GENERALIZED DUNKL OPERATOR

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Abstract. In the paper we introduce a generalized Dunkl operator acting in the space of entire functions on \( \mathbb{C} \). We study problems of harmonic analysis related with this operator and show its connection with the Gelfond-Leont’ev operator of generalized differentiation.

Keywords: Dunkl operator, eigenfunction, Dunkl convolution operator, Dunkl transform, characteristic function, hypercyclic operator.

1. Introduction

Let \( H(\mathbb{C}) \) be the space of entire functions with the topology of uniform convergence on compact sets, \( H^*(\mathbb{C}) \) is the strongly dual space for \( H(\mathbb{C}) \), \( P_\mathbb{C} \) is the space of entire functions of exponential type. It is known that space \( H^*(\mathbb{C}) \) is isomorphic to \( H_0(\{\infty\}) \), which is the space of functions analytic in the vicinity of the point at infinity and vanishing at the point \( \infty \) (see, for instance, [1]). Moreover, if \( F \in H^*(\mathbb{C}) \) and \( g_F \in H_0(\{\infty\}) \) is the associated function (according to the mentioned isomorphism), then

\[
(F, f) = \frac{1}{2\pi i} \int_C f(z)g_F(z) \, dz,
\]

where \( f \in H(\mathbb{C}) \), \( C \) is a closed rectifiable contour enveloping all the singularities of function \( g_F \) and located in the analyticity domain for this function.

Consider the generalized Dunkl operator \( \Lambda \) on \( H(\mathbb{C}) \)

\[
\Lambda f(z) = \frac{d}{dz}f(z) + e^{2\pi i/m} \sum_{j=0}^{m-1} \alpha_j f(\alpha_j z), \quad z \in \mathbb{C},
\]

where \( \alpha_j = e^{2\pi i j/m}, \ j = 0, \ldots, m-1, \ f \in H(\mathbb{C}) \), \( m \) is a fixed natural number obeying \( m \geq 2 \). Without loss of generality, in what follows we assume \( c = 1 \).

This operator generalizes the studied before in work [2] Dunkl operator

\[
Df(z) = \frac{d}{dz}f(z) + \frac{c}{z} (f(z) - f(-z)), \quad z \in \mathbb{C}.
\]

Dunkl operators (cf. [3]) are differential-difference operators with finite groups of reflections in some Euclidian spaces. These operators play an important role in various problems of mathematics and physics (cf., for instance, [4]). We study problems of harmonic analysis related with operator (1.2) (Dunkl shift operators, Dunkl convolution, Dunkl transform and so forth).

Consider the function \( g \in H_0(\{\infty\}) \)

\[
g(z) = \frac{1}{z^2} + \sum_{j=0}^{m-1} \frac{e^{2\pi i j}}{z - 2\pi i/m}.
\]

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According to the aforementioned isomorphism, this function is associated with a functional $F \in H^*(\mathbb{C})$. We take the Laplace transform of this functional $\hat{F}(\mu) = (F, e^{\mu z})$. Applying (1.1), the transform $\hat{F}$ can be written as

$$
\hat{F}(\mu) = \frac{1}{2\pi i} \int_{C} e^{\mu z} g(z) \, dz = \frac{1}{2\pi i} \int_{C} e^{\mu z} \left( \frac{1}{z} + \sum_{j=0}^{m-1} \frac{e^{\frac{2\pi i j}{m}}}{z - \frac{2\pi i j}{m}} \right) \, dz = \mu + \sum_{j=0}^{m-1} e^{\frac{2\pi i j}{m}(\mu+1)}.
$$

Here contour $C$ envelopes the origin and points $\frac{2\pi i j}{m}$, $j = 0, m - 1$. We introduce the function

$$
y(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)}.
$$

In the second section we study the properties of eigenfunctions $y_{\lambda}$ of operator $\Lambda$ associated with an eigenvalue $\lambda$ and obeying condition $y_{\lambda}(0) = 1$. We shall show that function $y_{\lambda}$ is determined by the solution $y_{\lambda}(z) = y(\lambda z)$, where function $y$ is defined by (1.4).

Then by (1.2) we construct Dunkl shift operator $\tau_{w}$ (Section 3):

$$
(\tau_{w} f)(z) = f(z) + \sum_{k=1}^{\infty} \frac{w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \Lambda^k f(z), \quad z, w \in \mathbb{C}.
$$

Then Dunkl convolution operator of a functional $T \in H^*(\mathbb{C})$ and a function $f \in H(\mathbb{C})$ is determined as

$$
M_T[f](z) = T * f(z) = \langle T_w, (\tau_w f)(z) \rangle, \quad z, w \in \mathbb{C}.
$$

In conclusion we introduce Dunkl transform of functional $T \in H^*(\mathbb{C})$

$$
\mathcal{D}(T)(\lambda) = \hat{T}(\lambda) = \langle T, y(\lambda z) \rangle, \quad \lambda \in \mathbb{C},
$$

which establishes a topological isomorphism between spaces $H^*(\mathbb{C})$ and $P_{\mathbb{C}}$. We also consider generalized convolution equations, both homogeneous and non-homogeneous.

2. Eigenfunction of Dunkl operator $\Lambda$

**Proposition 1.** a) The eigenfunction $y_{\lambda}$ of operator $\Lambda$ associated with an eigenvalue $\lambda$ and obeying $y_{\lambda}(0) = 1$ is unique and is determined by formula (1.4).

b) Function $y(z)$ is an entire function of exponential type, and its type is $\sigma = 1$.

**Proof.** a) Indeed, let $y$ is defined by (1.4), then

$$
\Lambda(y(\lambda z)) = \Lambda \left( 1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \right) = \sum_{k=1}^{\infty} \frac{\lambda^k \Lambda(z^k)}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)}.
$$

Taking into consideration that

$$
\Lambda(z^k) = \frac{d}{dz} z^k + \frac{1}{z} \sum_{j=0}^{m-1} \alpha_j (\alpha_j^k z^k) = k z^{k-1} + \frac{1}{z} \sum_{j=0}^{m-1} \alpha_j^{k+1} z^k =
$$

$$
= \left( k + \sum_{j=0}^{m-1} \alpha_j^{k+1} \right) z^{k-1} = \hat{F}(k) z^{k-1}, \quad k \in \mathbb{N}, \quad \hat{F}(0) = 0,
$$

we obtain

$$
\Lambda(y(\lambda z)) = \sum_{k=1}^{\infty} \frac{\lambda^k z^{k-1}}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \hat{F}(k) = \lambda \left( 1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \right) = \lambda y(\lambda z).
$$

Let us prove the uniqueness of eigenfunction. Let \( d \in H(\mathbb{C}) \) be such that \( \Lambda(d(\lambda z)) = \lambda d(\lambda z) \). If \( d(z) = \sum_{k=0}^{\infty} b_k z^k \), where \( b_0 = 1 \), then
\[
\Lambda(d(\lambda z)) = \sum_{k=0}^{\infty} b_k \lambda^k \Lambda(z^k) = \sum_{k=1}^{\infty} b_k \lambda^k \hat{F}(k) z^{k-1} = \frac{1}{z} \sum_{k=1}^{\infty} b_k \hat{F}(k)(\lambda z)^k. \tag{2.3}
\]
On the other hand,
\[
\Lambda(d(\lambda z)) = \lambda \sum_{k=0}^{\infty} b_k (\lambda z)^k. \tag{2.4}
\]
Since \( b_0 = 1 \), it follows from (2.3) and (2.4) that
\[b_k = \frac{1}{\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)}, \quad k = 1, 2, \ldots .\]
Hence, \( d(\lambda z) \equiv y(\lambda z) \).

b) We recall that function \( f \in H(\mathbb{C}) \) is entire of exponential type if
\[\exists C, a > 0: |f(z)| \leq Ce^{a|z|}, \quad z \in \mathbb{C}.
\]
It follows from (1.4) that
\[|y(z)| \leq 1 + \sum_{k=1}^{\infty} \frac{|z|^k}{|\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)|} . \tag{2.5}\]

Let us estimate \(|\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)|\). We have
\[
|\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)| = |\hat{F}(1)||\hat{F}(2)||\ldots||\hat{F}(k)| = \left(1 + \sum_{j=0}^{m-1} e^{\frac{4\pi i j}{m}}\right) \left(2 + \sum_{j=0}^{m-1} e^{\frac{6\pi i j}{m}}\right) \ldots \left(k + \sum_{j=0}^{m-1} e^{\frac{2(k+1)\pi i j}{m}}\right) \leq \left(1 + m\right)\left(2 + m\right)\ldots\left(k + m\right) = \frac{(k + m)!}{m!}.
\]

Since \( \hat{F}(k) \) takes the values:
\[
\hat{F}(k) = \begin{cases} k + m, & \text{if } k = lm - 1, \ l \in \mathbb{N}; \\ k, & \text{if } k \neq lm - 1, \ l \in \mathbb{N}, \end{cases}
\]
then it is obvious that
\[|\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)| \geq k!. \tag{2.6}\]

Thus,
\[k! \leq |\hat{F}(1) \hat{F}(2) \ldots \hat{F}(k)| \leq \frac{(k + m)!}{m!}. \tag{2.7}\]

By (2.6) we get
\[|y(z)| \leq 1 + \sum_{k=1}^{\infty} \frac{|z|^k}{k!} = e^{|z|}. \tag{2.8}\]

Consider the function \( \psi(z) = 1 + \sum_{k=1}^{\infty} \frac{m!}{(k + m)!} z^k \) and let us calculate its order. We recall that

the order of an arbitrary entire function \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) can be calculated by the formula (cf.
\[
\rho_f = \lim_{k \to \infty} \frac{k \ln k}{\ln |\frac{1}{a_k}|}.
\]

Therefore,
\[
\rho_\psi = \lim_{k \to \infty} \frac{k \ln k}{\ln (\frac{(k+m)!}{m!})} = \lim_{k \to \infty} \frac{k \ln k}{\ln (k+m)! - \ln m!} = \lim_{k \to \infty} \frac{k \ln k}{
\ln \sqrt{2\pi k + k(\ln k - 1)} = 1.
\]

Since \(\frac{1}{|\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)|} \geq \frac{m!}{(k+m)!}\), the orders of corresponding functions satisfy inequality \(\rho_y \geq \rho_\psi\). Employing estimate (2.8), we conclude \(1 \leq \rho_y \leq 1\). The latter means that the order of function \(y\) is also 1.

Let us calculate the type for \(y\). Since \(\rho_y = 1\), we can employ formula (cf. [5])
\[
\lim_{k \to \infty} k^{\frac{1}{\rho_y}} \sqrt{|a_k|} = (\sigma_f \rho_f)^{\frac{1}{\rho_f}},
\]
(2.9)

where \(a_k\) the coefficients of function \(f \in H(\mathbb{C})\), \(0 < \rho_f < \infty\) and \(\sigma_f\) is the order and type of \(f\), respectively.

In our case
\[
a_k = \frac{1}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)}, \quad k = 1, 2, \ldots, \quad a_0 = 1.
\]

Then, employing estimate (2.6) and Stirling approximation \(k! \approx \sqrt{2\pi k} (\frac{k}{e})^k\), we deduce
\[
\lim_{k \to \infty} k^{\frac{1}{\rho_y}} \sqrt{|a_k|} = \lim_{k \to \infty} k^{\frac{1}{\rho_y}} \sqrt{\frac{1}{|\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)|}} = \lim_{k \to \infty} \frac{k}{\sqrt{k!}} = \lim_{k \to \infty} \frac{e}{\frac{k!}{k(2\pi k)^{\frac{1}{2}}}} = e.
\]

Applying (2.9), we find \(\sigma_y e = e\). Therefore, \(\sigma_y = 1\). Thus, \(y \in P_\mathbb{C}\).

**Proposition 2.** The following product formula
\[
y(\lambda z) \cdot y(\lambda w) = \tau_w(y(\lambda.))(z), \quad z, w \in \mathbb{C},
\]
holds true.

**Proof.** Employing (1.4), we obtain
\[
y(\lambda z) \cdot y(\lambda w) = \left(1 + \sum_{k=1}^{\infty} \frac{\lambda^k w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)}\right) \cdot y(\lambda z) = y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \lambda^k y(\lambda z).
\]
Since \(\Lambda^k y(\lambda z) = \lambda^k y(\lambda z)\), then
\[
y(\lambda z) \cdot y(\lambda w) = y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \lambda^k y(\lambda z)
\]
\[
= y(\lambda z) + \sum_{k=1}^{\infty} \frac{w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \Lambda^k y(\lambda z) = \tau_w(y(\lambda.))(z).
\]
\}
3. DUNKL SHIFT OPERATOR. DUNKL CONVOLUTION

We consider first the properties of operator (1.2).

**Proposition 3.** Operator \( \Lambda \) is a continuous mapping from \( H(\mathbb{C}) \) into \( H(\mathbb{C}) \).

**Proof.** Let \( f \in H(\mathbb{C}) \). Without loss of generality we can let \( f(0) = 1 \). We write (1.2) as

\[
\Lambda f(z) = f'(z) + \frac{1}{z} \sum_{j=0}^{m-1} \alpha_j (f(\alpha_j z) - 1) + \frac{1}{z} \sum_{j=0}^{m-1} \alpha_j \int_0^z f'(\alpha_j z t) dt.
\]

Then employing Cauchy integral formula, for each \( R > 0 \) we obtain

\[
\|\Lambda f\|_R \leq \| f' \|_R + \sum_{j=0}^{m-1} |\alpha_j|^2 \| f' \|_R = (m+1) \| f' \|_R \leq (m+1) \frac{\| f \|_R^2}{R},
\]

where \( \| f \|_R = \max_{|z| \leq R} |f(z)| \). Thus, \( \Lambda : H(\mathbb{C}) \to H(\mathbb{C}) \) is a continuous operator.

**Theorem 1.** If \( f \in H(\mathbb{C}) \), \( f(0) = 1 \), then \( f \) can be represented as

\[
f(z) = 1 + \sum_{k=1}^{\infty} \frac{\left(\Lambda^k f\right)(0)}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} z^k, \quad z \in \mathbb{C}.
\]

**Proof.** Let

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1, \quad z \in \mathbb{C}.
\]

Then by the continuity of operator \( \Lambda \) for each \( k \in \mathbb{N} \)

\[
\left(\Lambda^k f\right)(z) = \sum_{n=1}^{\infty} a_n \Lambda^k (z^n),
\]

\[
\Lambda^k (z^n) = \hat{F}(n)\hat{F}(n-1)\ldots\hat{F}(n-k+1)z^{n-k}, \quad k = 1, n, \quad n = 1, 2, \ldots.
\]

In particular, \( \Lambda^k (z^k) = \hat{F}(k)\hat{F}(k-1)\ldots\hat{F}(1) \) and \( \Lambda^k (z^n) = 0 \) as \( n < k \) or \( n > k \). Hence,

\[
\left(\Lambda^k f\right)(0) = a_k \hat{F}(k)\hat{F}(k-1)\ldots\hat{F}(1).
\]

Thus,

\[
a_k = \frac{\left(\Lambda^k f\right)(0)}{\hat{F}(k)\hat{F}(k-1)\ldots\hat{F}(1)}, \quad k = 1, 2, \ldots
\]

Substituting the latter into (3.2), we complete the proof.

**Proposition 4.** Series \( (1.5) \) converges in \( H(\mathbb{C}) \) and \( \tau_w : H(\mathbb{C}) \to H(\mathbb{C}) \) is a continuous operator.
Proof. Let \( f \in H(\mathbb{C}) \). By (3.1) we obtain

\[
(A^2 f)(z) = f''(z) + \sum_{k=0}^{m-1} \alpha_k \int_0^1 (1 + \alpha_k t) f''(\alpha_k z) \, dt \\
+ \sum_{j=1}^{m-1} \alpha_j \int_0^1 t f''(\alpha_j z) \, dt.
\]

By induction we get

\[
(A^n f)(z) = f^{(n)}(z) + \sum_{k=1}^{n} \sum_{j=1}^{m-1} \alpha^{k+1}_j \alpha^k_j \ldots \alpha^2_j \int_0^1 \ldots \int_0^1 P_{k,n}(t_1, \ldots, t_k) f^{(n)}(\alpha_j \ldots \alpha_k z t_1 \ldots t_k) \, dt_1 \ldots dt_k,
\]

where \( P_{k,n} \) is a polynomial in \( t_1, \ldots, t_k, 1 \leq k \leq n \), satisfying inequality

\[
|P_{k,n}(t_1, \ldots, t_k)| \leq \binom{n}{k}, \quad t_1, \ldots, t_k \in [0, 1].
\]

Then taking into consideration that \(|\alpha_{j_1}| = |\alpha_{j_2}| = \ldots = |\alpha_{j_k}| = 1, \ k = 1, \ldots, n, \ n = 1, 2, \ldots, \)

we obtain

\[
\|A^n f\|_R \leq \|f^{(n)}\|_R + \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{m-1} |\alpha_{j_1}|^{k+1} \sum_{j_2=0}^{m-1} |\alpha_{j_2}|^k \ldots \sum_{j_k=0}^{m-1} |\alpha_{j_k}|^2 \|f^{(n)}\|_R
\]

\[
= \left(1 + \sum_{j=1}^{m-1} \sum_{j=0}^{m-1} (2^n - 1) \right) \|f^{(n)}\|_R = (1 + (2^n - 1)m^n) \|f^{(n)}\|_R.
\]

By Cauchy integral formula, for each \( R > 0 \)

\[
\|f^{(n)}\|_R \leq \frac{n!}{R^n} \|f\|_{2R}.
\]

Then

\[
\|A^n f\|_R \leq (1 + (2^n - 1)m^n) \frac{n!}{R^n} \|f\|_{2R}. \tag{3.3}
\]

Employing (3.3), we find that for each \( R > 0 \) and \(|z| \leq R\)

\[
\lim_{n \to \infty} \frac{w^n(A^n f)(z)}{F(1)F(2) \ldots F(n)} \left| \frac{1}{n!} \right| \leq \lim_{n \to \infty} \frac{w^n \|A^n f\|_R}{n!} \left| \frac{1}{n} \right| \leq \frac{2m|w|}{R}.
\]
It implies that for each \( z, w \in \mathbb{C} \) series (1.5) converges uniformly on each compact set \( \mathbb{C} \times \mathbb{C} \). Thus, \( \tau_w f \in H(\mathbb{C}) \). Let us show that \( \| \tau_w f \|_R \leq M(R, w)\| f \|_{2R+4m|w|} \). This estimate will imply the continuity of operator \( \tau_w \). Indeed, applying (3.3), we obtain

\[
\| \tau_w f \|_R \leq \| \tau_w f \|_{R+2m|w|} = \left\| f(z) + \sum_{k=1}^{\infty} \frac{w^k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \Lambda^k f(z) \right\|_{R+2m|w|} \\
\leq \left(1 + \sum_{k=1}^{\infty} \left| \frac{w^k}{(R+2m|w|)^k} \right| (1 + (2^k - 1)m^k) \right) \| f \|_{2(R+2m|w|)} \leq M(R, w)\| f \|_{2R+4m|w|}.
\]

The obtained series converges since

\[
\lim_{k \to \infty} \left(1 + (2^k - 1)m^k\right) \frac{|w|^k}{(R+2m|w|)^k} = \frac{2m|w|}{R+2m|w|} < 1.
\]

Therefore, \( \tau_w \) is a continuous operator.

The following statement characterizes the relation between generalized Dunkl operator and Gel’fond-Leont’ev operator.

**Theorem 2.** Operator (1.2) is a particular case of Gel’fond-Leont’ev generalized differentiation operator.

**Proof.** We take a function \( f \in H(\mathbb{C}) \):

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n.
\]

Gel’fond-Leont’ev generalized differentiation operator (cf. [6]) acts on function \( f \) as

\[
D^k[f](z) = \sum_{n=0}^{\infty} a_n b_{n-k} \frac{1}{b_n} z^{n-k},
\]

where \( b_n \) are the coefficients of some entire function \( F(z) \) of order \( \rho \) (\( 0 < \rho < \infty \)) and type \( \sigma \) (\( 0 < \sigma < \infty \)), at that \( b_n \neq 0, n \geq 0 \), and there exists

\[
\lim_{n \to \infty} n^\frac{1}{\sigma} \sqrt{n!} = (\sigma e \rho)^\frac{1}{\sigma}.
\]

(3.4)

Consider the function

\[
y(z) = \sum_{n=0}^{\infty} b_n z^n = 1 + \sum_{n=1}^{\infty} \frac{1}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(n)} z^n, \quad b_0 = 1.
\]

It is obvious that \( b_n \neq 0, n \geq 0 \). Since function \( y \) is of exponential type, taking into consideration that \( \sigma = 1 \), employing estimate (2.7) and Stirling’s approximation, we obtain

\[
\lim_{n \to \infty} n^\frac{1}{\sigma} \sqrt{n!} = \lim_{n \to \infty} n^\frac{1}{\sigma} \frac{1}{\sqrt{|\hat{F}(1)\hat{F}(2)\ldots\hat{F}(n)|}} = \lim_{n \to \infty} n^\frac{1}{\sigma} \frac{1}{\sqrt{n!}}
\]

\[
= \lim_{n \to \infty} n \frac{1}{n^{(2\pi n)^{1/2}}} = e \lim_{n \to \infty} \frac{1}{n^{(2\pi n)^{1/2}}} = e.
\]

Therefore, condition (3.4) is satisfied. We apply operator \( \Lambda \) to \( f \in H(\mathbb{C}) \):

\[
\Lambda f(z) = \sum_{n=1}^{\infty} a_n \Lambda(z^n).
\]
It follows from (2.2) that \( \Lambda(z^n) = \hat{F}(n)z^{n-1} \). Therefore,

\[
\Lambda f(z) = \sum_{n=1}^{\infty} a_n \hat{F}(n)z^{n-1} = \sum_{n=1}^{\infty} a_n \frac{b_{n-1}}{b_n} z^{n-1},
\]

\[
\Lambda^2 f(z) = \sum_{n=1}^{\infty} a_n \hat{F}(n)\Lambda(z^{n-1}) = \sum_{n=2}^{\infty} a_n \hat{F}(n)\hat{F}(n-1)z^{n-2} = \sum_{n=2}^{\infty} a_n \frac{b_{n-2}}{b_n} z^{n-2}.
\]

By induction in \( k \), we see that if identity

\[
\Lambda^k f(z) = \sum_{n=k-1}^{\infty} a_n \hat{F}(n)\hat{F}(n-1)\ldots\hat{F}(n-k+2)z^{n-k+1} = \sum_{n=k-1}^{\infty} a_n \frac{b_{n-k+1}}{b_n} z^{n-k+1}
\]

is satisfied, then

\[
\Lambda^k f(z) = \Lambda(\Lambda^{k-1} f(z)) = \sum_{n=k-1}^{\infty} a_n \hat{F}(n)\hat{F}(n-1)\ldots\hat{F}(n-k+2)\Lambda(z^{n-k+1})
\]

\[
= \sum_{n=k}^{\infty} a_n \hat{F}(n)\hat{F}(n-1)\ldots\hat{F}(n-k+2)\hat{F}(n-k+1)z^{n-k} = \sum_{n=k}^{\infty} a_n \frac{b_{n-k}}{b_n} z^{n-k}.
\]

Thus, we have obtained the required representation.

Let us provide some properties of Dunkl convolution operator (1.6). Let \( X \) be a topological vector space, \( L \) be a linear continuous operator in \( X \).

**Definition 1.** A linear continuous operator \( L: X \rightarrow X \) is called hypercyclic, if there exists an element \( x \in X \) (called hypercyclic vector of operator \( L \)), such that its orbit \( \{L^n x, \ n = 0, 1, 2, \ldots \} \) is dense in \( X \).

Each hypercyclic operator \( L \) is topologically transitive in the sense of dynamical system, i.e., for each pair of open and non-empty subsets \( U \) and \( V \) in \( X \) there exists \( n \in \mathbb{N} \) such that \( L^n(U) \cap V \neq \emptyset \).

**Definition 2.** Point \( x \in X \) is called periodic for \( L \) if \( L^n x = x \) for some \( n \in \mathbb{N} \).

**Definition 3.** Operator \( L: X \rightarrow X \) is called chaotic if it is topologically transitive and its has dense set of periodic points.

**Proposition 5.** Let \( T \in H^*(\mathbb{C}) \).

1) Operator (1.6) acts continuously from \( H(\mathbb{C}) \) into \( H(\mathbb{C}) \).

2) Dunkl convolution operator is hypercyclic and chaotic on \( H(\mathbb{C}) \).

**Proof.** 1) Consider sequence \( (f_n)_{n \in \mathbb{N}} \in H(\mathbb{C}) : \)

\[
f_n \rightarrow f, \quad M_T[f_n] \rightarrow g \text{ as } n \rightarrow \infty, \quad f, g \in H(\mathbb{C}).
\]

For each \( w \in \mathbb{C} \) it follows from Proposition 4 that

\[
\tau_w f_n \rightarrow \tau_w f \text{ as } n \rightarrow \infty \text{ in } H(\mathbb{C}).
\]

Then

\[
M_T[f_n](z) \rightarrow M_T[f](z) \text{ as } n \rightarrow \infty \text{ for each } z \in \mathbb{C}.
\]

Applying theorem on closed graph, we obtain that \( M_T: H(\mathbb{C}) \rightarrow H(\mathbb{C}) \) is a continuous operator.

2) Since Dunkl generalized differentiation operator is a particular case of Gel’fond-Leont’ev generalized differentiation operator, Theorem 1 in work [7] holds true which implies that Dunkl convolution operator (1.6) is hypercyclic and chaotic.

4. Corollaries of Theorem 2

Theorem 2 implies a series of important corollaries.
4.1. Dunkl transform. Denote by $P_\alpha(\mathbb{C})$ the space of entire functions of exponential type: 

$$|f(z)| \leq Ce^{\alpha|z|}, \quad z \in \mathbb{C}, \quad C, \alpha > 0,$$

where constant $C$ depends on $f$. In this space we introduce norm $p_\alpha$

$$p_\alpha(f) = \sup_{z \in \mathbb{C}} |f(z)|e^{-\alpha|z|}.$$

As it is known $P_\alpha(\mathbb{C})$ is a Banach space. Then

$$P_\mathbb{C} = \bigcup_{\alpha > 0} P_\alpha(\mathbb{C}).$$

Space $P_\mathbb{C}$ is equipped by the topology of inductive limit. We define Dunkl functional $T \in H^*(\mathbb{C})$ by the formula

$$\mathcal{D}(T)(z) = \langle T_w, y(wz) \rangle = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(n)} z^n,$$

where $a_n = \langle T_w, w^n \rangle$, $n \geq 0$, $z \in \mathbb{C}$.

Applying the result of work [8], we obtain

**Corollary 1.** Dunkl transform $\mathcal{D}$ makes a topological isomorphism between spaces $H^*(\mathbb{C})$ and $P(\mathbb{C})$.

4.2. Convolution equation generated by generalized Dunkl operator. Consider Dunkl convolution operator $M_T[f](z) = \langle T_w, (\tau_w f)(z) \rangle$, $z, w \in \mathbb{C}$. In view of (1.5) we rewrite it as

$$M_T[f](z) = a_0 f(z) + \sum_{k=1}^{\infty} \frac{a_k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \Lambda^k f(z) = \sum_{k=0}^{\infty} c_k \Lambda^k f(z), \quad (4.1)$$

where

$$c_0 = a_0, \quad c_k = \frac{a_k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)}, \quad a_k = \langle T_w, w^k \rangle, \quad k = 1, 2, \ldots.$$

4.2.1. Homogeneous convolution equation. Homogeneous convolution equation is an equation of the form $M_T[f](z) = 0$. By (4.1) we get

$$M_T[f](z) = \sum_{k=0}^{\infty} c_k \Lambda^k f(z) = 0. \quad (4.2)$$

The characteristic function of equation (4.2) is

$$\check{T}(\lambda) = \langle T, y(\lambda z) \rangle = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\hat{F}(1)\hat{F}(2)\ldots\hat{F}(k)} \lambda^k = \sum_{k=0}^{\infty} c_k \lambda^k.$$

Taking into consideration Theorem 2 and the result of work [9], we obtain that equation (4.2) has solutions of the form $z^m y^{(m)}(\lambda_n z)$, $m = 0, 1, \ldots, p_n - 1$, $n = 1, 2, \ldots$, where $\lambda_1, \lambda_2, \ldots$ are the zeroes of characteristic function $\check{T}(\lambda)$ of multiplicities $p_1, p_2, \ldots$, respectively.

We call the solution of the form $z^m y^{(m)}(\lambda_n z)$, $m = 0, 1, \ldots, p_n - 1$, $n = 1, 2, \ldots$ primitive solutions to equation (4.2). We indicate the set of such solutions by $E$. Let $W$ be a set of all entire solutions to equation (4.2). Then [9, Thm. 3.3.5] implies

**Corollary 2.** The closure of linear span of set $E$ in $H(\mathbb{C})$ coincides with $W$.

In $H(\mathbb{C})$ we consider the non-homogeneous convolution equation

$$M_T[f](z) = g(z), \quad g(z) \in H(\mathbb{C}). \quad (4.3)$$

**Corollary 3 ([9]).** Equation (4.3) is solvable in $H(\mathbb{C})$ for each function $g \in H(\mathbb{C})$. 

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