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ON INTEGRATION OF AUTOMORPHIC SYSTEMS OF FINITE-DIMENSIONAL LIE GROUPS

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Abstract. The present paper contains certain results on integration and order reducing of automorphic systems for finite-dimensional Lie groups.

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Introduction

A system of differential equations is called automorphic with respect to a Lie group G if each solution of this system is obtained from one fixed solution by action of transformations in the group [1, Sect. 25].

It was shown in work [2] that some finite extension (as a rule, of zero order) of each automorphic system finite-dimensional Lie group is a completely integrable system.

In the present paper for completely integrable Pfaff systems we prove theorems on order reducing for the system admitting Lie symmetry, on a sequential order reducing for the system admitting more than one-parametric Lie symmetries, and on the integrating factor for one-dimensional systems.

The first and the third theorems are direct generalization of similar theorems for ordinary differential equations described in [1, Sect. 8]. Other ways of employing an admissible Lie symmetry for the order reducing of ordinary differential equations are described in works [3], [4], [5].

Generally speaking, the calculation of admissible group for completely integrable Pfaff system (as for a system of ordinary differential equations) is not simple than integrating the original system. But the automorphic system of a finite-dimensional Lie group G admits a Lie group which is given "for free": it is the restriction of group G on the manifold determined by the system.

In what follows all the constructions are local and are made in neighborhoods of general points, and all the considered functions are smooth enough.

1. Symmetries for Pfaff system

We consider the completely integrable Pfaff system

$$\omega = dy - \varphi^1(x,y)\,dx^1 - \dots - \varphi^n(x,y)\,dx^n = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m. \tag{1}$$

System (1) is called completely integrable in some domain of space $\mathbb{R}^n \times \mathbb{R}^m$ if for each point of this domain an n-dimensional integral surface of system (1) passes through (in this case for each point just one surface passes through) [6, Sect. 23]. In accordance with Frobenius theorem [6, Sect. 26], system (1) is completely integrable if and only if the external derivatives

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$d\omega$ belongs to the ideal generated by forms ω. The latter is true if and only if $d\omega|_{\omega=0} = 0$. And this condition is equivalent to the identities

$$[D_i, D_j] = 0, \quad i, j = 1, \dots, n, \tag{2}$$

where

$$D_i = \partial_{x^i} + \varphi^i \cdot \partial_y, \quad i = 1, \dots, n. \tag{3}$$

Indeed,

$$d\omega|_{\omega=0} = -\sum_{i,j} D_j\varphi^i dx^j \wedge dx^i = -\sum_{j>i} (D_j\varphi^i - D_i\varphi^j)\, dx^j \wedge dx^i.$$

The one-parametric group with the infinitesimal operator

$$L = \xi(x, y) \cdot \partial_x + \eta(x, y) \cdot \partial_y \tag{4}$$

is admitted by system (1) if and only if

$$[\tilde{L}, D_i] = 0, \quad i = 1, \dots, n, \tag{5}$$

where

$$\tilde{L} = L - \xi^1 D_1 - \dots - \xi^n D_n = (\eta - \xi^1\varphi^1 - \dots - \xi^n\varphi^n) \cdot \partial_y = \tilde{\eta} \cdot \partial_y. \tag{6}$$

This fact can be established straightforwardly or one can employ a similar statement for the generalized symmetries [3, Lm. 5.12]. The matter is that the generalized symmetries of system (1) coincide with point ones.

It follows from relations (5) and (6) that, in particular, each linear combination of operators (3) with coefficients depending on variables x, y is admitted by system (1) and these linear combinations form an ideal of the algebra admitted by system (1). It is also clear that component ξ of field L is an arbitrary vector-function of variables x, y.

2. Order reducing for Pfaff system

Theorem 1. *Suppose Pfaff system (1) is completely integrable and admits the one-parametric Lie group with operator (4). Then once $m > 1$, the knowing of the universal invariant for operator (6) allows us to reduce the dimension of system (1) by one.*

Proof. Universal invariant of operator (6) can be chosen as

$$x^1, \dots, x^n, \quad J^1(x, y), \dots, J^{m-1}(x, y).$$

If an one-dimensional mapping $\psi(x, y)$ is so that $\tilde{L}\psi = 1$, then $\operatorname{rank} \partial(J, \psi)/\partial y = m$ [1], and hence, for each x the mapping

$$y' = J(x, y) = (J^1(x, y), \dots, J^{m-1}(x, y)), \quad y'' = \psi(x, y) \tag{7}$$

is a local transformation of space $\mathbb{R}^m$. By (5), operators (3) are the operators of invariant differentiation for the algebra generated by the operator (6). Therefore the mapping DJ is an invariant of this algebra and is expressed by its universal invariant, i.e., there exists a mapping θ such that $DJ = \theta(x, J)$. In view of this fact, in terms of variables x, y', y'' system (1) is written as

$$dy' = J_x dx + J_y dy = DJdx = \theta(x, y')dx, \tag{8}$$

$$dy'' = D\psi dx = \vartheta(x, y', y'')dx. \tag{9}$$

System (8) consists of $m - 1$ equations for $m - 1$ functions. Equation (9) should be integrated after the integration of system (8). $\square$

Remark 2.1. If function ψ satisfies additional conditions

$$D_i\psi = 0, \quad i = 1, \dots, n, \tag{10}$$

then equation (9) casts into the form $dy'' = 0$, i.e., it can be written as the closed relation $y'' = \text{const}$. It is clear that there always exists a function ψ satisfying conditions (10). The question is just how to find its analytic representation. But the same concerns the universal invariant.

3. Further order reducing

Lemma 1. *Suppose the operators $L_1 = \xi_1 \cdot \partial_z$, $L_2 = \xi_2 \cdot \partial_z$, $D = \eta \cdot \partial_z$ acts in space $\mathbb{R}^k$ and satisfy the conditions*

$$[L_1, D] = 0, \quad [L_2, D] = 0.$$

Let also the mappings $\phi : \mathbb{R}^k \to \mathbb{R}^{k-1}$, $\psi : \mathbb{R}^k \to R$ obey the conditions

$$L_1\phi = 0, \quad L_1\psi = 1, \quad D\psi = 0, \quad rank\,\partial\phi/\partial x = k - 1.$$

Then

$$[L_2|_M, D|_M] = [L_2, D]|_M = 0,$$

where $M = \{z \in \mathbb{R}^k : \psi(z) = 0\}$.

Proof. After the change of variables

$$z' = \phi(z), \quad z'' = \psi(z)$$

operators L_2 and D are written as

$$L_2' = \xi_2'(z', z'')\partial_{z'} + \xi_2''(z', z'')\partial_{z''}, \quad D' = \eta'(z')\partial_{z'}.$$

Indeed, $D\phi$ is the universal invariant of operator L_1 and thus it is expressed in terms of universal invariant ϕ. Then

$$\begin{aligned}[L_2', D']|_M &= ((\eta_{z'}'\xi_2'(z', z'') - \xi_{2z'}'(z', z'')\eta') \cdot \partial_{z'} + (\dots)\partial_{z''})|_M = \\ &= (\eta_{z'}'\xi_2'(z', 0) - \xi_{2z'}'(z', 0)\eta') \cdot \partial_{z'} = [L_2'|_M, D'|_M].\end{aligned}$$

It is clear that $[L_2', D'] = 0$ and therefore the restriction of this commutator on manifold M vanishes. □

Lemma 1 implies the theorem which allows one to reduce sequentially the order of the system once the group is wide enough.

Theorem 2. *Each symmetry of system (1) transformed by (7) and restricted then to the space of variables (x, y') is admitted by system (8).*

4. First order $(m = 1)$

Theorem 3. *Suppose that Pfaff system (1) is completely integrable, admits one-parametric Lie group with infinitesimal operator (4) and $m = 1$, i.e. we are given one Pfaff equation. Then $\mu = 1/\tilde{\eta}$ is an integrating factor for this Pfaff equation.*

Proof. To prove the theorem, we need to establish the existence of a function $u(x, y)$ such that

$$\frac{\partial u}{\partial y} = \frac{1}{\tilde{\eta}}, \quad \frac{\partial u}{\partial x_i} = -\frac{\varphi^i}{\tilde{\eta}}, \quad i = 1, \dots, n.$$

In its turn, such function u exists if

$$\frac{\partial}{\partial x_i}\frac{1}{\tilde{\eta}} + \frac{\partial}{\partial y}\frac{\varphi^i}{\tilde{\eta}} = 0, \quad \frac{\partial}{\partial x_j}\frac{\varphi^i}{\tilde{\eta}} - \frac{\partial}{\partial x_i}\frac{\varphi^j}{\tilde{\eta}} = 0, \quad i, j = 1, \dots, n. \tag{11}$$

These conditions are satisfied by (2), (5). Indeed, the left hand side of the former expression in (11) after the differentiation is written as $\tilde{\eta}^{-2}(\tilde{\eta}_{x_i} + \tilde{\eta}_y \varphi^i - \varphi^i_y \tilde{\eta})$ and up to a factor it coincides with the left hand side of expression (5). The left hand side of the latter expression (11) after the differentiation is written as $\tilde{\eta}^{-2}(\varphi^i_{x_j}\tilde{\eta} - \varphi^i \tilde{\eta}_{x_j} - \varphi^j_{x_i}\tilde{\eta} + \varphi^j \tilde{\eta}_{x_i})$. Substituting then the expressions for $\tilde{\eta}_{x_i}$, $\tilde{\eta}_{x_j}$ in (5) and for $\varphi^i_{x_j}$ in (2), we see that it vanishes. □

5. Automorphic systems

The restriction of Pfaff system

$$dy_\alpha - \sum_{j=1}^{n} y_{\alpha+\gamma_j} dx_j = 0, \quad 0 \leqslant |\alpha| < k,$$

on the manifold determined by an automorphic (with respect to r-parametric Lie group G^r) system of order k gives completely integrable Pfaff system [2]. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ are multi-indices, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $|\gamma_i| = 1$, at that, the component with index i is equal to one.

The restriction of k-th prolongation of group G^r on the manifold determined by an automorphic system is obviously admitted by Pfaff system. Application of Theorem 1–3 allows us to reduce the order of Pfaff system or completely integrate under the sufficient rank of the group.

6. Example 1

Pfaff system

$$\begin{aligned} du &= -\left(u + c\sqrt{p/\rho}\right) u_1 dt + u_1 dx, \\ d\rho &= -\left(\rho + u(c^2 - \gamma + 1)/c\sqrt{\rho^3/p}\right) u_1 dt + \\ &+ (c^2 - \gamma + 1)/c\sqrt{\rho^3/p}\, u_1 dx, \\ dp &= -\left(\gamma p + uc\sqrt{\rho p}\right) u_1 dt + c\sqrt{\rho p} u_1 dx, \\ du_1 &= -(\gamma + 1)/2 u_1^2 dt \end{aligned} \tag{12}$$

is equivalent to the automorphic system

$$\begin{gathered} u_t + uu_x + \rho^{-1} p_x = 0, \quad \rho_t + u\rho_x + \rho u_x = 0, \quad p_t + up_x + \gamma p u_x = 0, \\ p_x = c\sqrt{\rho p}\, u_x, \quad \rho_x = (c^2 - \gamma + 1)/c\sqrt{\rho^3/p}\, u_x, \quad u_{xx} = 0 \end{gathered}$$

from Example 1 in work [2]. Here constant c_4 ($c_4^2 \neq \gamma$) in [2] is replaced by constant c. Below we treat the case $\gamma \neq 1$.

The restrictions of first prolongation of the operators in the admissible group on the manifold determined by the system read as

$$\begin{gathered} L_1 = \partial_t, \quad L_2 = \partial_x, \quad L_3 = t\partial_x + \partial_u, \quad L_4 = t\partial_t + x\partial_x - u_1\partial_{u_1}, \\ L_5 = t\partial_t - u\partial_u + 2\rho\partial_\rho - u_1\partial_{u_1}, \quad L_6 = p\partial_p + \rho\partial_\rho. \end{gathered}$$

In what follows, for the order reducing of the system we employ sequentially operators L_6, L_3, and L_2. In each step we indicate the universal invariant, function ψ, new representations of operators D_t, D_x, and remained operators $\tilde{L}$. New representation for the Pfaff system is not written since it is completely determined by operators D_t, D_x. The superscript indicates the step. While writing out the universal invariant, we introduce the notations for new variables. In each step function ψ obeys condition (10).

First step (operator $\tilde{L}_6$). Universal invariant is $t, x, u, u_1, p_1 = p/\rho$.

$$\begin{aligned}
\psi^1 &= \ln\rho - 2(c^2-\gamma+1)/(\gamma-1)\ln\left(2c\sqrt{p/\rho}\right) + 2(c^2-\gamma)/(\gamma+1)\ln u_1,\\
D_t^1 &= \partial_t - u_1(c\sqrt{p_1}+u)\partial_u - u_1^2(\gamma+1)/2\partial_{u_1} + p_1u_1(1-\gamma)(1+u/(c\sqrt{p_1}))\partial_{p_1},\\
D_x^1 &= \partial_x + u_1\partial_u + p_1u_1(\gamma-1)/(c\sqrt{p_1})\partial_{p_1},\\
\tilde{L}_3^1 &= (1-tu_1)\partial_u + (p_1tu_1(1-\gamma))/(c\sqrt{p_1})\partial_{p_1},\\
\tilde{L}_2^1 &= -u_1\partial_u - p_1u_1(\gamma-1)/(c\sqrt{p_1})\partial_{p_1}.
\end{aligned}$$

Second step (operator $\tilde{L}_3^1$). Universal invariant is $t, x, u_1, p_2 = 2\sqrt{p_1} - tuu_1(\gamma-1)/(c(tu_1-1))$.

$$\begin{aligned}
\psi^2 &= u - 2c\sqrt{p_1}/(\gamma-1),\\
D_t^2 &= \partial_t - u_1^2(\gamma+1))/2\partial_{u_1} + (p_2u_1(\gamma-1))/(2(tu_1-1))\partial_{p_2},\\
D_x^2 &= \partial_x + (u_1(-\gamma+1))/(c(tu_1-1))\partial_{p_2},\\
\tilde{L}_2^2 &= (u_1(\gamma-1))/(c(tu_1-1))\partial_{p_2}.
\end{aligned}$$

Third step (operator $\tilde{L}_2^2$). Universal invariant is t, x, u_1.

$$\begin{aligned}
\psi^3 &= (2cp_2tu_1 - 2cp_2 - \gamma^2tu_1 + 2\gamma u_1x + 2\gamma + tu_1 - 2u_1x - 2)/(2u_1(\gamma-1)),\\
D_t^3 &= \partial_t - u_1^2(\gamma+1)/2\partial_{u_1},\\
D_x^3 &= \partial_x.
\end{aligned}$$

Finally we obtain one-dimensional Pfaff system

$$du_1 + u_1^2(\gamma+1)/2\,dt = 0,$$

which is easily integrated $u_1 = 2/((\gamma+1)t + c_4)$, where c_4 is some constant.

Returning back to the original variables and employing the expressions for the invariants and relations $\psi^i = c_i$, $i = 1, 2, 3$, where c_i are some constants, we obtain the general solution of system (12):

$$u = (c_2(\gamma-1)t + 2x + c_4 - 2c_3 + c_2c_4)/((\gamma+1)t + c_4),$$

$$\rho = c_1(4c^2p_1)^{c^2/(\gamma-1)-1}u_1^{2(\gamma-c^2)/(\gamma+1)}, \quad p = p_1\rho, \quad u_1 = 2/((\gamma+1)t + c_4),$$

where $p_1 = (\gamma-1)^2(x - c_2t + c_4/2 - c_3)^2/(c(c_4 + (\gamma+1)t))^2$.

7. Example 2

In work [7] there was found an admissible group and there was constructed a group foliation for the Karman-Gooderley equation

$$-u_xu_{xx} + u_{yy} + u_{zz} = 0 \tag{13}$$

with respect to the infinite-dimensional part of the admissible group. To construct the automorphic system, here there was used a finite-dimensional subgroup of the admissible group generated by the operators

$$\begin{gathered}
L_1 = \partial_x, \quad L_2 = \partial_y, \quad L_3 = \partial_z, \quad L_4 = z\partial_y - y\partial_z,\\
L_5 = y\partial_y + z\partial_z - 2u\partial_u, \quad L_6 = x\partial_x + 3u\partial_u.
\end{gathered} \tag{14}$$

The solutions to equation (13) which form six-dimensional orbits in the extended spaces under the action of the group generated by operators (14) satisfy one of two following automorphic system:

$$\begin{aligned}
&u_x = (3/2)^{2/3}(u_y^2+u_z^2)^{1/3}, \quad u_{xx} = (3/2)^{1/3}(u_y^2+u_z^2)^{2/3}/u,\\
&u_{xy} = (3/2)^{2/3}(u_y^2+u_z^2)^{1/3}u_y/u, \quad u_{xz} = (3/2)^{2/3}(u_y^2+u_z^2)^{1/3}u_z/u,\\
&u_{yy} = 3/2\,u_y^2/u, \quad u_{yz} = 3/2\,u_yu_z/u, \quad u_{zz} = 3/2\,u_z^2/u;
\end{aligned} \tag{15}$$

$$\begin{gathered}u_x=(3/2)^{1/3}(u_y^2+u_z^2)^{1/3},\quad u_{xx}=(3/2)^{-1/3}(u_y^2+u_z^2)^{2/3}/u,\\ u_{xy}=(3/2)^{1/3}(u_y^2+u_z^2)^{1/3}u_y)/u,\quad u_{xz}=(3/2)^{1/3}(u_y^2+u_z^2)^{1/3}u_z/u,\\ u_{yy}=(3u_y^2-u_z^2)/(2u),\quad u_{yz}=(2u_yu_z)/u,\quad u_{zz}=(-u_y^2+3u_z^2)/(2u).\end{gathered}\tag{16}$$

In what follows, to reduce the order of automorphic system (15) we employ sequentially operators L_2, L_4. Pfaff system equivalent to system (15) can be written as

$$\begin{gathered}du-(3/2)^{2/3}(v^2+w^2)^{1/3}dx-vdy-wdz=0,\\ dv-(3/2)^{2/3}(v^2+w^2)^{1/3}vdx-3/(2u)v^2dy-3/(2u)vwdz=0,\\ dw-(3/2)^{2/3}(v^2+w^2)^{1/3}wdx-3/(2u)vwdy-3/(2u)w^2dz=0.\end{gathered}$$

Then the restrictions of first prolongation of operators L_2, L_4 on the manifold determined by system (15) followed by the factorization with respect to the ideal are written as follows:

$$\begin{aligned}\tilde L_2 &= -v\partial_u-3/(2u)(v^2\partial_v+vw\partial_w),\\ \tilde L_4 &= (-vz+wy)\partial_u+(2uw-3v^2z+3vwy)/(2u)\partial_v+\\ &+ (-2uv-3vwz+3w^2y)/(2u)\partial_w.\end{aligned}$$

First step (operator $\tilde L_2$). The universal invariant is $x,y,z,v_1=v/u^{3/2},w_1=w/u^{3/2}$.

$$\begin{aligned}\psi^1 &= 2u/v+y+zw/v,\\ D_x^1 &= \partial_x+(3/2)^{2/3}(v_1^2+w_1^2)^{1/3}(\partial_u+v_1\partial_{v_1}+w_1\partial_{w_1}),\\ D_y^1 &= \partial_y,\quad D_z^1 = \partial_z,\quad \tilde L_4^1 = w_1\partial_{v_1}-v_1\partial_{w_1}.\end{aligned}$$

Second step (operator $\tilde L_4^1$). The universal invariant is $x,y,z,v_2=\sqrt{v_1^2+w_1^2}$.

$$\begin{aligned}\psi^2 &= \operatorname{arctg}(v_1/w_1)+3^{2/3}2^{-5/3}x-3/2(v_1^2+w_1^2)^{-1/3},\\ D_x^2 &= \partial_x-(3/2)^{2/3}/2v_2^{5/3}\partial_{v_2},\quad D_y^2 = \partial_y,\quad D_z^2 = \partial_z.\end{aligned}$$

As a result we get one-dimensional Pfaff system $dv_2+(3/2)^{2/3}/2v_2^{5/3}dx=0$, which can be easily integrated $v_2=(x/\sqrt[3]{12}+c_3)^{-3/2}$. Returning back to the original variables and employing the expressions for the invariants and relations $\psi^i=c_i$, $i=1,2$, where c_i are some constants, we obtain the general solution to system (15):

$$u=4(x/\sqrt[3]{12}+c_3)^3/(y\sin(c_2)+z\cos(c_2)+c_1)^2.\tag{17}$$

In the same way one can obtain the general solution to system (16):

$$u=2/9(x+c_1)^3/((y+c_2)^2+(z+c_3)^2).\tag{18}$$

Remark 7.1. The above examples of solutions demonstrate application of the results of this paper and we do not claim that other known methods can not be applied here. In particular, solutions (17), (18) as $c_1=c_2=c_3=0$ equal $x^3/(3z^2)$, $2/9x^3/(y^2+z^2)$, respectively, and they are self-similar solutions to equation (13). Hence, solutions (17), (18) can be obtained from self-similar solutions by means of translations and rotations.

For bulky calculations we employed system of analytic calculations "Reduce 3.8" (http://reduce-algebra.sourceforge.net).

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