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PROBLEM OF MULTIPLE INTERPOLATION IN CLASS OF ANALYTICAL FUNCTIONS OF ZERO ORDER IN HALF-PLANE

O.A. BOZHENKO, K.G. MALYUTIN

Abstract. In the paper we consider the problem of multiple interpolation in a class of functions of a zero order and type not exceeding normal in the upper half-plane of the complex variable. This problem belongs to the class of problems of free interpolation considered initially by A.F. Leont'ev. We find necessary and sufficient solvability conditions for this problem. The found criteria are formulated in terms of the canonical products constructed on interpolation nodes, and in terms of the Nevanlinna measure determined by these nodes. The work is a continuation of researches of the second author considered similar problems in classes of analytic functions in the upper half-plane of a nonzero order.

Keywords: zero proximate order, divisor, canonical product, multiple interpolation, Levin condition, Nevanlinna measure.

Mathematics Subject Classification: 30E05, 30D15

1. Introduction

The classical interpolation problem is to find a function in a given class (analytic functions with restrictions for the growth, in particular, entire functions, functions analytic in the upper half-plane, functions of class $\mathbf{H}^\infty$ and so forth) taken given values in given points, *interpolation nodes*.

In 1948, A.F. Leont'ev [1] considered for the first time an interpolation problem in the class of entire functions $[\rho, \infty]$ of finite order $\rho > 0$ called later *free interpolation problem.* These studies were continued by A.F. Leont'ev in works [2, 3] in classes $[\rho, \infty)$ of entire functions of normal type and order ρ. In a more general class $[\rho(r), \infty)$, where $\rho(r)$ is a given proximate order, the free interpolation problem was solved by O.S. Firsakova [5]. G.P. Lapin [4] extended the results of A.F. Leont'ev on free interpolation in the class $[\rho, \infty)$ to the problem on multiple interpolation. Multiple interpolation theory in the spaces of entire functions described by a proximate order $\rho(r)$ had a further development in works by A.V. Bratishchev [6], A.V. Bratishchev and Yu.F. Korobeinika [7]. Similar problems in the classes of functions analytic in upper half-plane were not studied well enough. We mention only the work [8], where there was solved the multiple interpolation problem in the class of analytic functions of non-zero finite order and of normal type. The complete results for the half-plane are known for the class $\mathbf{H}^\infty$ starting from the famous Carleson theorem and numerous works devoted to this subject. The present work is a continuation of the studies of the second author [8, 10].

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2. Classes of analytic function in half-plane

We shall make use of the terminology of works [8, 10]

We indicate by $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ the upper half-plane. By $C(a, r)$ we denote an open ball of radius r centered at a, while $B(a, r)$ stands for the similar closed ball. Let Ω_+ be the intersection of a set Ω with the half-plane $\mathbb{C}_+$: $\Omega_+ = \Omega \cap \mathbb{C}_+$.

Let $D = \{a_n, q_n\}_{n=1}^{\infty}$ be a divisor, i.e., the set of different complex numbers $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$ taken counting multiplicity $\{q_n\}_{n=1}^{\infty} \subset \mathbb{N}$. Given a divisor $D = \{a_n, q_n\}_{n=1}^{\infty}$, $a_n = r_n e^{i\theta_n}$, we define measures $n_D(G) = \sum_{a_n \in G} q_n$, $\tilde{n}_D^+(G) = \sum_{a_n \in G} q_n \operatorname{Im} a_n$, $n_D^+(G) = \sum_{a_n \in G \setminus B(0,1)} q_n \sin\theta_n + \tilde{n}_D^+(G \cap B(0,1))$. If it does not lead to ambiguity, we shall omit subscript D. The divisor of roots for an arbitrary function f will be indicated as D_f. We denote $n_f = n_{D_f}$, $n_f^+ = n_{D_f}^+$, $n_{f,a}(r) = n_f(C(a, r))$, $n_{f,a}^+(r) = n_f^+(C(a, r))$, $n_{D,a}(r) = n_D(C(a, r))$, $n_{D,a}^+(r) = n_D^+(C(a, r))$. In particular, we let $n_f(r) = n_{f,0}(r)$, $n_f^+(r) = n_{f,0}^+(r)$, $n_D(r) = n_{D,0}(r)$, $n_D^+(r) = n_{D,0}^+(r)$. All considered measured will be supposed to be continued into the complex plane assuming there restrictions on $\mathbb{C}_-$ being zero measure. Once we deal with internal measures defined on $\mathbb{C}_+$, there restriction on the real axis is zero measure.

Dealing with divisor $D_f = \{a_n, q_n\}_{n=1}^{\infty}$ of zeroes of some function f, we sometimes denote it by $\{z_n\}_{n=1}^{\infty}$, where in the sequence $\{z_n\}_{n=1}^{\infty}$ the point a_n appears exactly q_n times.

A differentiable on semi-axis $(0, +\infty)$ differentiable function $\rho(r)$ is called *proximate order* if the conditions

1) $\lim\limits_{r\to\infty} \rho(r) = \rho$,
2) $\lim\limits_{r\to\infty} r\rho'(r) \ln r = 0$.

hold true. The detailed exposition on the properties of proximate order can be found in works [11, 12, 13]. In the paper we employ the notation $V(r) = r^{\rho(r)}$. In addition we assume that $V(r) \equiv 1$ as $r \in [0, 1]$. This assumption does not lead to the loss of generality but simplifies some arguments.

Within the work we shall actively make use of a well-known property of proximate order which we formulate as the next lemma.

Lemma 1. *Let $\rho(r)$ be a proximate order. Then for each $t > 0$*

$$\lim_{r\to\infty} \frac{V(rt)}{V(r)} = t^{\rho}, \tag{1}$$

and the limit is uniform on a fixed segment $[a, b] \subset (0, +\infty)$.

In the case number ρ in the definition of the proximate order vanishes, proximate order $\rho(r)$ called *zero proximate order*. In fact, we make no assumptions for the zero proximate order $\rho(r)$. However, we assume the following addition condition:

$$\lim_{r\to+\infty} \frac{V(r)}{\ln r} = +\infty\,. \tag{2}$$

proximate order $\rho(r)$ is called *formal order* of a function f if there exists a constant M_f depending on f only such for each $z \in \mathbb{C}_+$ the inequality

$$\log|f(z)| < M_f V(|z|) \tag{3}$$

holds true.

We shall employ the symbol $[\rho(r), \infty)_+$ to indicate the class of analytic in $\mathbb{C}_+$ functions f of formal order $\rho(r)$.

proximate order $\rho(r)$ is called *semi-formal order* of an analytic in $\mathbb{C}_+$ function f if $\rho(r)$ is a formal order of a function f and there holds the following Levin condition [11]: there exist numbers $q \in (0,1)$, $\delta \in (0, \pi/2)$ such that in each domain

$$D(R, q, \delta) = \{z : qR \leqslant |z| \leqslant \frac{1}{q}R,\ \delta < \arg z < \pi - \delta\}$$

there exists a point z, at which one has the inequality

$$\log|f(z)| \geqslant -M_f V(|z|)\,.$$

The class of analytic in $\mathbb{C}_+$ functions having $\rho(r)$ as the semi-formal order is denoted by $[\rho(r), \infty)_+^h$. This terminology is due to A.F. Grishin. It is clear that $[\rho(r), \infty)_+^h \subset [\rho(r), \infty)_+$.

If $\rho = \lim\limits_{r\to\infty} \rho(r) > 1$ and $\rho(r)$ is the formal order of a function f in $\mathbb{C}_+$, then $\rho(r)$ is also the semi-formal order for this function [14]. On the other hand, $\rho(r) \equiv 0$ is the formal order for the function e^{iz}, and $\rho(r) \equiv 1$ is the semi-formal order of this function. Indeed, function e^{iz} is bounded in the half-plane, and for each $z \in \mathbb{C}_+$ the inequality $|e^{iz}| \geqslant e^{-|z|}$ is satisfied.

Thus, the difference between formal and semi-formal order occurs in the half-plane only as $\rho \leqslant 1$, and in particular, as $\rho = 0$.

Functions f in class $[\rho(r), \infty)_+$ possess the following propertiesl [15]:

a) $\log|f(z)|$ has a non-tangential limit $\log|f(t)|$, $t \in \mathbb{R}$, almost everywhere on the real axis, $\log|f(t)| \in L^1_{loc}(-\infty, \infty)$;

b) on the real axis there exists a signed measure (charge) ν such that

$$\lim_{y\to+0} \int_a^b \log|f(t+iy)|\,dt = \nu([a,b]) - \frac{1}{2}(\nu(\{a\}) + \nu(\{b\}))\,.$$

Measure ν is called boundary measure of a function f;

c) $d\nu(t) = \log|f(t)|\,dt + d\sigma(t)$, where σ is a singular measure with respect to Lebesgue measure.

Following [15], for a function $f \in [\rho(r), \infty)_+$ we define the full measure λ as

$$\lambda(G) = 2\pi \int\limits_{\mathbb{C}_+\cap G} \operatorname{Im}\zeta\, d\mu(\zeta) - \nu(G)\,,$$

where μ is the Riesz measure for the subharmonic in the upper half-plane function $\log|f|$. Measure λ possesses the following properties

1) λ is a finite measure on each compact set $G \subset \mathbb{C}$,
2) λ is a nonnegative measure outside $\mathbb{R}$,
3) λ vanishes in the half-plane $\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}$.

We shall make use of the following lemma [15].

Lemma 2. *Let λ_f be the full measure for a function $f \in [\rho(r), \infty)_+$. Then inequality*

$$\iint\limits_{B_+(0,R)} \frac{d|\lambda_f|(\xi)}{1+|\xi|^2} \leqslant M_f \left(\int_0^R \frac{V(t)}{1+t^2}dt + \frac{V(R)}{R} \right) \tag{4}$$

holds true with some constant $M_f > 0$ independent of R.

3. Formulation of interpolation problem in class $[\rho(r), \infty)_+$ (in class $[\rho(r), \infty)_+^h$)

We denote $\Lambda_z = \min\{1; \operatorname{Im} z\}$, $\Lambda_n = \Lambda_{a_n}$. Let $f \in [\rho(r), \infty)_+$ ($f \in [\rho(r), \infty)_+^h$). The Cauchy formula for the derivatives implies easily the inequality

$$|f^{(k-1)}(z)| \leqslant \frac{(k-1)!}{\Lambda_z^{k-1}} \exp[M_f V(|z|)], \quad k \in \mathbb{N}\,.$$

This inequality leads us to a reasonability of introducing of the following definition.

Definition 1. *Divisor $D = \{a_n, q_n\}_{n=1}^{\infty}$ is called interpolation in class $[\rho(r), \infty)_+$ (in class $[\rho(r), \infty)_+^h$), if for each sequence of complex numbers $b_{n,k}$, $k = 1, 2, \dots, q_n$, $n \in \mathbb{N}$ satisfying condition*

$$\sup_n \frac{1}{V(|a_n|)} \sup_{1\leqslant k\leqslant q_n} \log^+ \frac{|b_{n,k}|\Lambda_n^{k-1}}{(k-1)!} < \infty \tag{5}$$

there exists a function $F \in [\rho(r), \infty)_+$ ($F \in [\rho(r), \infty)_+^h$) with property

$$F^{(k-1)}(a_n) = b_{n,k}, \quad k = 1, 2, \dots, q_n,\ n \in \mathbb{N}\,. \tag{6}$$

Given a divisor D, we define the families of functions

$$\Phi_D^+(z, \alpha) = \frac{n_D^+(C(z, \alpha|z|) \setminus \{a_n\})}{V(|z|)}, \quad \alpha > 0\,,$$

where a_n is the point in the support of divisor D closest to point z (if there exist several such points, we choose any of them). We let

$$I_D^+(z, \delta) = \sin\theta \int_0^{\delta} \frac{\Phi_D^+(z, \alpha)\, d\alpha}{\alpha(\alpha + \sin\theta)^2}, \quad \theta = \arg z\,.$$

We formulate the main theorem of our work.

Theorem. *Let $\rho(r)$ be the zero proximate order. Then the following three statements are equivalent.*

1) *Divisor D is an interpolation one in the class $[\rho(r), \infty)_+$ (in class $[\rho(r), \infty)_+^h$).*

2) *The conditions hold:*

2.1)

$$\sum_{n=1}^{\infty} \frac{q_n \operatorname{Im} a_n}{1 + |a_n|^2} < \infty\,, \tag{7}$$

2.2) *The canonical product*

$$E(z) = \prod_{|a_n|\leqslant 1} \left(\frac{z - a_n}{z - \bar{a}_n}\right)^{q_n} \prod_{|a_n|>1} \left(\frac{z - a_n}{z - \bar{a}_n} \cdot \frac{\bar{a}_n}{a_n}\right)^{q_n}$$

of divisor D satisfies condition:

$$\sup_n \frac{1}{V(|a_n|)} \log \frac{|\gamma_{n,1}|}{\Lambda_n^{q_n}} < \infty\,, \tag{8}$$

where

$$\gamma_{n,k} = \frac{1}{(k-1)!} \left(\frac{d}{dz}\right)^{k-1} \frac{(z - a_n)^{q_n}}{E(z)}\Bigg|_{z=a_n},\ k = 1, \dots, q_n, n \in \mathbb{N}\,.$$

3) *Conditions (7) are satisfied and*

3.1) *for each $\delta > 0$*

$$\sup_{z\in\mathbb{C}_+} I_D^+(z, \delta) < \infty\,; \tag{9}$$

3.2) *it holds*

$$\sup_{n\in\mathbb{N}} \frac{q_n}{V(|a_n|)} \log \frac{2\operatorname{Im} a_n}{\Lambda_n} < \infty\,. \tag{10}$$

4. Necessary solvability conditions for interpolation problem

Theorem. *Let $D=\{a_n,q_n\}_{n=1}^{\infty}$ be an interpolation divisor in class $[\rho(r),\infty)_+$ (in class $[\rho(r),\infty)_+^h$) and $\rho(r)$ is the zero proximate order. Then condition* (7) *is satisfied.*

Proof. Let F be a function in class $[\rho(r),\infty)_+$ solving interpolation problem $F(a_1)=1$, $F^{(k-1)}(a_1)=0$, $k=2,\dots,q_1$, $F^{(k-1)}(a_n)=0$, $k=1,\dots,q_n$ as $n\geq 2$. By the hypothesis of theorem such function exists. Since divisor D except point a_1 is a part of zeroes of function F, it follows from inequality (4) in Lemma 2 that

$$\sum_{|a_n|\leqslant R}\frac{q_n\mathrm{Im}\, a_n}{1+|a_n|^2}\leqslant M_F\left(\int\limits_0^R\frac{V(t)}{1+t^2}dt+\frac{V(R)}{R}\right) \tag{11}$$

with some constant $M_F>0$ independent of R.

Since $\rho(r)$ is a zero proximate order, then $V(R)\leqslant M_1R^{1/2}$ with some constant $M_1>0$ independent of R. This is why $\lim\limits_{R\to\infty}\dfrac{V(R)}{R}=0$ and integral $\int\limits_0^\infty\dfrac{V(t)}{1+t^2}dt$ converges. Then by (11) it implies (7). The proof is complete. □

Theorem. *Let $D=\{a_n,q_n\}_{n=1}^{\infty}$ be the interpolation divisor in class $[\rho(r),\infty)_+$ (in class $[\rho(r),\infty)_+^h$) and $\rho(r)$ is the zero proximate order. Then Item* 2) *of Theorem 3 holds true.*

Proof. Condition 2.1) follows from Theorem 4. The proof of Condition 2.2) reproduces word-by-word the proof of similar condition in work [8]. □

Theorem. *Let $D=\{a_n,q_n\}_{n=1}^{\infty}$ be an interpolation divisor in class $[\rho(r),\infty)_+$ (in class $[\rho(r),\infty)_+^h$) and $\rho(r)$ is the zero proximate order. Then Item* 3) *of Theorem 3.*

The proof of Conditions 3.1) and 3.2) was made in work [8] for $\rho>1$. The analysis of these statements shows that they are valid also for $0\leqslant\rho\leqslant 1$.

Theorem. *Let $D=\{a_n,q_n\}_{n=1}^{\infty}$ be a divisor such that Condition* (7) *holds true and $\rho(r)$ is the zero proximate order. Then Items* 2) *and* 3) *Theorem 3 are equivalent.*

In work [8] the equivalence of these conditions was proven $\rho>1$. And again one can make sure that it is also true for $0\leqslant\rho\leqslant 1$.

We shall make use of the following lemma in [8].

Lemma 3. *Let divisor $D=\{a_n,q_n\}_{n=1}^{\infty}$ be interpolation in class $[\rho(r),\infty)_+$ (in class $[\rho(r),\infty)_+^h$) and $\rho(r)$ is the zero proximate order. Then*

$$\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}\frac{q_k\mathrm{Im}\, a_k\mathrm{Im}\, a_n}{|a_n-\bar a_k|^2(1+r_k^2)^{\frac32}}<\infty\,. \tag{12}$$

We note [8] that if divisor D satisfies condition (8), then condition (10) holds true. Moreover, the following lemma is valid.

Lemma 4. *Suppose that divisor D satisfies condition* (8)*, then*

$$\sup_{n\in\mathbb{N}}\frac{1}{V(r_n)}\max_{1\leqslant k\leqslant q_n}\frac{|\gamma_{n,k}|}{\Lambda_n^{q_n-k+1}}<\infty\,. \tag{13}$$

Proof. In the proof we shall employ the following statement in work [9].

Let a function $G(\zeta)$ be analytic in a circle $C(0,r)$, $|G(\zeta)|\leqslant M$, and let $G(\zeta)$ have the zero of multiplicity m at the point $\zeta=0$ and the zero of multiplicity q at the point $\zeta=a$. Then

$$|a|^q\geqslant\frac{G^{(m)}(0)}{m!}\cdot\frac{r^{m+q}}{M}\,. \tag{14}$$

We denote by l_n the quantity

$$l_n = \min\left\{\Lambda_n/2, \operatorname{dist}\left(\{a_i\}_{i=1}^{\infty} \setminus \{a_n\}; \{a_n\}\right)\right\}, \quad n \in \mathbb{N},$$

where dist denotes the distance between the sets. Suppose, for instance, $l_n = |a_k - a_n|$. We let $G(\zeta) = E(a_k + \zeta)$, $r = \Lambda_k$. Noticing that in this case $\Lambda_k > \Lambda_n/2 \geqslant l_n = |a_k - a_n|$, we apply inequality (14) to function $G(\zeta)$. We have

$$l_n^{q_n} \geqslant \frac{E^{(q_k)}(a_k)}{q_k!} \cdot \frac{\Lambda_k^{q_k+q_n}}{\max\limits_{|\zeta - a_k| \leqslant \Lambda_k} |E(\zeta)|}.$$

This inequality, the boundedness of function $E(\zeta)$ ($|E(\zeta)| \leqslant 1$, $\zeta \in \mathbb{C}_+$), conditions (8), (10), and the properties of proximate order (1) yield that

$$l_n^{q_n} \geqslant \Lambda_n^{q_n} \exp(-M_1 V(|a_n|)), \quad n \in \mathbb{N} \tag{15}$$

for some $M_1 > 0$. By condition (10), this inequality is valid also for $l_n = \Lambda_n/2$, $n \in \mathbb{N}$.

We define an analytic in the circle $C(0,1)$ function $\psi(t)$ by the identity $\psi(t)t^{q_n} = E(a_n + l_n t)$. Applying l'Hospital rule as well as inequalities (8) and (15), we get

$$|\psi(0)| = l_n^{q_n} \frac{|E^{(q_n)}(a_n)|}{q_n!} \geqslant \exp(-M_2 V(|a_n|))$$

for some $M_2 > 0$. Moreover, as $|t| \leqslant 1$, function $\psi(t)$ is bounded since

$$|\psi(t)| \leqslant \max_{|t|=1} |\psi(t)| = \max_{|t|=1} |\psi(t)t^{q_n}| = \max_{|t|=1} |E(a_n + l_n t)| \leqslant 1.$$

We then apply the following theorem [11, Thm. 9, Ch. I, Sec. 6].

Theorem. *Suppose that a holomorphic in the circle $C(0,R)$ function $f(z)$ has no zeroes. Then in a circle $C(0,r)$, $r < R$, the inequality*

$$\log|f(z)| \geqslant \frac{-2r}{R-r} \max_{|\zeta| \leqslant R} \log|f(\zeta)| \tag{16}$$

holds true.

We let $g(\zeta) = \psi(\zeta)\psi^{-1}(0)$. Since function $g(\zeta)$ has no zeroes in the circle $C(0,1/2)$ and $g(0) = 1$, we can apply inequality (16), which for $|\zeta| \leqslant r = 1/4$ and $R = 1/2$ implies $g(\zeta) \geqslant \exp(-2M_2 V(|a_n|))$. Thus,

$$|E(a_n + \tau)| \geqslant \frac{|\tau|^{q_n}}{|l_n|^{q_n}} \exp(-M_3 V(|a_n|)), \quad |\tau| \leqslant \frac{l_n}{4} \tag{17}$$

for some $M_3 > 0$.

Then by the definition we have

$$\gamma_{n,k} = \frac{1}{2\pi i} \int\limits_{|\zeta - a_n| = l_n/4} \frac{(\zeta - a_n)^{q_n - k}}{E(\zeta)}\, d\zeta, \quad k \in \overline{1, q_n},\ n \in \mathbb{N}.$$

Inequality (13) now follows from this relation, definition l_n, (17), and (10). The proof is complete. $\square$

5. Proof of implication 2) ⇒ 1) in Theorem 3

Denote

$$\alpha_{n,m}=\frac{(-1)^{m-1}}{(m-1)!}\sum_{i=o}^{q_n-m}\frac{1}{i!}\gamma_{n,q_n+1-m-i}b_{n,i+1},\quad m\in\overline{1,q_n},\ n\in\mathbb{N}\,. \tag{18}$$

Re-ordering, if needed, the points of divisor D, we can assume that

$$\frac{\operatorname{Im}a_{n+1}}{1+r_{n+1}^2}\leqslant\frac{\operatorname{Im}a_n}{1+r_n^2},\quad n\in\mathbb{N}\,. \tag{19}$$

We let

$$\beta_n(z)=\sum_{k=n}^{\infty}\frac{1+\bar{a}_k(z+i\Lambda_n)}{i(\bar{a}_k-z-i\Lambda_n)}\frac{\operatorname{Im}a_k}{(1+r_k^2)^{\frac{3}{2}}},\quad n\in\mathbb{N}\,. \tag{20}$$

The series determining functions $\beta_n(z)$ in (20) converges uniformly in each of domains

$$D_{r,\delta}^n=\{z:|z|\leqslant r,\operatorname{Im}z\geqslant-\Lambda_n+\delta,\ \delta>0\}\,,$$

since as $z\in D_{r,\delta}^n$, $r\geqslant 2$,

$$\left|\frac{1+\bar{a}_k(z+i\Lambda_n)}{i(\bar{a}_k-z-i\Lambda_n)}\right|\frac{\operatorname{Im}a_k}{(1+r_k^2)^{\frac{3}{2}}}\leqslant\frac{\sqrt{(1+r)(1+r_k)}}{\delta}\frac{\operatorname{Im}a_k}{(1+r_k^2)^{\frac{3}{2}}}\,,$$

and series (7) converges.

Let us estimate $\operatorname{Re}\beta_n(z)$. We have

$$\operatorname{Re}\beta_n(z)=\sum_{k=n}^{\infty}\frac{(\operatorname{Im}a_k+\operatorname{Im}z+\Lambda_n+r_k^2(\operatorname{Im}z+\Lambda_n)+|z+i\Lambda_n|^2\operatorname{Im}a_k)}{|\bar{a}_k-z-i\Lambda_n|^2}\frac{\operatorname{Im}a_k}{(1+r_k^2)^{\frac{3}{2}}}\,. \tag{21}$$

Since $\operatorname{Im}a_n>0$, $\operatorname{Im}\bar{a}_k<0$, then $|\bar{a}_k-a_n-i\Lambda_n|>|\bar{a}_k-a_n|$. By Lemma 3, inequality (19), and (21) we obtain, in particular, that

$$\begin{aligned}\operatorname{Re}\beta_n(a_n)&\leqslant\sum_{k=n}^{\infty}\frac{\operatorname{Im}a_k(\operatorname{Im}a_k(1+|a_n+i\Lambda_n|^2)+2\operatorname{Im}a_n(1+r_k^2))}{|\bar{a}_k-a_n|^2(1+r_k^2)^{\frac{3}{2}}}\\&\leqslant\sum_{k=n}^{\infty}\left(\frac{\operatorname{Im}a_k}{1+r_k^2}+\frac{2\operatorname{Im}a_n}{1+4r_n^2}\right)\frac{\operatorname{Im}a_k(1+r_k^2)(1+4r_n^2)}{|\bar{a}_k-a_n|^2(1+r_k^2)^{\frac{3}{2}}}\\&\leqslant 5\frac{1+4r_n^2}{1+r_n^2}\sum_{k=n}^{\infty}\frac{\operatorname{Im}a_n}{|\bar{a}_k-a_n|^2}\frac{\operatorname{Im}a_k}{(1+r_k^2)^{\frac{1}{2}}}\leqslant K_1<\infty\,.\end{aligned} \tag{22}$$

And also

$$\operatorname{Re}\beta_n(z)\geqslant\sum_{k=n}^{\infty}\frac{(\operatorname{Im}a_k)^2}{(1+r_k^2)^{\frac{3}{2}}}\frac{1}{|\bar{a}_k-z-i\Lambda_n|^2}\,. \tag{23}$$

We then let

$$P_n(z)=\sum_{m=1}^{q_n}\alpha_{n,m}\left[\frac{\varphi_n(z)}{z-a_n}\right]^{(m-1)}\,, \tag{24}$$

where

$$\varphi_n(z)=\left(\frac{1+z\bar{a}_n}{1+r_n^2}\right)^3\frac{g(z)}{g(a_n)}\left(\frac{2\operatorname{Im}a_n}{z-\bar{a}_n}\right)^2\exp[\beta_n(a_n)-\beta_n(z)]\,,$$

$g(z)$ is an entire function of class $[\rho(r),\infty)_+$ (class $[\rho(r),\infty)_+^h$) which will be determined below.

We note that

$$\varphi_n(a_n)=1,\quad n\in\mathbb{N}\,. \tag{25}$$

Moreover, employing an elementary inequality $1+x \leqslant \sqrt{2(1+x^2)}$, for $|z| \geqslant 1$ we obtain

$$\left|\frac{1+z\bar{a}_n}{1+r_n^2}\right| \leqslant \frac{|z|(1+r_n)}{1+r_n^2} \leqslant \frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}}\,.$$

It yields that

$$|\varphi_n(z)| \leqslant 4\left(\frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}}\right)^3 \frac{|g(z)|}{|g(a_n)|}\frac{(\operatorname{Im} a_n)^2}{|z-\bar{a}_n|^2}\exp\{\operatorname{Re}[\beta_n(a_n)-\beta_n(z)]\}, \quad n\in\mathbb{N}\,. \tag{26}$$

The formal series

$$F(z) = E(z)\sum_{n=1}^{\infty} P_n(z) \tag{27}$$

solves interpolation problem (6) [8].

Let us show that under an appropriate choice choice of function $g(z)$, function $F(z)$ belongs to class $[\rho(r),\infty)_+$ ($[\rho(r),\infty)_+^h$). It follows from condition (5), inequality (13) and identity (18) that for each $m=1,\ldots,q_n$, $n\in\mathbb{N}$,

$$|\alpha_{n,m}| \leqslant \frac{q_n-m+1}{(m-1)!}\Lambda_n^m \exp[K_2 V(r_n)]\,. \tag{28}$$

We denote

$$u_{n,m}(z) = \left[\frac{\varphi_n(z)}{z-a_n}\right]^{(m-1)}, \ m=1,\ldots,q_n, n\in\mathbb{N}\,.$$

Let us estimate $u_{n,m}(z)$ for $z\in\mathbb{C}_+$, $z\notin C(a_n,\Lambda_n/2)$. We note that if $|t-z|=\Lambda_n/4$, then first,

$$|t-a_n| \geqslant \Lambda_n/4, \quad n\in\mathbb{N}\,, \tag{29}$$

and second, $|t-\bar{a}_n| \geqslant \operatorname{Im} a_n - \Lambda_n/4 \geq 3\operatorname{Im} a_n/4$ ($n\in\mathbb{N}$), $|z-\bar{a}_n| \leqslant |z-t|+|t-\bar{a}_n| = \Lambda_n/4+|t-\bar{a}_n| \leqslant \operatorname{Im} a_n/4 + |t-\bar{a}_n| \leqslant 7|t-\bar{a}_n|/3$, and $|t-\bar{a}_n| \leqslant |z-t|+|z-\bar{a}_n| = \Lambda_n/4+|z-\bar{a}_n| \leqslant \operatorname{Im} a_n/4+|z-\bar{a}_n| \leqslant 5|z-\bar{a}_n|/4$, i.e.,

$$3|z-\bar{a}_n|/7 \leqslant |t-\bar{a}_n| \leqslant 5|z-\bar{a}_n|/4\,. \tag{30}$$

Moreover, if $|z-t|=\Lambda_n/4$, then

$$|t+i\Lambda_n-\bar{a}_n| \geqslant 3\Lambda_n/4+\operatorname{Im} z+\operatorname{Im} a_n\,. \tag{31}$$

Employing integral Cauchy formula for the derivatives for the circle $C_{z,n}=\{t : |t-z|=\Lambda_n/4\}$, by (26), (29), (30), and (31) we get

$$|u_{n,m}(z)| = \frac{(m-1)!}{2\pi}\left|\int\limits_{C_{z,n}} \frac{\varphi_n(t)\,dt}{(t-a_n)(t-z)^m}\right| \leqslant \frac{4^m(m-1)!}{\Lambda_n^m}\max_{t\in C_{z,n}}|\varphi_n(t)| \leqslant$$

$$\leqslant \frac{4^m 49(m-1)!(\operatorname{Im} a_n)^2}{9\Lambda_n^m|z-\bar{a}_n|^2}\left(\frac{\sqrt{2}(|z|+1/4)}{\sqrt{1+r_n^2}}\right)^3 \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|}\max_{t\in C_{z,n}}\exp[\operatorname{Re}(\beta_n(a_n)-\beta_n(t))]\,.$$

In view of (22), (23), and (31) it finally follows that

$$\begin{aligned} |u_{n,m}(z)| \leqslant & \frac{4^m 49(m-1)! e^{K_1}(\sqrt{2}(|z|+1/4))^3}{9\Lambda_n^m|z-\bar{a}_n|^2}\frac{(\operatorname{Im} a_n)^2}{(1+r_n^2)^{\frac{3}{2}}} \\ & \times \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|}\exp\left[-\sum_{k=n}^{\infty}\frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4+\operatorname{Im} z+\operatorname{Im} a_k)^2(1+r_k^2)^{\frac{3}{2}}}\right], \end{aligned} \tag{32}$$

$m=1,\ldots,q_n$, $n\in\mathbb{N}$.

Then by (24), (28), and (32) we get as $z \in \mathbb{C}_+$, $z \notin C(a_n, \Lambda_n/2)$, inequality

$$
\begin{aligned}
|P_n(z)| \leqslant \sum_{m=1}^{q_n} |\alpha_{nm}||u_{nm}(z)| \leqslant \frac{49}{9} \exp[K_3 V(r_n)] \frac{(\sqrt{2}(|z|+1/4))^3 (\operatorname{Im} a_n)^2}{(1+r_n^2)^{\frac{3}{2}} |z-\bar{a}_n|^2} \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|} \\
\times \sum_{m=1}^{q_n} 4^m (q_n - m + 1) \exp\left[-\sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4 + \operatorname{Im} z + \operatorname{Im} a_k)^2 (1+r_k^2)^{\frac{3}{2}}} \right] \\
\leqslant \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|} \frac{(\operatorname{Im} a_n)^2}{|z-\bar{a}_n|^2 (1+r_n^2)^{\frac{3}{2}}} \frac{49}{18} q_n (q_n+1) \exp[K_3 V(r_n) + q_n \ln 4] \times \\
\times (\sqrt{2}(|z|+1/4))^3 \exp\left[-\sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4 + \operatorname{Im} z + \operatorname{Im} a_k)^2 (1+r_k^2)^{\frac{3}{2}}} \right], \quad n \in \mathbb{N},
\end{aligned}
\tag{33}
$$

holds true. Employing (10), by (33) for $z \in \mathbb{C}_+$, $z \notin C(a_n, \Lambda_n/2)$ we obtain

$$
\begin{aligned}
|P_n(z)| \leqslant \exp[K_4 V(r_n)] (\sqrt{2}(|z|+1/4))^3 \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|} \frac{(\operatorname{Im} a_n)^2}{|z-\bar{a}_n|^2 (1+r_n^2)^{\frac{3}{2}}} \times \\
\times \exp\left[-\sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4 + \operatorname{Im} z + \operatorname{Im} a_k)^2 (1+r_k^2)^{\frac{3}{2}}} \right], \ n \in \mathbb{N}\,.
\end{aligned}
\tag{34}
$$

We then note that once $|t - a_n| \leqslant \Lambda_n/2$, and $|z - a_n| = \Lambda_n/2$, we have

$$
|z| \leqslant |t| + 1 \tag{35}
$$

and

$$
3|t - \bar{a}_n|/5 \leqslant |z - \bar{a}_n| \leqslant 5|t - \bar{a}_n|/3\,. \tag{36}
$$

Applying maximum modulus principle to an analytic in $\mathbb{C}_+$ function $\Phi_n(z) = E(z)P_n(z)$, employing inequalities (34), (35), (36) and Lemma 1, and taking into consideration that $\operatorname{Im} t \geqslant \operatorname{Im} z/4$ for $t \in C(a_n, \Lambda_n/2)$, we get

$$
\begin{aligned}
|\Phi_n(t)| \leqslant \max_{|z-a_n|=\Lambda_n/2} |E(z)||P_n(z)| \leqslant \exp[K_5(V(r_n) + V(|z|))] \frac{|g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|} \\
\frac{25 (\operatorname{Im} a_n)^2}{9|t-\bar{a}_n|^2 (1+r_n^2)^{\frac{3}{2}}} \exp\left[-\sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4 + 4\operatorname{Im} t + \operatorname{Im} a_k)^2 (1+r_k^2)^{\frac{3}{2}}} \right].
\end{aligned}
\tag{37}
$$

By (34) inequality (37) holds true for each $t \in \mathbb{C}_+$. We denote

$$
\lambda_n(z) = \sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(3\Lambda_n/4 + 4\operatorname{Im} z + \operatorname{Im} a_k)^2 (1+r_k^2)^{\frac{3}{2}}}\,,
$$

so that

$$
\lambda_n(z) - \lambda_{n+1}(z) = \frac{(\operatorname{Im} a_n)^2}{(3\Lambda_n/4 + 4\operatorname{Im} z + \operatorname{Im} a_n)^2 (1+r_n^2)^{\frac{3}{2}}}\,, \quad n \in \mathbb{N}\,.
$$

It is clear that $\lambda_n(z) \downarrow 0$ as $n \to \infty$, $z \in \mathbb{C}_+$. Noticing that as $z \in \mathbb{C}_+$ inequality

$$
3\Lambda_n/4 + 4\operatorname{Im} z + \operatorname{Im} a_n \leqslant 4\operatorname{Im} z + 7\operatorname{Im} a_n/4 \leqslant 4(\operatorname{Im} z + \operatorname{Im} a_n) \leqslant 4|z - \bar{a}_n|
$$

holds true, by (37) we obtain

$$
|\Phi_n(z)| \leqslant 16 \exp[-\lambda_n(z)][\lambda_n(z) - \lambda_{n+1}(z)] \frac{\exp[MV(r_n) + MV(|z|)] |g(\sqrt{2}(|z|+1/4))|}{|g(a_n)|}.
$$

Applying an elementary inequality $t \leqslant e^t - 1$, $t \geqslant 0$, as $t = \lambda_n(z) - \lambda_{n+1}(z)$, we get

$$|\Phi_n(z)| \leqslant \exp[K_5(V(r_n) + V(|z|))][\exp[-\lambda_{n+1}(z)] - \exp[-\lambda_n(z)]]\frac{|g(\sqrt{2}(|z| + 1/4))|}{|g(a_n)|}\,. \qquad (38)$$

We choose a function $g(z)$ so that function $F(z)$ defined by series (27) belongs to class $[\rho(r), \infty)_+$. As $g(z)$ we take an entire function of completely regular growth of order $\rho(r)$, whose indicator equals $K_5 + 1$ and whose zeroes are located on the negative imaginary semi-axis $i\mathbb{R}_- = \{z : \operatorname{Im} z \leqslant -1\}$. Since $\rho(r)$ is the zero proximate order, this function exists [16].

Outside C^0-set the asymptotic identity [16]

$$\ln |g(z)| \approx (K_5 + 1)V(|z|)$$

holds true.

Since zeroes of function $g(z)$ are located on the semi-axis $i\mathbb{R}_-$, we can assume that exceptional circles forming C^0-set are located in the lower half-plane. Then inequality

$$\ln |g(a_n)| \geqslant K_5V(r_n)$$

holds true for all sufficiently large n. Multiplying if needed function $g(z)$ for a sufficiently large positive number, we can assume this inequality is satisfied for each natural n.

By (38) we then obtain for each natural $N \geqslant 1$

$$\begin{aligned}|E(z)\sum_{n=1}^{N} P_n(z)| \leqslant \sum_{n=1}^{N} |E(z)P_n(z)| &\leqslant \exp[K_6V(|z|)]\{\exp[-\lambda_{N+1}(z)] - \exp[-\lambda_1(z)]\}\\ &\leqslant \exp[K_6V(|z|)]\,.\end{aligned}$$

It follows the convergence of series (27) on compact sets in $\mathbb{C}_+$ and the belonging of function F to class $[\rho(r), \infty)_+$. In order function F to belong to class$[\rho(r), \infty)_+^h$, we need to obey also B.Ya. Levin's condition. We note that the canonical function E belongs to class $[\rho(r), \infty)_+^h$. The results of work [16] follow that outside the set C_η with an arbitrarily small upper density $\eta > 0$ everywhere in the half-plane $\mathbb{C}_+$ the inequality

$$\log |E(z)| \geqslant -M_\eta V(|z|)$$

holds true.

Let $g_1(z)$ be an entire function of completely regular growth of order $\rho(r)$, whose indicator equals $2K_5 + M_\eta + 1$. Then outside C^0-set the inequality

$$\log |g_1(z)| \geqslant (2K_5 + M_\eta)V(|z|)$$

holds true.

The set $\tilde{C}_\eta = C_\eta \cup C^0$ has the upper density not exceeding η. Outside $\tilde{C}_\eta$ inequality

$$\log |g_1(z)E(z)| \geqslant 2K_5V(|z|)$$

holds everywhere in $\mathbb{C}_+$.

The function

$$F_1(z) = F(z) + g_1(z)E(z)$$

possesses property (6) and outside $\tilde{C}_\eta$-set the estimate

$$\begin{aligned}\log |F_1(z)| = \log |g_1(z)E(z)| + \log \left|1 + \frac{F(z)}{g_1(z)E(z)}\right| \geqslant\\ \geqslant 2K_5V(|z|) + \log(1 - 1/e)\end{aligned}$$

holds true. Therefore, function F_1 belongs to class $[\rho(r), \infty)_+^h$. The implication 2) $\Rightarrow$ 1) in Theorem 3 is proven.

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Oksana Anatol'evna Bozhenko,
Sumy State University,
Rymskogo-Korsakova, 2,
40007, Sumy, Ukraine
Konstantin Gennad'evich Malyutin,
Sumy State University,
Rymskogo-Korsakova, 2,
40007, Sumy, Ukraine
E-mail: malyutinkg@yahoo.com