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GENERALIZED DUNKL OPERATOR

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Abstract. In the paper we introduce a generalized Dunkl operator acting in the space of entire functions on $\mathbb{C}$. We study problems of harmonic analysis related with this operator and show its connection with the Gelfond-Leont'ev operator of generalized differentiation.

Keywords: Dunkl operator, eigenfunction, Dunkl convolution operator, Dunkl transform, characteristic function, hypercyclic operator.

1. Introduction

Let $H(\mathbb{C})$ be the space of entire functions with the topology of uniform convergence on compact sets, $H^*(\mathbb{C})$ is the strongly dual space for $H(\mathbb{C})$, $P_{\mathbb{C}}$ is the space of entire functions of exponential type. It is known that space $H^*(\mathbb{C})$ is isomorphic to $H_0(\{\infty\})$, which is the space of functions analytic in the vicinity of the point at infinity and vanishing at the point ∞ (see, for instance, [1]). Moreover, if $F \in H^*(\mathbb{C})$ and $g_F \in H_0(\{\infty\})$ is the associated function (according to the mentioned isomorphism), then

$$(F, f) = \frac{1}{2\pi i} \int\limits_C f(z) g_F(z)\, dz, \tag{1.1}$$

where $f \in H(\mathbb{C})$, C is a closed rectifiable contour enveloping all the singularities of function g_F and located in the analyticity domain for this function.

Consider the generalized Dunkl operator Λ on $H(\mathbb{C})$

$$\Lambda f(z) = \frac{d}{dz} f(z) + \frac{c}{z} \sum_{j=0}^{m-1} \alpha_j f(\alpha_j z), \quad z \in \mathbb{C}, \tag{1.2}$$

where $\alpha_j = e^{\frac{2\pi i j}{m}}$, $j = \overline{0, m-1}$, $f \in H(\mathbb{C})$, m is a fixed natural number obeying $m \geqslant 2$. Without loss of generality, in what follows we assume $c = 1$.

This operator generalizes the studied before in work [2] Dunkl operator

$$Df(z) = \frac{d}{dz} f(z) + \frac{c}{z}(f(z) - f(-z)), \quad z \in \mathbb{C}.$$

Dunkl operators (cf. [3]) are differential-difference operators with finite groups of reflections in some Euclidian spaces. These operators play an important role in various problems of mathematics and physics (cf., for instance, [4]). We study problems of harmonic analysis related with operator (1.2) (Dunkl shift operators, Dunkl convolution, Dunkl transform and so forth).

Consider the function $g \in H_0(\{\infty\})$

$$g(z) = \frac{1}{z^2} + \sum_{j=0}^{m-1} \frac{e^{\frac{2\pi i j}{m}}}{z - \frac{2\pi i j}{m}}.$$

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According to the aforementioned isomorphism, this function is associated with a functional $F \in H^*(\mathbb{C})$. We take the Laplace transform of this functional $\widehat{F}(\mu) = (F, e^{\mu z})$. Applying (1.1), the transform $\widehat{F}$ can be written as

$$\widehat{F}(\mu) = \frac{1}{2\pi i}\int\limits_C e^{\mu z} g(z)\,dz = \frac{1}{2\pi i}\int\limits_C e^{\mu z}\left(\frac{1}{z^2} + \sum_{j=0}^{m-1} \frac{e^{\frac{2\pi i j}{m}}}{z - \frac{2\pi i j}{m}}\right) dz = \mu + \sum_{j=0}^{m-1} e^{\frac{2\pi i j}{m}(\mu+1)}. \quad (1.3)$$

Here contour C envelopes the origin and points $\frac{2\pi i j}{m}$, $j = \overline{0, m-1}$. We introduce the function

$$y(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)}. \quad (1.4)$$

In the second section we study the properties of eigenfunctions y_λ of operator Λ associated with an eigenvalue λ and obeying condition $y_\lambda(0) = 1$. We shall show that function y_λ is determined by the solution $y_\lambda(z) = y(\lambda z)$, where function y is defined by (1.4).

Then by (1.2) we construct Dunkl shift operator τ_w (Section 3):

$$(\tau_w f)(z) = f(z) + \sum_{k=1}^{\infty} \frac{w^k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)} \Lambda^k f(z), \quad z, w \in \mathbb{C}. \quad (1.5)$$

Then Dunkl convolution operator of a functional $T \in H^*(\mathbb{C})$ and a function $f \in H(\mathbb{C})$ is determined as

$$M_T[f](z) = T * f(z) = \langle T_w, (\tau_w f)(z)\rangle, \quad z, w \in \mathbb{C}. \quad (1.6)$$

In conclusion we introduce Dunkl transform of functional $T \in H^*(\mathbb{C})$

$$\mathfrak{D}(T)(\lambda) = \breve{T}(\lambda) = \langle T, y(\lambda z)\rangle, \quad \lambda \in \mathbb{C}, \quad (1.7)$$

which establishes a topological isomorphism between spaces $H^*(\mathbb{C})$ and $P_{\mathbb{C}}$. We also consider generalized convolution equations, both homogeneous and non-homogeneous.

2. Eigenfunction of Dunkl operator Λ

Proposition 1. *a) The eigenfunction y_λ of operator Λ associated with an eigenvalue λ and obeying $y_\lambda(0) = 1$ is unique and is determined by formula* (1.4).

b) Function $y(z)$ is an entire function of exponential type, and its type is $\sigma = 1$.

Proof. a) Indeed, let y is defined by (1.4), then

$$\Lambda(y(\lambda z)) = \Lambda\left(1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)}\right) = \sum_{k=1}^{\infty} \frac{\lambda^k \Lambda(z^k)}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)}. \quad (2.1)$$

Taking into consideration that

$$\Lambda(z^k) = \frac{d}{dz} z^k + \frac{1}{z}\sum_{j=0}^{m-1} \alpha_j \left(\alpha_j^k z^k\right) = k z^{k-1} + \frac{1}{z}\sum_{j=0}^{m-1} \alpha_j^{k+1} z^k =$$

$$= \left(k + \sum_{j=0}^{m-1} \alpha_j^{k+1}\right) z^{k-1} = \widehat{F}(k) z^{k-1}, \quad k \in \mathbb{N}, \ \widehat{F}(0) = 0, \quad (2.2)$$

we obtain

$$\Lambda(y(\lambda z)) = \sum_{k=1}^{\infty} \frac{\lambda^k z^{k-1}}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)} \widehat{F}(k) = \lambda\left(1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)}\right) = \lambda y(\lambda z).$$

Let us prove the uniqueness of eigenfunction. Let $d \in H(\mathbb{C})$ be such that $\Lambda(d(\lambda z)) = \lambda d(\lambda z)$. If $d(z) = \sum_{k=0}^{\infty} b_k z^k$, where $b_0 = 1$, then

$$\Lambda(d(\lambda z)) = \sum_{k=0}^{\infty} b_k \lambda^k \Lambda(z^k) = \sum_{k=1}^{\infty} b_k \lambda^k \widehat{F}(k) z^{k-1} = \frac{1}{z} \sum_{k=0}^{\infty} b_k \widehat{F}(k) (\lambda z)^k. \tag{2.3}$$

On the other hand,

$$\Lambda(d(\lambda z)) = \lambda \sum_{k=0}^{\infty} b_k (\lambda z)^k. \tag{2.4}$$

Since $b_0 = 1$, it follows from (2.3) and (2.4) that

$$b_k = \frac{1}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}, \ k = 1, 2, \dots .$$

Hence, $d(\lambda z) \equiv y(\lambda z)$.

b) We recall that function $f \in H(\mathbb{C})$ is entire of exponential type if

$$\exists\, C, a > 0\colon \ |f(z)| \leqslant C e^{a|z|}, \quad z \in \mathbb{C}.$$

It follows from (1.4) that

$$|y(z)| \leqslant 1 + \sum_{k=1}^{\infty} \frac{|z|^k}{|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)|}. \tag{2.5}$$

Let us estimate $|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)|$. We have

$$|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)| = |\widehat{F}(1)||\widehat{F}(2)|\dots|\widehat{F}(k)| =$$

$$= \left|1 + \sum_{j=0}^{m-1} e^{\frac{4\pi ij}{m}}\right| \left|2 + \sum_{j=0}^{m-1} e^{\frac{6\pi ij}{m}}\right| \dots \left|k + \sum_{j=0}^{m-1} e^{\frac{2(k+1)\pi ij}{m}}\right| \leqslant$$

$$\leqslant \left(1 + \sum_{j=0}^{m-1} \left|e^{\frac{4\pi ij}{m}}\right|\right) \left(2 + \sum_{j=0}^{m-1} \left|e^{\frac{6\pi ij}{m}}\right|\right) \dots \left(k + \sum_{j=0}^{m-1} \left|e^{\frac{2(k+1)\pi ij}{m}}\right|\right) =$$

$$= (1+m)(2+m)\dots(k+m) = \frac{(k+m)!}{m!}.$$

Since $\widehat{F}(k)$ takes the values:

$$\widehat{F}(k) = \begin{cases} k+m, \ \text{if } k = lm-1, \ l \in \mathbb{N}; \\ k, \ \text{if } k \neq lm-1, \ l \in \mathbb{N}, \end{cases}$$

then it is obvious that

$$|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)| \geqslant k!. \tag{2.6}$$

Thus,

$$k! \leqslant |\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)| \leqslant \frac{(k+m)!}{m!}. \tag{2.7}$$

By (2.6) we get

$$|y(z)| \leqslant 1 + \sum_{k=1}^{\infty} \frac{|z|^k}{k!} = e^{|z|}. \tag{2.8}$$

Consider the function $\psi(z) = 1 + \sum_{k=1}^{\infty} \frac{m!}{(k+m)!} z^k$ and let us calculate its order. We recall that the order of an arbitrary entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ can be calculated by the formula (cf.

[5])

$$\rho_f = \overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln|\frac{1}{a_k}|}.$$

Therefore,

$$\rho_\psi = \overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln\frac{(k+m)!}{m!}} = \overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln(k+m)! - \ln m!}$$
$$= \overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln k!} = \overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln\sqrt{2\pi k} + k(\ln k - 1)} = 1.$$

Since $\dfrac{1}{|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)|} \geqslant \dfrac{m!}{(k+m)!}$, the orders of corresponding functions satisfy inequality $\rho_y \geqslant \rho_\psi$. Employing estimate (2.8), we conclude $1 \leqslant \rho_y \leqslant 1$. The latter means that the order of function y is also 1.

Let us calculate the type for y. Since $\rho_y = 1$, we can employ formula (cf. [5])

$$\overline{\lim_{k\to\infty}} k^{\frac{1}{\rho_f}} \sqrt[k]{|a_k|} = (\sigma_f e\rho_f)^{\frac{1}{\rho_f}}, \tag{2.9}$$

where a_k the coefficients of function $f \in H(\mathbb{C})$, $0 < \rho_f < \infty$ and σ_f is the order and type of f, respectively.

In our case

$$a_k = \frac{1}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}, \quad k = 1, 2, \dots, \quad a_0 = 1.$$

Then, employing estimate (2.6) and Stirling approximation $k! \approx \sqrt{2\pi k}(\frac{k}{e})^k$, we deduce

$$\overline{\lim_{k\to\infty}} k^{\frac{1}{\rho_y}} \sqrt[k]{|a_k|} = \overline{\lim_{k\to\infty}} k \sqrt[k]{\frac{1}{|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)|}} = \overline{\lim_{k\to\infty}} k\frac{1}{\sqrt[k]{k!}}$$
$$= \overline{\lim_{k\to\infty}} k\frac{e}{k(2\pi k)^{\frac{1}{2k}}} = e\,\overline{\lim_{k\to\infty}} \frac{1}{(2\pi k)^{\frac{1}{2k}}} = e.$$

Applying (2.9), we find $\sigma_y e = e$. Therefore, $\sigma_y = 1$. Thus, $y \in P_{\mathbb{C}}$. □

Proposition 2. *The following product formula*

$$y(\lambda z)\cdot y(\lambda w) = \tau_w(y(\lambda.))(z), \quad z, w \in \mathbb{C}. \tag{2.10}$$

holds true.

Proof. Employing (1.4), we obtain

$$y(\lambda z)\cdot y(\lambda w) = \left(1 + \sum_{k=1}^{\infty} \frac{\lambda^k w^k}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}\right)\cdot y(\lambda z) = y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}\lambda^k y(\lambda z).$$

Since $\Lambda^k y(\lambda z) = \lambda^k y(\lambda z)$, then

$$y(\lambda z)\cdot y(\lambda w) = y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}\lambda^k y(\lambda z)$$
$$= y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)}\Lambda^k y(\lambda z) = \tau_w(y(\lambda.))(z).$$

□

3. Dunkl shift operator. Dunkl convolution

We consider first the properties of operator (1.2).

Proposition 3. *Operator Λ is a continuous mapping from $H(\mathbb{C})$ into $H(\mathbb{C})$.*

Proof. Let $f \in H(\mathbb{C})$. Without loss of generality we can let $f(0) = 1$. We write (1.2) as

$$\begin{aligned}\Lambda f(z) =& f'(z) + \frac{1}{z}\sum_{j=0}^{m-1} \alpha_j (f(\alpha_j z) - 1) + \frac{1}{z}\sum_{j=0}^{m-1}\alpha_j \\ =& f'(z) + \sum_{j=0}^{m-1}\alpha_j \frac{(f(\alpha_j z) - 1)}{z} = f'(z) + \sum_{j=0}^{m-1}\alpha_j^2 \int_0^1 f'(\alpha_j z t)dt.\end{aligned} \tag{3.1}$$

Then employing Cauchy integral formula, for each $R > 0$ we obtain

$$\|\Lambda f\|_R \leqslant \|f'\|_R + \sum_{j=0}^{m-1} |\alpha_j|^2 \|f'\|_R = (m+1)\|f'\|_R \leqslant (m+1)\frac{\|f\|_{2R}}{R},$$

where $\|f\|_R = \max\limits_{|z|\leqslant R} |f(z)|$. Thus, Λ: $H(\mathbb{C}) \to H(\mathbb{C})$ is a continuous operator. □

Theorem 1. *If $f \in H(\mathbb{C}), f(0) = 1$, then f can be represented as*

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{(\Lambda^k f)(0)}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)} z^k, \quad z \in \mathbb{C}.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ a_0 = 1, \ z \in \mathbb{C}. \tag{3.2}$$

Then by the continuity of operator Λ for each $k \in \mathbb{N}$

$$(\Lambda^k f)(z) = \sum_{n=1}^{\infty} a_n \Lambda^k(z^n),$$
$$\Lambda^k(z^n) = \widehat{F}(n)\widehat{F}(n-1)\dots\widehat{F}(n-k+1)z^{n-k}, \quad k = \overline{1,n}, \quad n = 1, 2, \dots.$$

In particular, $\Lambda^k(z^k) = \widehat{F}(k)\widehat{F}(k-1)\dots\widehat{F}(1)$ and $\Lambda^k(z^n) = 0$ as $n < k$ or $n > k$. Hence,

$$(\Lambda^k f)(0) = a_k \widehat{F}(k)\widehat{F}(k-1)\dots\widehat{F}(1).$$

Thus,

$$a_k = \frac{(\Lambda^k f)(0)}{\widehat{F}(k)\widehat{F}(k-1)\dots\widehat{F}(1)}, \ k = 1, 2, \dots$$

Substituting the latter into (3.2), we complete the proof. □

Proposition 4. *Series (1.5) converges in $H(\mathbb{C})$ and τ_w: $H(\mathbb{C}) \to H(\mathbb{C})$ is a continuous operator.*

Proof. Let $f \in H(\mathbb{C})$. By (3.1) we obtain

$$\begin{aligned}(\Lambda^2 f)(z) =& f''(z) + \sum_{j_1=0}^{m-1} \alpha_{j_1}^2 \int_0^1 (1+\alpha_{j_1} t) f''(\alpha_{j_1} z t)\, dt \\ &+ \sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1} \alpha_{j_1}^3 \alpha_{j_2}^2 \int_0^1\int_0^1 t_1 f''(\alpha_{j_1}\alpha_{j_2} z t_1 t_2)\, dt_1 dt_2, \\ (\Lambda^3 f)(z) =& f'''(z) + \sum_{j_1=0}^{m-1} \alpha_{j_1}^2 \int_0^1 (1+\alpha_{j_1} t + \alpha_{j_1}^2 t^2) f'''(\alpha_{j_1} z t)\, dt \\ &+ \sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1} \alpha_{j_1}^3 \alpha_{j_2}^2 \int_0^1\int_0^1 t_1 (1+\alpha_{j_1} t_1 + \alpha_{j_1}\alpha_{j_2} t_1 t_2) f'''(\alpha_{j_1}\alpha_{j_2} z t_1 t_2)\, dt_1 dt_2 \\ &+ \sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}\sum_{j_3=0}^{m-1} \alpha_{j_1}^4 \alpha_{j_2}^3 \alpha_{j_3}^2 \int_0^1\int_0^1\int_0^1 t_1^2 t_2 f'''(\alpha_{j_1}\alpha_{j_2}\alpha_{j_3} z t_1 t_2 t_3)\, dt_1 dt_2 dt_3.\end{aligned}$$

By induction we get

$$\begin{aligned}(\Lambda^n f)(z) = f^{(n)}(z) + \sum_{k=1}^{n} \sum_{j_1=0}^{m-1} \dots \sum_{j_k=0}^{m-1} \alpha_{j_1}^{k+1} \alpha_{j_2}^{k} \dots \alpha_{j_k}^2 \\ \cdot \int_0^1 \dots \int_0^1 P_{k,n}(t_1,\dots,t_k) f^{(n)}(\alpha_{j_1}\dots\alpha_{j_k} z t_1 \dots t_k)\, dt_1 \dots dt_k,\end{aligned}$$

where $P_{k,n}$ is a polynomial in $t_1,\dots,t_k$, $1 \leqslant k \leqslant n$, satisfying inequality

$$|P_{k,n}(t_1,\dots,t_k)| \leqslant \binom{n}{k}, \quad t_1,\dots,t_k \in [0,1].$$

Then taking into consideration that $|\alpha_{j_1}| = |\alpha_{j_2}| = \dots = |\alpha_{j_k}| = 1,\ k = 1,\dots,n,\ n = 1,2,\dots$, we obtain

$$\begin{aligned}\|\Lambda^n f\|_R \leqslant& \|f^{(n)}\|_R + \sum_{k=1}^{n} \binom{n}{k} \sum_{j_1=0}^{m-1} \dots \sum_{j_k=0}^{m-1} |\alpha_{j_1}|^{k+1} |\alpha_{j_2}|^k \dots |\alpha_{j_k}|^2 \|f^{(n)}\|_R \\ =& \left(1 + \sum_{j_1=0}^{m-1} \dots \sum_{j_k=0}^{m-1} (2^n - 1)\right) \|f^{(n)}\|_R = (1 + (2^n-1)m^n) \|f^{(n)}\|_R\end{aligned}$$

By Cauchy integral formula, for each $R > 0$

$$\|f^{(n)}\|_R \leqslant \frac{n!}{R^n} \|f\|_{2R}.$$

Then

$$\|\Lambda^n f\|_R \leqslant (1 + (2^n-1)m^n) \frac{n!}{R^n} \|f\|_{2R}. \tag{3.3}$$

Employing (3.3), we find that for each $R > 0$ and $|z| \leqslant R$

$$\varlimsup_{n\to\infty} \left| \frac{w^n (\Lambda^n f)(z)}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(n)} \right|^{\frac{1}{n}} \leqslant \varlimsup_{n\to\infty} \left| \frac{w^n \|\Lambda^n f\|_R}{n!} \right|^{\frac{1}{n}} \leqslant \frac{2m|w|}{R}.$$

It implies that for each $z, w \in \mathbb{C}$ series (1.5) converges uniformly on each compact set $\mathbb{C} \times \mathbb{C}$. Thus, $\tau_w f \in H(\mathbb{C})$. Let us show that $\|\tau_w f\|_R \leqslant M(R,w)\|f\|_{2R+4m|w|}$. This estimate will imply the continuity of operator τ_w. Indeed, applying (3.3), we obtain

$$\|\tau_w f\|_R \leqslant \|\tau_w f\|_{R+2m|w|} = \left\| f(z) + \sum_{k=1}^{\infty} \frac{w^k}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(k)} \Lambda^k f(z) \right\|_{R+2m|w|}$$

$$\leqslant \left(1 + \sum_{k=1}^{\infty} \frac{|w|^k}{(R+2m|w|)^k}(1+(2^k-1)m^k)\right) \|f\|_{2(R+2m|w|)} \leqslant M(R,w)\|f\|_{2R+4m|w|}.$$

The obtained series converges since

$$\overline{\lim_{k\to\infty}} \left(\frac{(1+(2^k-1)m^k)|w|^k}{(R+2m|w|)^k}\right)^{\frac{1}{k}} = \frac{2m|w|}{R+2m|w|} < 1.$$

Therefore, τ_w is a continuous operator. □

The following statement characterizes the relation between generalized Dunkl operator and Gel'fond-Leont'ev operator.

Theorem 2. *Operator* (1.2) *is a particular case of Gel'fond-Leont'ev generalized differentiation operator.*

Proof. We take a function $f \in H(\mathbb{C})$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Gel'fond-Leont'ev generalized differentiation operator (cf. [6]) acts on function f as

$$D^k[f](z) = \sum_{n=k}^{\infty} a_n \frac{b_{n-k}}{b_n} z^{n-k},$$

where b_n are the coefficients of some entire function $F(z)$ of order ρ ($0 < \rho < \infty$) and type σ ($0 < \sigma < \infty$), at that $b_n \neq 0$, $n \geqslant 0$, and there exists

$$\lim_{n\to\infty} n^{\frac{1}{\rho}} \sqrt[n]{|b_n|} = (\sigma e \rho)^{\frac{1}{\rho}}. \tag{3.4}$$

Consider the function

$$y(z) = \sum_{n=0}^{\infty} b_n z^n = 1 + \sum_{n=1}^{\infty} \frac{1}{\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(n)} z^n, \quad b_0 = 1.$$

It is obvious that $b_n \neq 0$, $n \geqslant 0$. Since function y is of exponential type, taking into consideration that $\sigma = 1$, employing estimate (2.7) and Stirling's approximation, we obtain

$$\lim_{n\to\infty} n\sqrt[n]{|b_n|} = \lim_{n\to\infty} n \frac{1}{\sqrt[n]{|\widehat{F}(1)\widehat{F}(2)\dots\widehat{F}(n)|}} = \lim_{n\to\infty} n \frac{1}{\sqrt[n]{n!}}$$

$$= \lim_{n\to\infty} n \frac{e}{n(2\pi n)^{\frac{1}{2n}}} = e \lim_{n\to\infty} \frac{1}{(2\pi n)^{\frac{1}{2n}}} = e.$$

Therefore, condition (3.4) is satisfied. We apply operator Λ to $f \in H(\mathbb{C})$:

$$\Lambda f(z) = \sum_{n=1}^{\infty} a_n \Lambda(z^n).$$

It follows from (2.2) that $\Lambda(z^n) = \widehat{F}(n)z^{n-1}$. Therefore,

$$\Lambda f(z) = \sum_{n=1}^{\infty} a_n \widehat{F}(n) z^{n-1} = \sum_{n=1}^{\infty} a_n \frac{b_{n-1}}{b_n} z^{n-1},$$

$$\Lambda^2 f(z) = \sum_{n=1}^{\infty} a_n \widehat{F}(n)\Lambda(z^{n-1}) = \sum_{n=2}^{\infty} a_n \widehat{F}(n)\widehat{F}(n-1) z^{n-2} = \sum_{n=2}^{\infty} a_n \frac{b_{n-2}}{b_n} z^{n-2}.$$

By induction in k, we see that if identity

$$\Lambda^{k-1} f(z) = \sum_{n=k-1}^{\infty} a_n \widehat{F}(n)\widehat{F}(n-1)\ldots\widehat{F}(n-k+2) z^{n-k+1} = \sum_{n=k-1}^{\infty} a_n \frac{b_{n-k+1}}{b_n} z^{n-k+1}$$

is satisfied, then

$$\begin{aligned}\Lambda^k f(z) =& \Lambda(\Lambda^{k-1} f(z)) = \sum_{n=k-1}^{\infty} a_n \widehat{F}(n)\widehat{F}(n-1)\ldots\widehat{F}(n-k+2)\Lambda(z^{n-k+1}) \\ &= \sum_{n=k}^{\infty} a_n \widehat{F}(n)\widehat{F}(n-1)\ldots\widehat{F}(n-k+2)\widehat{F}(n-k+1) z^{n-k} = \sum_{n=k}^{\infty} a_n \frac{b_{n-k}}{b_n} z^{n-k}.\end{aligned}$$

Thus, we have obtained the required representation. □

Let us provide some properties of Dunkl convolution operator (1.6). Let X be a topological vector space, L be a linear continuous operator in X.

Definition 1. *A linear continuous operator L: $X \to X$ is called hypercyclic, if there exists an element $x \in X$ (called hypercyclic vector of operator L), such that its orbit $\{L^n x,\ n = 0, 1, 2, \ldots\}$ is dense in X.*

Each hypercyclic operator L is topologically transitive in the sense of dynamical system, i.e., for each pair of open and non-empty subsets U and V in X there exists $n \in \mathbb{N}$ such that $L^n(U) \cap V \neq \varnothing$.

Definition 2. *Point $x \in X$ is called periodic for L if $L^n x = x$ for some $n \in \mathbb{N}$.*

Definition 3. *Operator L: $X \to X$ is called chaotic if it is topologically transitive and its has dense set of periodic points.*

Proposition 5. *Let $T \in H^*(\mathbb{C})$.*
1) Operator (1.6) acts continuously from $H(\mathbb{C})$ into $H(\mathbb{C})$.
2) Dunkl convolution operator is hypercyclic and chaotic on $H(\mathbb{C})$.

Proof. 1) Consider sequence $(f_n)_{n\in\mathbb{N}} \in H(\mathbb{C})$:

$$f_n \to f,\ M_T[f_n] \to g \text{ as } n \to \infty,\ f, g \in H(\mathbb{C}).$$

For each $w \in \mathbb{C}$ it follows from Proposition 4 that

$$\tau_w f_n \to \tau_w f \text{ as } n \to \infty \text{ in } H(\mathbb{C}).$$

Then

$$M_T[f_n](z) \to M_T[f](z) \text{ as } n \to \infty \text{ for each } z \in \mathbb{C}.$$

Applying theorem on closed graph, we obtain that M_T: $H(\mathbb{C}) \to H(\mathbb{C})$ is a continuous operator.

2) Since Dunkl generalized differentiation operator is a particular case of Gel'fond-Leont'ev generalized differentiation operator, Theorem 1 in work [7] holds true which implies that Dunkl convolution operator (1.6) is hypercyclic and chaotic. □

4. Corollaries of Theorem 2

Theorem 2 implies a series of important corollaries.

4.1. Dunkl transform. Denote by $P_a(\mathbb{C})$ the space of entire functions of exponential type:

$$|f(z)| \leqslant Ce^{a|z|}, \quad z \in \mathbb{C}, \quad C, a > 0,$$

where constant C depends on f. In this space we introduce norm p_a

$$p_a(f) = \sup_{z\in\mathbb{C}} |f(z)|e^{-a|z|}.$$

As it is known $P_a(\mathbb{C})$ is a Banach space. Then

$$P_{\mathbb{C}} = \bigcup_{a>0} P_a(\mathbb{C}).$$

Space $P_{\mathbb{C}}$ is equipped by the topology of inductive limit. We define Dunkl functional $T \in H^*(\mathbb{C})$ by the formula

$$\mathfrak{D}(T)(z) = \langle T_w, y(wz)\rangle = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(n)} z^n,$$

where $a_n = \langle T_w, w^n\rangle$, $n \geq 0$, $z \in \mathbb{C}$.

Applying the result of work [8], we obtain

Corollary 1. *Dunkl transform $\mathfrak{D}$ makes a topological isomorphism between spaces $H^*(\mathbb{C})$ and $P(\mathbb{C})$.*

4.2. Convolution equation generated by generalized Dunkl operator. Consider Dunkl convolution operator $M_T[f](z) = \langle T_w, (\tau_w f)(z)\rangle$, $z, w \in \mathbb{C}$. In view of (1.5) we rewrite it as

$$M_T[f](z) = a_0 f(z) + \sum_{k=1}^{\infty} \frac{a_k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)} \Lambda^k f(z) = \sum_{k=0}^{\infty} c_k \Lambda^k f(z), \tag{4.1}$$

where

$$c_0 = a_0, \quad c_k = \frac{a_k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)}, \ a_k = \langle T_w, w^k\rangle, \ k = 1, 2, \ldots.$$

4.2.1. Homogeneous convolution equation. Homogeneous convolution equation is an equation of the form $M_T[f](z) = 0$. By (4.1) we get

$$M_T[f](z) = \sum_{k=0}^{\infty} c_k \Lambda^k f(z) = 0. \tag{4.2}$$

The characteristic function of equation (4.2) is

$$\breve{T}(\lambda) = \langle T, y(\lambda z)\rangle = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\widehat{F}(1)\widehat{F}(2)\ldots\widehat{F}(k)} \lambda^k = \sum_{k=0}^{\infty} c_k \lambda^k.$$

Taking into consideration Theorem 2 and the result of work [9], we obtain that equation (4.2) has solutions of the form $z^m y^{(m)}(\lambda_n z)$, $m = 0, 1, \ldots, p_n - 1$, $n = 1, 2, \ldots$, where λ_1, λ_2, ... are the zeroes of characteristic function $\breve{T}(\lambda)$ of multiplicities p_1, p_2, ..., respectively.

We call the solution of the form $z^m y^{(m)}(\lambda_n z)$, $m = 0, 1, \ldots, p_n - 1$, $n = 1, 2, \ldots$ primitive solutions to equation (4.2). We indicate the set of such solutions by E. Let W be a set of all entire solutions to equation (4.2). Then [9, Thm. 3.3.5] implies

Corollary 2. *The closure of linear span of set E in $H(\mathbb{C})$ coincides with W.*

In $H(\mathbb{C})$ we consider the non-homogeneous convolution equation

$$M_T[f](z) = g(z), \quad g(z) \in H(\mathbb{C}). \tag{4.3}$$

Corollary 3 ([9])**.** *Equation (4.3) is solvable in $H(\mathbb{C})$ for each function $g \in H(\mathbb{C})$.*

BIBLIOGRAPHY

1. B.Ya. Levin. *Distribution of zeroes of entire functions.* GITTL, Moscow (1956). [Amer. Math. Soc., Providence, RI. (1964).]
2. J.J. Betancor, M. Sifi, K. Trimeche. *Hypercyclic and chaotic convolution operators associated with the Dunkl operators on* $\mathbb{C}$. Acta Math. Hungar. **106**:1-2, 101-116 (2005).
3. C.F. Dunkl. *Differential-difference operators associated with reflections groups.* Trans. Amer. Math. Soc. **311**:1, 167-183 (1989).
4. M. Rösler. *Dunkl operators: theory and applications.* in "Orthogonal Polynomials and Special Functions (Leuven, 2002)". Lecture Notes in Math. Springer-Verlag, Berlin, 93-135 (2003).
5. A.F. Leont'ev. *Entire functions. Exponential series.* Nauka, Moscow (1983).
6. A.O. Gel'fond, A.F. Leont'ev. *On a generalization of Fourier series.* Matem. sbornik. **29**:3, 477-500 (1951). (in Russian).
7. V.É. Kim. *Hypercyclicity and chaotic character of generalized convolution operators generated by Gel'fond-Leont'ev operators.* Matem. zametki. **85**:6, 849-856 (2009). [Math. Notes. **85**:6, 807-813 (2009).]
8. S.V. Panyushkin. *Generalized fourier transform and its applications.* Matem. zametki. **79**:4, 581-596 (2006). [Math. Notes. **79**:4, 537-550 (2006).]
9. A.F. Leont'ev. *Generalization of series of exponentials.* Moscow, Nauka (1981). (in Russian).

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