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CAUCHY–HADAMARD THEOREM FOR EXPONENTIAL SERIES

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Abstract. In this paper we study the connection between the growth of coefficients of an exponentials series with its convergence domain in finite-dimensional real and complex spaces. Among the first results of the subject is the well-known Cauchy–Hadamard formula. We obtain exact conditions on the exponentials and a convex region in which there is a generalization of the Cauchy–Hadamard theorem. To the sequence of coefficients of exponential series we associate a space of sequences forming a commutative ring with unit. A study of the properties of this ring allows us to obtain the results on solvability of non-homogeneous systems of convolution equations.

Keywords: convex domains, series of exponentials, Cauchy–Hadamard formula.

Mathematics Subject Classification: 40B05

1. Introduction

In this paper we study the relation between the growth of the coefficients of exponential series with its convergence domain.

The main result in this direction is the well-known Cauchy-Hadamard theorem after an appropriate change of variables. An analogue of Cauchy-Hadamard theorem for exponential series of one variable with non-negative exponents was proven by Valiron [1]. For the series of exponential monomials of one complex variable it was studied in paper [2]. The case of exponential series of many complex variables with complex exponents was considered in paper [3].

2. Cauchy-Hadamard theorem

In what follows we shall make use of several notations. Let $\mathbb{E}$ be either space $\mathbb{R}^m$ or $\mathbb{C}^m$. For elements $z,\ w \in \mathbb{E}$ we indicate

$$zw = \sum_{j=1}^{m} z_j w_j.$$

By $\mathbb{B}$ we denote a closed unit ball in space $\mathbb{E}$ centered at the origin.

The support function of a set $M \subset \mathbb{E}$ is determined by the formula

$$H(\lambda, M) = \sup_{z\in M} \operatorname{Re} \lambda z,\ \ \lambda \in \mathbb{E}.$$

It is a homogeneous convex function. At it is known, if set M is non-empty, this function is continuous.

S.G. Merlyakov, Cauchy–Hadamard theorem for exponential series.

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In the present paper we provide exact conditions for an arbitrary convex domain $U \subset \mathbb{E}$ and sequence последовательность $\Lambda = \{\lambda_n \in \mathbb{E} : n \in \mathbb{N}\}$, whose terms not necessary differ, under those Cauchy-Hadamard statement holds true, namely, the absolute convergence of the series

$$\sum_{n=1}^{\infty} c_n e^{\lambda_n z}, \tag{1}$$

$c_n \in \mathbb{C}$, in domain U is equivalent to the relation

$$\varlimsup_{n\to\infty} \frac{\ln|c_n| + H(\lambda_n, U)}{|\lambda_n|} \leqslant 0. \tag{2}$$

If this equivalence is satisfied, we call sequence Λ a Cauchy-Hadamard system for domain U.

It is easy to show there always exist coefficients $c_n > 0$, $n \in \mathbb{N}$, for which series (1) converges absolutely in the whole space. In this case it is obvious that Cauchy-Hadamard statement does not hold once $H(\lambda_n, U) = \infty$ or $\lambda_n = 0$ for infinitely many indices. This is why without loss of generality we assume that $H(\lambda_n, U) < \infty$, $\lambda_n \neq 0$, $n \in \mathbb{N}$.

We note that each subsequence of Cauchy-Hadamard system for domain U is also similar system. Indeed, the exponential series with exponents in this subsequence can be complemented by terms zero coefficients that makes influence neither on the convergence no on the mentioned upper limit.

We let

$$S = \left\{ \frac{\lambda}{|\lambda|} : \lambda \in \Lambda \right\}. \tag{3}$$

In what follows we shall employ several results on relation between the behavior of series (1) with relations (2).

Lemma 1. *Suppose that the coefficients of series* (1) *satisfy condition* (2) *and a compact set $R \subset U$ and a point $z \in U$ obey the inclusion $z + R \subset U$. Then inequality*

$$\ln|c_n e^{\lambda_n z}| \leqslant -\delta(z)|\lambda_n| - H(\lambda_n, R), \ \delta(z) > 0, \quad n \geqslant N(\delta(z)), \tag{4}$$

holds true.

Доказательство. As it is easy to show, there exists a number $\delta(z) > 0$ for which $z + 2\delta(z)\mathbb{B} + R \subset U$ and thus $\operatorname{Re}\lambda z \leqslant H(\lambda, U) - 2\delta(z)|\lambda| - H(\lambda, R), \lambda \in \mathbb{E}$. It follows from inequality (2) that for each $\varepsilon > 0$ the formula $\ln|c_n| \leqslant -H(\lambda_n, U) + \varepsilon|\lambda_n|$, $n \geqslant N(\varepsilon)$, holds true that completes the proof. □

Corollary 1. *Suppose that inequality* (2) *and relation*

$$\forall\varepsilon > 0 \ \sum_{n=1}^{\infty} e^{-\varepsilon|\lambda_n| - H(\lambda_n, R)} < \infty \tag{5}$$

hold true. Then series (1) *converges absolutely in each point $z \in \mathbb{E}$ with the property $z + R \subset U$.*

Corollary 2. *Suppose that relation*

$$\forall\varepsilon > 0 \ \sum_{n=1}^{\infty} e^{-\varepsilon|\lambda_n|} < \infty \tag{6}$$

holds true and the left hand side of formula (2) equals $-\delta$, $\delta > 0$. Then series (1) converges absolutely in domain $U + \delta\mathbb{B}$.

Indeed, in this case, as one can see easily, there hold inequality (2) with domain U replaced by $U + \delta\mathbb{B}$ and relation (5) for $R = \{0\}$.

Corollary 3. *Inequality* (2) *implies pointwise boundedness for the terms of series* (1) *in domain U.*

Indeed, we obtain the desired statement once we let $R = \{0\}$.

Under additional assumption the opposite holds.

Proposition 1. *If function $H(s, U)$ is continuous on the closure of a set S, the terms of series* (1) *are pointwise bounded in domain U, and the identity*

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{7}$$

is satisfied, then relation (2) *holds true.*

Доказательство. Domain U can be exhausted by an increasing sequence of compact sets K_p, $p \in \mathbb{N}$. At that, as one can show easily, relation

$$H(s, K_p) \uparrow H(s, U)$$

holds true. By Dini theorem, this convergence is uniform on the compact set $\overline{S}$, so for each number $\delta > 0$ there exists a compact set $K \subset U$ obeying $H(s, U) \leqslant H(s, K) + \delta, \in S$.

Compact set K is contained in the convex hull of some points $z_1, \ldots, z_k$ in domain U (cf. [4]), and thus for each element $z \in K$ the relation

$$\operatorname{Re} sz \leqslant \max\{\operatorname{Re} sz_1, \ldots, \operatorname{Re} sz_k\}, \quad s \in S, \tag{8}$$

holds true. Hence, we obtain

$$\ln|c_n| + H(\lambda_n, U) \leqslant \ln|c_n| + H(\lambda_n, K) + \delta|\lambda_n| \leqslant$$
$$\max\{\ln|c_n| + \operatorname{Re}\lambda_n z_1, \ldots, \ln|c_n| + \operatorname{Re}\lambda_n z_k\} + \delta|\lambda_n|.$$

The assumption for the terms of series (1) and the arbitrariness of number $\delta > 0$ complete the proof. □

Corollary. *Suppose that the identity*

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{|\lambda_n|} \leqslant d < \infty \tag{9}$$

holds true, and the terms of series (1) *are pointwise bounded in some ball U of a radius greater than d. Then series* (1) *converges absolutely at the center of this series.*

Indeed, it is obvious that condition (7) is satisfied and thus, the same is true for inequality (2).

We let $R = d\,\mathbb{B}$. In this case $H(\lambda, R) = d|\lambda|$, $\lambda \in \mathbb{E}$, and as one can easily see, relation (5) is satisfied. Hence, the desired statement follows from Corollary 1 of Lemma 1.

The obtained result generalized Theorem 3.1.2 in monograph [5].

Exponential series possess the following property.

Proposition 2. *Absolutely convergent in U series* (1) *converges normally on compact sets in this domain.*

Доказательство. As it was mentioned above, for each compact set K of domain U there exist points $z_1, \ldots, z_k$ in this domain satisfying relation (8) for points in compact set K and we get

$$\max_{z\in K} |c_n e^{\lambda_n z}| \leqslant |c_n| e^{\max\{\operatorname{Re}\lambda_n z_1, \ldots, \operatorname{Re}\lambda_n z_k\}} \leqslant |c_n| \left(|e^{\lambda_n z_1}| + \cdots + |e^{\lambda_n z_k}|\right).$$

Finally,

$$\sum_{n=1}^{\infty} \max_{z\in K} |c_n e^{\lambda_n z}| < \infty. \tag{10}$$

The proof is complete. □

Let us adduce an example proving the exactness of Corollary of Proposition 1.

Example. Let

$$U = \{z \in \mathbb{E} : |z| < 2\}, \quad \lim_{n\to\infty} \frac{\ln n}{|\lambda_n|} = 1, \quad \ln|c_n| = -2|\lambda_n|.$$

The terms of series (1) are obviously pointwise bounded in ball U and by Corollary of Proposition 1, it converges absolutely in the ball $\{z \in \mathbb{E} : |z| < 1\}$ but has no such property in a ball of a bigger radius.

Indeed, suppose series (1) converges absolutely in the ball

$$\{z \in \mathbb{E} : |z| < 1 + \delta\}, \quad \delta > 0.$$

By the just proven result this series should converge normally on the ball $\mathbb{B}$ and it is equivalent to the convergence of harmonic series. The obtained contradiction proves the desired statement.

Proposition 3. *Suppose that the identity $U = R + V$ holds true for a convex compact set $R \subset \mathbb{E}$ and a convex domain $V \subset \mathbb{E}$, function $H(s,U)$ is closed on the closure of set S and for each sequence of coefficients satisfying inequality* (2) *series* (1) *converges absolutely in domain V. Then relation* (5) *holds true.*

Доказательство. The representation for domain U implies the identity

$$H(\lambda, U) = H(\lambda, R) + H(\lambda, V), \ \lambda \in \mathbb{E},$$

cf. [6]. By assumption, series (1) with coefficients

$$c_n = e^{-H(\lambda_n,U)}, \quad n \in \mathbb{N},$$

converges absolutely in domain V and in accordance with Proposition 2 this convergence is normal on each compact set in domain V. As it was shown above, for arbitrary number $\varepsilon > 0$ there exists a compact set $K \subset V$ such that $H(s,V) \leqslant H(s,K) + \varepsilon$, $s \in S$.

Thus,

$$\sum_{n=1}^{\infty} e^{-\varepsilon|\lambda_n| - H(\lambda_n,R)} \leqslant \sum_{n=1}^{\infty} e^{H(\lambda_n,K) - H(\lambda_n,U)} = \sum_{n=1}^{\infty} \max_{z\in K} |c_n e^{\lambda_n z}| < \infty,$$

and it completes the proof. □

In literature condition (5) is often used and this is why let us find out its connection with condition (5) for the case $R = d\mathbb{B}$, i.e.,

$$\forall \varepsilon > 0 \ \sum_{n=1}^{\infty} e^{-\varepsilon|\lambda_n| - d|\lambda_n|} < \infty. \tag{11}$$

It is obvious that the latter condition follows from relation (5). The opposite is not true, as one can show easily, points λ_n can be transposed by a permutation $r : \mathbb{N} \to \mathbb{N}$ so that

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{|\lambda_{r(n)}|} = \infty,$$

and condition (11) remains unchanged. But under additional assumption these conditions are equivalent.

Proposition 4. *Suppose that a sequence $\{|\lambda_n| : n \in \mathbb{N}\}$ is monotonically increasing and satisfies relation* (11)*. Then inequality* (5) *holds true.*

Доказательство. For each $\varepsilon > 0$ the terms of series (11) decrease monotonically and as it is well known it implies identity

$$\lim_{n\to\infty} n e^{-(d+\varepsilon)|\lambda_n|} = 0$$

that yields

$$\lim_{n\to\infty} \ln n - (d+\varepsilon)|\lambda_n| = -\infty.$$

Finally,

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{|\lambda_n|} \leqslant d+\varepsilon.$$

The statement follows from the arbitrariness of ε. □

Let us provide examples on applications of obtained results.

Example 1. Given a series

$$\sum_{\|k\|=0}^{\infty} c_k(z-a)^k, \quad c_k \in \mathbb{C}, \quad k \in \mathbb{N}_0^m, \quad z \in \mathbb{C}^m,$$

let $r \in \mathbb{R}_+^m$ be its dual convergence radii (see [7]), where by $\mathbb{N}_0$ we denote the set of non-negative integers and $\|b\| = \sum_{j=1}^m |b_j|, b \in \mathbb{E}$.

Making the change

$$z - a = (e^{w_1}, \ldots, e^{w_m}),$$

we obtain the series of exponentials with exponents $\{k : k \in \mathbb{N}_0^m\}$, whose terms are bounded in domain

$$U = \{w \in \mathbb{E} : \operatorname{Re} w_1 < \ln r_1, \ldots, \operatorname{Re} w_m < \ln r_m\}.$$

As one can see easily, the support function of this domain is defined by the identities $H(\lambda, U) = \lambda \ln r$, if $\lambda_1 \geqslant 0, \ldots, \lambda_m \geqslant 0$, and $H(\lambda, U) = \infty$ otherwise. It is clear that function H is continuous on set S, the sequence $\{k : k \in \mathbb{N}_0^m\}$ tends to infinity, and for an arbitrary point $b \in \mathbb{E}$ inequalities

$$\frac{\|b\|}{\sqrt{m}} \leqslant |b| \leqslant \sqrt{m}\|b\|$$

hold true so that

$$\sum_{\|k\|=0}^{\infty} e^{-\varepsilon|k|} \leqslant \sum_{\|k\|=0}^{\infty} e^{-\varepsilon\|k\|/\sqrt{m}} = \frac{e^{\varepsilon\sqrt{m}}}{(1-\varepsilon/\sqrt{m})^m}, \quad \varepsilon > 0.$$

In this case by Proposition 1 we obtain

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k| r^k} \leqslant 1.$$

The left hand side of this relation can not be less than one otherwise by Corollary 2 of Lemma 1 the power series converges absolutely on some swelling of the polycircle $\{z \in \mathbb{E} : |z_1 - a_1| < r_1, \ldots, |z_m - a_m| < r_m\}$ that contradicts to the definition of dual convergence radii.

It proves classical Cauchy-Hadamard formula.

Example 2. Let $\mathbb{E} = \mathbb{C}^m$, set D be a domain in space $\mathbb{R}^m$, $\operatorname{Re}\lambda_n \neq 0$, $H(\lambda_n, D) \neq \infty$, $n \in \mathbb{N}$, function $H(\operatorname{Re} s, D)$ is continuous on the closure of the set

$$\left\{\frac{\operatorname{Re}\lambda}{|\operatorname{Re}\lambda|} : \lambda \in \Lambda\right\},$$

and

$$\forall \varepsilon > 0 \ \sum_{n=1}^{\infty} e^{-\varepsilon|\operatorname{Re}\lambda_n|} < \infty.$$

Then absolute convergence of series (1) on set D is equivalent to the inequality

$$\overline{\lim_{n\to\infty}} \frac{\ln|c_n| + H(\operatorname{Re}\lambda_n, D)}{|\operatorname{Re}\lambda_n|} \leqslant 0.$$

Indeed, as one can easily make sure, it is reduced to applying of Proposition 1 and Corollary 1 of Lemma 1 for the case $\mathbb{E} = \mathbb{R}^m$.

Proposition 1 and Corollary 1 of Lemma1 for the case $R = \{0\}$ strengthen one of the results of paper [3], namely,

Theorem. *Let U be a bounded convex domain in space $\mathbb{C}^m$, $c_n \in \mathbb{C}$, $\lambda_n \in \mathbb{C}^m$, $n \in \mathbb{N}$. If series* (1) *converges uniformly on compact sets in domain U, and sequence $\{\lambda_n : n \in \mathbb{N}\}$ satisfies condition* (7)*, then relation* (2) *holds true.*

Vice versa, if inequality (2) *is obeyed and*

$$\lim_{n\to\infty} \frac{\ln n}{|\lambda_n|} = 0, \tag{12}$$

then series (1) *converges normally on compact sets in domain U.*

We note that Cauchy-Hadamard formula for power series is not implied from this results since after the change of variables we always get unbounded domains.

We are in position to formulate the main result of the present paper.

Theorem 1. *Sequence Λ is a Cauchy-Hadamard system for domain U if and only if*

1. *The restriction of function $H(s, U)$ on the closure of set* (3) *is continuous.*
2. *Relation* (6) *holds true.*

Доказательство. It is obvious that sufficiency follow from Proposition 1 and Corollary 1 of Lemma 1 for the case $R = \{0\}$.

Suppose that sequence is a Cauchy-Hadamard system for domain U and let us show that Conditions 1 and 2 of Theorem are satisfied.

First we prove identity (7). Indeed, by assumption, series (1) with $\ln |c_n| = -H(\lambda_n, U)$, $n \in \mathbb{N}$, converges absolutely in domain U, and this is why the series

$$\sum_{n=1}^{\infty} e c_n e^{\lambda_n z}$$

satisfies the same. By assumption it implies inequality

$$\overline{\lim_{n\to\infty}} \frac{1}{|\lambda_n|} \leqslant 0,$$

which is equivalent to the original identity.

We take an arbitrary point $s \in \overline{S}$ and prove the identity

$$\lim_{\lambda\to s, \lambda\in S} H(\lambda, U) = H(s, U). \tag{13}$$

Function $H(s, U)$ is lower semi-continuous as an upper envelope for a family of continuous function (cf. [8]), and hence it is sufficient to prove inequality

$$\overline{\lim_{\lambda\to s, \lambda\in S}} H(\lambda, U) \leqslant H(s, U). \tag{14}$$

We indicate the left hand side of this formula as b.

There exists a mapping $r : \mathbb{N} \to \mathbb{N}$ obeying condition

$$\lim_{n\to\infty} \frac{\lambda_{r(n)}}{|\lambda_{r(n)}|} = s, \ \lim_{n\to\infty} \frac{H\left(\lambda_{r(n)}, U\right)}{|\lambda_{r(n)}|} = b.$$

If function r is bounded, then obviously $b = H(s, U)$, and we shall assume that this function is unbounded.

It is easy to construct the mapping $p : \mathbb{N} \to \mathbb{N}$ such that the superposition $l = r \circ p : \mathbb{N} \to \mathbb{N}$ is strictly increasing and, taking into consideration the identity (7), it satisfies the condition

$$\lim_{n\to\infty} \frac{\ln n}{|\lambda_{l(n)}|} = 0.$$

We let $\mu_n = \lambda_{l(n)}, n \in \mathbb{N}$. Thus, $\{\mu_n : n \in \mathbb{N}\}$ is a subsequence of sequence $\{\lambda_n, n \in \mathbb{N}\}$. Hence, it is a Cauchy-Hadamard system for domain U and it satisfies the relations

$$\lim_{n\to\infty} \frac{\mu_n}{|\mu_n|} = s, \ \lim_{n\to\infty} \frac{H(\mu_n, U)}{|\mu_n|} = b, \ \lim_{n\to\infty} \frac{\ln n}{|\mu_n|} = 0. \tag{15}$$

Let us show that $b < \infty$. Indeed, otherwise we let $\ln |c_n| = |\mu_n| - H(\mu_n, U)$ and estimate the terms of the series

$$\sum_{n=1}^{\infty} c_n e^{\mu_n z}. \tag{16}$$

For each fixed number z we have

$$\ln |c_n| + \operatorname{Re} \mu_n z = |\mu_n| - H(\mu_n, U) + \operatorname{Re} \mu_n z \leqslant -|\mu_n|,$$

if number $n \in \mathbb{N}$ is large enough. By the third identity in relations (15) we conclude that series (16) converges absolutely in whole space $\mathbb{E}$, and on the other hand,

$$\frac{\ln |c_n| + H(\mu_n, U)}{|\mu_n|} = 1, \quad n \in \mathbb{N}.$$

The obtained contradiction proves the desired statement.

The lower semi-continuity of function H implies the estimate $H(s, U) \leqslant b$, so that $H(s, U) < \infty$. Let us show the convergence of series (16) with coefficients $\ln |c_n| = -H(s, U)|\mu_n|$ in domain U.

We fix a point z in domain U. This point lies in the domain with a neighborhood and hence, there exists $\varepsilon > 0$ satisfying

$$H(s, U) \geqslant \operatorname{Re} sz + \varepsilon.$$

The first formula in identities (15) implies that for $\delta > 0$ the inequality

$$\operatorname{Re} \mu_n z \leqslant |\mu_n| \operatorname{Re} sz + \delta|\mu_n|, \quad n \geqslant N(\delta)$$

holds true. Thus, we have

$$\ln |c_n e^{\mu_n z}| \leqslant (\delta - \varepsilon)|\mu_n|, \quad n \geqslant N(\delta),$$

and taking $\delta < \varepsilon$, we obtain the convergence of series (16) at point z. The analogue of inequality (2) for the sequence $\{\mu_n : \ n \in \mathbb{N}\}$ and the second identity in formulae (15) implies the relation $b \leqslant H(s, U)$ and it completes the proof of identity (13).

Thus, we have shown that for each point $s \in \overline{S}$ identity (13) is satisfied and $H(s, U) < \infty$ that implies the continuity of function $H(s, U)$ on set $\overline{S}$, see. [9].

Condition 2 of Theorem follows from Proposition 3. The proof is complete. □

Let us provide an example of a sequence not being Cauchy-Hadamard system.

Example. Let

$$U = \left\{z = x + iy \in \mathbb{C} : 2x + y^2 < 0\right\}, \quad \lambda_n = 1 + in, \quad n \in \mathbb{N}.$$

In this for the points $\lambda = u + iv \in \mathbb{C}$ we have

$$H(\lambda, U) = \begin{cases} v^2/2u, & \text{as } u > 0, \\ 0, & \text{as } u = 0, \quad v = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

As one can easily make sure, the support function of domain U is discontinuous будет разрывной на множестве $\overline{S}$.

3. Spaces of sequences

We introduce the space of sequences

$$A = \left\{ c \in \mathbb{C}^{\infty} : \forall z \in U \ \sum_{n=1}^{\infty} |c_n e^{\lambda_n z}| < \infty \right\},$$

and the space L with elements satisfying inequality (2).

In space A we introduce a topology by means of the family of semi-norms

$$\|c\|_{A,K} = \sum_{n=1}^{\infty} |c_n| e^{H(\lambda_n, K)}, \ c \in A,$$

where K is a compact set in domain U, while for space L we let

$$\|c\|_{L,\varepsilon} = \sup_{n \in \mathbb{N}} |c_n| e^{H(\lambda_n, U) - \varepsilon |\lambda_n|}, \quad c \in L, \quad \varepsilon > 0.$$

As one can show easily, these are Fréchet spaces.

Spaces A and L are related as follows.

Proposition 5. *Once relation* (6) *is satisfied, the inclusion $L \subset A$ is valid and the embedding $L \to A$ is continuous.*

If the restriction of function $H(s, U)$ on the closure of set (3) *is continuous, then the embedding $A \to L$ is continuous as well.*

Доказательство. As it mentioned above, for an arbitrary compact set $K \subset U$ the inequality $H(\lambda_n, K) \leqslant H(\lambda_n, U) - 2\varepsilon|\lambda_n|$, $n \in \mathbb{N}, \varepsilon > 0$, holds true, and this is why

$$\|c\|_{A,K} \leqslant \|c\|_{L,\varepsilon} \sum_{n=1}^{\infty} e^{-\varepsilon|\lambda_n|}.$$

It proves the first part of the proposition.

Under the hypothesis of the second part for each $\varepsilon > 0$ there exists a compact set $K \subset U$ such that $H(\lambda_n, U) - \varepsilon|\lambda_n| \leqslant H(\lambda_n, K)$, and therefore,

$$\|c\|_{L,\varepsilon} \leqslant \|c_{A,K}\|.$$

The proof is complete. □

With exponential series the space of sequences

$$\mathbb{K} = \left\{ a \in \mathbb{C}^{\infty} : \varlimsup_{n \to \infty} \frac{\ln |a_n|}{\alpha_n} \leqslant 0 \right\},$$

is closely related. Here $\{\alpha_n : n \in \mathbb{N}\}$ is a fixed sequence of positive numbers tending to infinity. It is obvious that set $\mathbb{K}$ is invariant with respect to term-wise multiplication.

Elements $a, b \in \mathbb{K}$ satisfy inequalities

$$\ln |a_n| \leqslant \varepsilon\alpha_n, \quad \ln |b_n| \leqslant \varepsilon\alpha_n, \varepsilon > 0, \quad n \geqslant N(\varepsilon),$$

and hence

$$\ln |a_n + b_n| \leqslant \ln(|a_n| + |b_n|) \leqslant \ln 2 + \varepsilon\alpha_n, \quad n \geqslant N(\varepsilon).$$

Thus, sense $\mathbb{K}$ is invariant also with respect to term-wise summation. Therefore, set $\mathbb{K}$ is obviously is a commutative ring with unity. It is easy to see that the multiplication in this ring is directly connected with the convolution.

In ring $\mathbb{K}$ we introduce the topology by means of the family of semi-norms

$$\|a\|_\varepsilon = \sup_{n\in\mathbb{N}} |a_n| e^{-\alpha_n \varepsilon}, \quad a \in \mathbb{K}, \quad \varepsilon \in \mathbb{N}.$$

It is easy to show that with this topology ring $\mathbb{K}$ is a Fréchet space and ring operations are continuous.

Under condition (7), space of sequences A is obviously topologically isomorphic to ring $\mathbb{K}$ for $\alpha_n = |\lambda_n|$, $n \in \mathbb{N}$.

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