the shifts of the arguments: $\mathbf{u}_{p,q} = D^p_n D^q_m \mathbf{u}(n, m) = \mathbf{u}(n+p, m+q)$, where D_m, D_n are the shift operators. The set of dynamical variables comprises variables $\mathbf{u}$ and their shifts $\mathbf{u}_{p,0}$, $\mathbf{u}_{0,q}$, where $p, q \in \mathbb{Z}$. We also assume that system of equations (1) is solvable with respect to the variables $\mathbf{u}_{-1,-1}$, $\mathbf{u}_{-1,1}$, $\mathbf{u}_{1,-1}$.

Definition 1. *i) Function $I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \mathbf{u}_{2,0}, \dots, \mathbf{u}_{k,0})$, $\sum_{i=1}^N \left(\frac{\partial I}{\partial u^i_{k,0}}\right)^2 \neq 0$ (function $F(n,$ m, $\mathbf{u}$, $\mathbf{u}_{0,1}$, $\mathbf{u}_{0,2}, \dots,$ $\mathbf{u}_{0,l})$, $\sum_{i=1}^N \left(\frac{\partial F}{\partial u^i_{0,l}}\right)^2 \neq 0$) is called m-integral (respectively, n-integral) for system of equations (1), if the identity $D_m I = I$ (or $D_n F = F$) holds true.*

ii) Integrals $I = I(n)$ ($F = F(m)$) are called trivial.

iii) A system of equation possessing N non-trivial independent integrals in each direction is called Darboux integrable.

iv) Integrals are called independent if none of them is expressed in terms of the others and their shifts.

Issues on Darboux integrability for hyperbolic differential equations and systems of such equations are studied actively during many decades [1], [2]. It was proven in works [3], [4] that a hyperbolic system is Darboux integrable if and only if its characteristic Lie ring has a finite dimension (see also [5]). In works [6], [7] the notion of characteristic ring was introduced for discrete equations and by means of this notion the classification of Darboux integrable differential-difference equations $u_{1,x} = f(u, u_1, u_x)$ was made. In works [8], [9] there was studied the problem on constructing the complete set of integrals for a hyperbolic system. The structure of the integrals for differential equations was studied in [5]. In works [10], [11] there was discussed the connection between Darboux integrability and breaking of the series of Laplace

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invariants for the linearized equation. Works [12], [13] were devoted to studying Darboux integrable discrete equations in the framework of symmetry approach.

1.1. Conditions for completeness of set of integrals for system of discrete equations. The order of the integral $I = I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \mathbf{u}_{2,0}, \dots, \mathbf{u}_{k,0})$ for system of discrete equations (1) is number k. It is assumed that there exist i, j such that $\frac{\partial I}{\partial u^i} \neq 0$, $\frac{\partial I}{\partial u^j_{k,0}} \neq 0$.

Suppose system (1) has N independent m-integrals $I^1, I^2, \dots, I^N$ of minimal orders $k_1 \leqslant k_2 \leqslant \dots \leqslant k_N$, $(ordI^i = k_i)$. It means the validity of the following conditions

1. $D_m I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k,0}) = I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k,0}), k \leqslant k_1$ if and only if $I = I(n)$;
2. $D_m I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k,0}) = I(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k,0}), k \in [k_{j-1}, k_j), j \in [2; N]$ if and only if I is a function depending only on n, $I^1, I^2, \dots, I^{j-1}$ and their shifts with respect to n.

We have

Lemma 1. *Suppose that system of equations (1) has N independent m-integrals of minimal order k. Then any other m-integral is a function of n, integrals $I^1, I^2, \dots, I^N$, and their shifts with respect to n.*

Proof. Assume that $\frac{\partial I^1}{\partial u^1_{k,0}} \neq 0$. If it is not true, we can achieve it by re-denoting the variables. We express variable $u^1_{k,0}$ in terms of other variables and integral I^1:

$$u^1_{k,0} = g^1(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}, u^2_{k,0}, \dots, u^N_{k,0}, I^1). \tag{2}$$

Then we pass from variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\dots$, $\mathbf{u}_{k,0}$ to new variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\dots$, $\mathbf{u}_{k-1,0}$, $u^2_{k,0}$, $\dots$, $u^N_{k,0}$, I^1. As a result, all the integrals I^s, $s = 2, 3, \dots, N$, will be rewritten as $I^s = I^s(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}, u^2_{k,0}, \dots, u^N_{k,0}, I^1)$. Hence, we have $\sum_{i=2}^N \left(\frac{\partial I^2}{\partial u^i_{k,0}}\right)^2 \neq 0$. Indeed, otherwise in the vicinity of the point $I^1 = i_0$, where i_0 is a complex number, the second given integral can be expanded into the power series

$$I^2 = \sum_{j=0}^{\infty} I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0})(I^1 - i_0)^j. \tag{3}$$

Since I^1, I^2 are m-integrals, the coefficients of series (3) are also m-integrals of order not exceeding $k-1$. Indeed, we have

$$D_m I^2 = D_m \left(\sum_{j=0}^{\infty} I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0})(I^1 - i_0)^j \right),$$

$$D_m I^2 = \sum_{j=0}^{\infty} D_m I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}) D_m (I^1 - i_0)^j,$$

$$I^2 = \sum_{j=0}^{\infty} D_m I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0})(I^1 - i_0)^j. \tag{4}$$

Comparing series (3) and (4) and employing the uniqueness of Taylor series, we obtain the identity

$$D_m I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}) = I_j(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}),$$

and integral I^2 is a function of variables n, I^1 that contradicts to the independence of considered set of integrals.

By analogy with the above arguments we can assume that $\frac{\partial I^2}{\partial u^2_{k,0}} \neq 0$. Then we express variable $u^2_{k,0}$ in terms of other variables and integrals I^1, I^2

$$u^2_{k,0} = g^2(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{k-1,0}, u^3_{k,0}, \dots, u^N_{k,0}, I^1, I^2). \tag{5}$$

We pass from variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\ldots$, $\mathbf{u}_{k-1,0}$, $u^2_{k,0}$, $\ldots$, $u^N_{k,0}$, I^1 to variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\ldots$, $\mathbf{u}_{k-1,0}$, $u^3_{k,0}$, $\ldots$, $u^N_{k,0}$, I^1, I^2 in integrals I^s, $s = 3, 4, \ldots, N$.

Arguing as above for the case of variable $u^3_{k,0}$ and integral I^3, one can show that $\sum_{i=3}^N \left(\frac{\partial I^3}{\partial u^i_{k,0}}\right)^2 \neq 0$. Assuming $\frac{\partial I^3}{\partial u^3_{k,0}} \neq 0$, we find

$$u^3_{k,0} = g^3(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{k-1,0}, u^4_{k,0}, \ldots, u^p_{k,0}, I^1, I^2, I^3). \tag{6}$$

On the i-th step we obtain the relations

$$u^i_{k,0} = g^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{k-1,0}, u^{i+1}_{k,0}, \ldots, u^N_{k,0}, I^1, \ldots, I^i), i \leqslant N. \tag{7}$$

As a result of made transformations we arrive at the formulae

$$u^i_{k,0} = \tilde{g}^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{k-1,0}, I^1, \ldots, I^N), i = 1, 2, \ldots, N. \tag{8}$$

Applying the shift operator D^j_n to the latter relations, we get

$$\begin{aligned} &u^i_{k+j,0} = \phi^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{k-1,0}, I^1, \ldots, I^N, \\ &D_n I^1, \ldots, D_n I^N, \ldots, D^j_n I^1, \ldots, D^j_n I^N), \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots. \end{aligned} \tag{9}$$

Let G be an arbitrary m-integral of order $q \geq k$:

$$G = G(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{q,0}).$$

We pass from variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\ldots$, $\mathbf{u}_{q,0}$ to variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\ldots$, $\mathbf{u}_{k-1,0}$, I^1, $\ldots$, I^N, $D_n I^1$, $\ldots$, $D_n I^N$, $\ldots$, $D^{q-k}_n I^1$, $\ldots$, $D^{q-k}_n I^N$ and in the vicinity of the point

$$\left(\xi^{(1)}, \xi^{(1)}_{1,0}, \ldots, \xi^{(1)}_{q-k,0}, \xi^{(2)}, \xi^{(2)}_{1,0}, \ldots, \xi^{(2)}_{q-k,0}, \ldots, \xi^{(N)}, \xi^{(N)}_{1,0}, \ldots, \xi^{(N)}_{q-k,0}, \right)$$

we expand function G into the power series

$$\begin{aligned} G = &\sum_{\alpha[1,0]+\alpha[1,1]+\cdots+\alpha[1,q-k]=0}^{\infty} \sum_{\alpha[2,0]+\alpha[2,1]+\cdots+\alpha[2,q-k]=0}^{\infty} \cdots \sum_{\alpha[N,0]+\alpha[N,1]+\cdots+\alpha[N,q-k]=0}^{\infty} \\ &G_{\alpha^1\alpha^2\ldots\alpha^N}(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \ldots, \mathbf{u}_{k-1,0}) \left(I^1 - \xi^{(1)}\right)^{\alpha[1,0]} \cdot \left(D_n I^1 - \xi^{(1)}_1\right)^{\alpha[1,1]} \cdots \\ &\left(D^{q-k}_n I^1 - \xi^{(1)}_{q-k}\right)^{\alpha[1,q-k]} \cdot \left(I^2 - \xi^{(2)}\right)^{\alpha[2,0]} \cdot \left(D_n I^2 - \xi^{(2)}_1\right)^{\alpha[2,1]} \cdots \\ &\left(D^{q-k}_n I^2 - \xi^{(2)}_{q-k}\right)^{\alpha[2,q-k]} \cdot \left(I^N - \xi^{(N)}\right)^{\alpha[N,0]} \cdots \left(D^{q-k}_n I^N - \xi^{(N)}_{q-k}\right)^{\alpha[N,q-k]}, \end{aligned}$$

where $\alpha^j = (\alpha[k,0], \alpha[j,1], \alpha[j,2], \ldots, \alpha[j,q-k])$, $j = 1, 2, \ldots, N$. Since functions G, I^1, $\ldots$, I^N, $D_n I^1$, $\ldots$, $D_n I^N$, $\ldots$, $D^{q-k}_n I^1$, $\ldots$, $D^{q-k}_n I^N$ are m-integrals, all the coefficients of this series are also m-integrals of order not exceeding $k-1$. Since number k is the minimal order of the integral, $G_{\alpha^1\alpha^2\ldots\alpha^N}$ is a function of variable n.

As a result we obtain that an arbitrary m-integral G is expressed in terms of the given integrals and their shifts with respect to n

$$G = G(n, I^1, \ldots, I^N, D_n I^1, \ldots, D_n I^N, \ldots, D^{q-k}_n I^1, \ldots, D^{q-k}_n I^N).$$

The proof is complete. □

In what follows we prove a theorem on structure of integrals for system (1) in the general situation.

Theorem 1. *Assume that system of equations (1) possesses p independent m-integrals I^j, $j = 1, 2, \ldots, N$, of minimal orders $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_N$. Then any other m-integrals is a function of variables n, integrals I^1, $\ldots$, I^N and their shifts with respect to n.*

Proof. Denote $K = k_N$. We reduce the given integrals I^j, $j = 1, 2, \dots, N$, to the same order K by applying the shift operator $D_n^{K-k_j}$ and adding I^j. We get

$$D_n^{K-k_j} I^j + I^j = \tilde{I}^j(n, m, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_{K-k_j,0}, \mathbf{u}_{K-k_j+1,0}, \dots, \mathbf{u}_{K,0}), \quad j = 1, 2, \dots, N-1. \quad (10)$$

Thus, we have N integrals $\tilde{I}^1$, $\tilde{I}^2$, $\dots$, $\tilde{I}^{N-1}$, I^N of the same order K. We note that the constructed integrals are independent. Then arguing as in the proof of Lemma 1, we obtain variables $u^i_{K,0}$, $i = 1, 2, \dots, N$, can be expressed in terms of dynamical variables $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\dots$, $\mathbf{u}_{K-1,0}$ and integrals $\tilde{I}^1$, $\tilde{I}^2$, $\dots$, $\tilde{I}^{N-1}$, I^N

$$u^i_{K,0} = g^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{K-1}, \tilde{I}^1, \tilde{I}^2, \dots, \tilde{I}^{N-1}, I^N), \quad i = 1, 2, \dots, N$$

or, taking into consideration (10),

$$u^i_{K,0} = g^i(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{K-1}, I^1, \dots, I^{N-1}, I^N, D_n^{K-k_1} I^1, \dots, D_n^{K-k_{N-1}} I^{N-1}), \quad i = 1, 2, \dots, N.$$

We apply the shift operator D_n^r to the latter relations and obtain the formual

$$\begin{aligned} u^i_{K+r,0} = \psi^i\big(&n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{K-1,0}, D_n^{K-k_1} I^1, D_n^{K-k_1+1} I^1, \dots, D_n^{K-k_1+r} I^1, \\ &D_n^{K-k_2} I^2, D_n^{K-k_2+1} I^2, \dots, D_n^{K-k_2+r} I^2, \dots, D_n^{K-k_{N-1}} I^{N-1}, D_n^{K-k_{N-1}+1} I^{N-1}, \dots, \\ &D_n^{K-k_{N-1}+r} I^{N-1}, I^N, D_n I^N, \dots, D_m^r I^N\big), \quad i = 1, 2, \dots, N, \quad k = 1, 2, \dots. \end{aligned}$$

Let G be an arbitrary m-integral of order $s \geq K$:

$$G = G(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \mathbf{u}_{2,0}, \dots, \mathbf{u}_{s,0}).$$

Then, as above, we pass from variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\mathbf{u}_{2,0}$, $\dots$, $\mathbf{u}_{s,0}$ to variables n, m, $\mathbf{u}$, $\mathbf{u}_{1,0}$, $\mathbf{u}_{2,0}$, $\dots$, $\mathbf{u}_{K-1,0}$, $D_n^{K-k_1} I^1$, $D_n^{K-k_1+1} I^1$, $\dots$, $D_n^{K-k_1+r} I^1$, $D_n^{K-k_2} I^2$, $D_n^{K-k_2+1} I^2$, $\dots$, $D_n^{K-k_2+r} I^2$, $\dots$, $D_n^{K-k_{N-1}} I^{N-1}$, $D_n^{K-k_{N-1}+1} I^{N-1}$, $\dots$, $D_n^{K-k_{N-1}+r} I^{N-1}$, I^N, $D_n I^N$, $\dots$, $D_m^r I^N$. In the vicinity of the point

$$\begin{aligned} \big(&\xi^{(1)}_{K-k_1,0}, \xi^{(1)}_{K-k_1+1,0}, \dots, \xi^{(1)}_{r-k_1,0}, \xi^{(2)}_{K-k_2,0}, \xi^{(2)}_{K-k_2+1,0}, \dots, \xi^{(2)}_{r-k_2,0}, \dots, \\ &\xi^{(N-1)}_{K-k_{N-1},0}, \xi^{(N-1)}_{K-k_{N-1}+1,0}, \dots, \xi^{(N-1)}_{s-k_{N-1},0}, \xi^{(N)}, \xi^{(N)}_{1,0}, \dots, \xi^{(N)}_{s-K,0}\big) \end{aligned}$$

the latter function G can be represented as the power series

$$\begin{aligned} G = &\sum_{\alpha[1,K-k_1]+\dots+\alpha[1,s-k_1]=0}^{\infty} \dots \sum_{\alpha[N-1,K-k_{N-1}]+\dots+\alpha[N-1,s-k_{N-1}]=0}^{\infty} \sum_{\alpha[N,0]+\alpha[N,1]+\dots+\alpha[N,s-K]=0}^{\infty} \\ &G_{\alpha^1\alpha^2\dots\alpha^N}(n, m, \mathbf{u}, \mathbf{u}_{1,0}, \dots, \mathbf{u}_{K-1,0}) \left(D_n^{K-k_1} I^1 - \xi^{(1)}_{K-k_1}\right)^{\alpha[1,K-k_1]} \\ &\left(D_n^{K-k_1+1} I^1 - \xi^{(1)}_{K-k_1+1}\right)^{\alpha[1,K-k_1+1]} \dots \left(D_n^{s-k_1} I^1 - \xi^{(1)}_{s-k_1}\right)^{\alpha[1,s-k_1]} \dots \\ &\left(D_n^{K-k_{N-1}} I^{N-1} - \xi^{(N-1)}_{K-k_{N-1}}\right)^{\alpha[N-1,K-k_{N-1}]} \left(D_n^{K-k_{N-1}+1} I^{N-1} - \xi^{(N-1)}_{K-k_{N-1}+1}\right)^{\alpha[N-1,K-k_{N-1}+1]} \dots \\ &\left(D_n^{s-k_{N-1}} I^{N-1} - \xi^{(N-1)}_{s-k_{N-1}}\right)^{\alpha[N-1,s-k_{N-1}]} \left(I^N - \xi^{(N)}\right)^{\alpha[N,0]} \left(D_n I^N - \xi^{(N)}_1\right)^{\alpha[N,1]} \dots \\ &\left(D_n^{s-K-1} I^N - \xi^{(N)}_{s-K-1}\right)^{\alpha[N,s-K-1]} \left(D_n^{s-K} I^N - \xi^{(N)}_{s-K}\right)^{\alpha[N,s-K]}. \end{aligned}$$

Here $\alpha^j = (\alpha[k, K-k_j], \alpha[j, K-k_j+1], \dots, \alpha[j, s-k_j])$, $j = 1, 2, \dots, N-1$, $\alpha^N = (\alpha[N, 0], \alpha[N, 1], \dots, \alpha[N, s-K])$. Since G and all the functions $D_n^{K-k_1} I^1$, $D_n^{K-k_1+1} I^1$, $\dots$, $D_n^{s-k_1} I^1$, $D_n^{K-k_2} I^2$, $D_n^{K-k_2+1} I^2$, $\dots$, $D_n^{s-k_2} I^2$, $\dots$, $D_n^{K-k_{N-1}} I^{N-1}$, $D_n^{K-k_{N-1}+1} I^{N-1}$, $\dots$, $D_n^{s-k_{N-1}} I^{N-1}$, I^N, $D_n I^N$, $\dots$, $D_n^{s-K} I^N$ are m-integrals, all the coefficients of this series are to be m-integrals of order not exceeding $K-1$. The definition of minimal order integrals implies that $G_{\alpha^1\alpha^2\dots\alpha^N}$ is a function of variables n, I^1, I^2, $\dots$, I^{N-1}, I^N, and their shifts with respect to n.

Thus, an arbitrary m-integral G read as

$$G = G(n, I^1, \ldots, I^N, D_n I^1, \ldots, D_n I^N, \ldots, D_n^{s-k_1} I^1, \ldots, D_n^{s-k_N} I^N),$$

and thus, integrals I^1, I^2, ..., I^N form the complete basis. The proof is complete. $\square$

1.2. Example of a system of discrete equations with complete sets of integrals. Consider the discrete Toda chain corresponding to simple Lie algebra of series A_2

$$\begin{aligned} a_{0,0}a_{1,1} - a_{1,0}a_{0,1} &= b_{1,0}, \\ b_{0,0}b_{1,1} - b_{1,0}b_{0,1} &= a_{0,1}. \end{aligned} \tag{11}$$

Here $a = a(n,m)$, $b = b(n,m)$ are unknown functions of two discrete variables n, m. In work [14] there was constructed the characteristic Lie algebra L_m with the basis

$$X_1 = \frac{\partial}{\partial a_{0,-1}}, \quad X_2 = \frac{\partial}{\partial b_{0,-1}},$$

$$\begin{aligned} Y_1 = D_m^{-1}\frac{\partial}{\partial a_{0,1}}D_m = \frac{\partial}{\partial a_{0,0}} &+ \left(\frac{a_{1,0}}{a_{0,0}} - \frac{b_{1,0}b_{0,-1}}{a_{0,0}b_{0,0}a_{0,-1}} + \frac{1}{a_{0,-1}b_{0,0}}\right)\frac{\partial}{\partial a_{1,0}} \\ &+ \left(\frac{a_{-1,0}}{a_{0,0}} + \frac{b_{0,-1}}{a_{0,0}a_{0,-1}}\right)\frac{\partial}{\partial a_{-1,0}} + \cdots + \frac{1}{b_{0,-1}}\frac{\partial}{\partial b_{1,0}} - \left(\frac{a_{-1,0}}{a_{0,0}b_{0,-1}} + \frac{1}{a_{0,0}a_{0,-1}}\right)\frac{\partial}{\partial b_{-1,0}} + \ldots, \end{aligned}$$

$$Y_2 = D_m^{-1}\frac{\partial}{\partial b_{0,1}}D_m = \frac{\partial}{\partial b_{0,0}} + \left(\frac{b_{1,0}}{b_{0,0}} - \frac{a_{0,0}}{b_{0,0}b_{0,-1}}\right)\frac{\partial}{\partial b_{1,0}} + \left(\frac{b_{-1,0}}{b_{0,0}} + \frac{a_{-1,0}}{b_{0,0}b_{0,-1}}\right)\frac{\partial}{\partial b_{-1,0}} + \ldots,$$

$P_1 = [X_1, Y_1]$, $P_2 = [X_2, Y_1]$, $P_3 = [X_2, Y_2]$. In each of these operators we neglect the terms containing the variables with negative shifts and we solve the system of equations

$$\begin{aligned} &X_1(F) = 0, \quad X_2(F) = 0, \quad Y_1(F) = 0, \quad Y_2(F) = 0, \\ &P_1(F) = 0, \quad P_2(F) = 0, \quad P_3(F) = 0. \end{aligned} \tag{12}$$

To solve system (12), it is sufficiently to assume that F depends on $b_{0,0}$, $a_{0,0}$, $b_{1,0}$, $a_{1,0}$, $b_{2,0}$, $a_{2,0}$ or on $a_{0,0}$, $b_{1,0}$, $a_{1,0}$, $b_{2,0}$, $a_{2,0}$, $b_{3,0}$. If we assume that F depends on a smaller set of variables, we obtain trivial integrals only. As a result we get two systems of first order linear partial differential equations Solving these systems, we obtain two independent m-integrals of this system

$$I^1 = \frac{b_{0,0}}{b_{1,0}} + \frac{a_{0,0}b_{2,0}}{a_{1,0}b_{1,0}} + \frac{a_{2,0}}{a_{1,0}}, \qquad I^2 = \frac{a_{0,0}}{a_{1,0}} + \frac{a_{2,0}b_{1,0}}{a_{1,0}b_{2,0}} + \frac{b_{3,0}}{b_{2,0}}.$$

Thus, we have found the complete set m-integrals of minimal orders. The hypothesis of Theorem 1 is satisfied and thus, any other m-integral is a function of n, I^1, I^2, and their shifts.

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