WIENER’S THEOREM FOR PERIODIC AT INFINITY FUNCTIONS WITH SUMMABLE WEIGHTED FOURIER SERIES

I.I. STRUKOVA

Abstract. In the article we define a Banach algebra of periodic at infinity functions. For this class of functions we introduce the notions of a Fourier series, its absolute convergence, and invertibility. We obtain an analogue of Wiener’s theorem on absolutely convergent Fourier series for periodic at infinity functions whose Fourier coefficients are summable with a weight.

Keywords: Banach space, slowly varying at infinity functions, periodic at infinity functions, Wiener’s theorem, absolutely convergent Fourier series, invertibility.

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1. Introduction

Let $l^1(\mathbb{Z})$ be the Banach space of two-sided summable sequences $a : \mathbb{Z} \to \mathbb{C}$ with the norm $\|a\|_1 = \sum_{k \in \mathbb{Z}} |a(k)| < \infty$.

By the symbol $C_\omega(\mathbb{R})$ we shall indicate the Banach space of all continuous $\omega$-periodic functions $f : \mathbb{R} \to \mathbb{C}$.

We say that a function $f \in C_\omega(\mathbb{R})$ has an absolutely convergent Fourier series if it can be represented as the series $f(t) = \sum_{k \in \mathbb{Z}} a(k)e^{i2\pi k \omega t}, \ t \in \mathbb{R}$, where $a \in l^1(\mathbb{Z})$. We denote the set of all such functions by $AC_\omega(\mathbb{R})$. We observe that $AC_\omega(\mathbb{R})$ is a Banach algebra (cf. [1]) with the pointwise multiplication and the norm

$$\|f\|_{AC} = \|a\|_1 = \sum_{k \in \mathbb{Z}} |a(k)|.$$

In terms of the introduced notations Wiener’s theorem reads as follows.

Theorem 1. If a function $f$ belongs to $AC_\omega(\mathbb{R})$ and $f(t) \neq 0$ for each $t \in \mathbb{R}$, then $1/f \in AC_\omega(\mathbb{R})$, i.e. $1/f(t) = \sum_{k \in \mathbb{Z}} b(k)e^{i2\pi k \omega t}$, where $b \in l^1(\mathbb{Z})$.

The proof of Theorem 1 is given in [2].

Wiener’s theorem was generalized in several directions. We mention Bochner-Fillips theorem [3] for the functions with values in a Banach algebra, as well as papers [4], [5], where Wiener’s theorem was proved for the operators whose matrices have absolutely summable diagonals. The references to the studies related with applications of the results are given in [6].

In the present paper we extend Wiener’s theorem for the class of periodic at infinity functions.
We introduce the set of periodic at infinite functions. Let \( X \) be a complex Banach space, \( \text{End} \, X \) be the Banach algebra of linear bounded operators acting in \( X \).

By the symbol \( C_{b,u} = C_{b,u}(\mathbb{R}, X) \) we denote the Banach space of continuous and bounded on \( \mathbb{R} \) functions with values in \( X \), the norm in this space is \( \|x\|_\infty = \sup_{t \in \mathbb{R}} \|x(t)\|_X \). The symbol \( C_0 = C_0(\mathbb{R}, X) \) will be employed to indicate the closed subspace of \( C_{b,u} \) consisting of the functions decaying at infinity.

In the Banach space \( C_{b,u} \) we consider an isometric group of operators (or representation) \( S : \mathbb{R} \to \text{End} \, C_{b,u} \) acting by the rule
\[
(S(\alpha)x)(t) = x(t + \alpha), \quad \alpha \in \mathbb{R}.
\]

**Definition 1.** A function \( x \in C_{b,u}(\mathbb{R}, X) \) is called **slowly varying or stationary at infinity** if
\[
S(\alpha)x - x \in C_0(\mathbb{R}, X) \quad \text{for each} \quad \alpha \in \mathbb{R}.
\]
For instance, a function \( f \in C_{b,u}(\mathbb{R}, \mathbb{C}) \) being \( f(t) = \sin \ln(1 + t^2) \) is slowly varying at infinity.

**Definition 2.** A function \( x \in C_{b,u}(\mathbb{R}, X) \) is called **periodic at infinity of period \( \omega > 0 \)** if
\[
S(\omega)x - x \in C_0(\mathbb{R}, X).
\]
The definition of periodic at infinity function was suggested by A.G. Baskakov and was employed in paper [7].

We denote the set of slowly varying at infinity functions by the symbol \( C_{sl} = C_{sl}(\mathbb{R}, X) \), while the symbol \( C_{\omega, \infty} = C_{\omega, \infty}(\mathbb{R}, X) \) stands for the functions periodic at infinity of period \( \omega \).

In case \( X = \mathbb{C} \), the considered spaces will be indicated as \( C_{b,u}(\mathbb{R}), C_0(\mathbb{R}), C_{sl}(\mathbb{R}), C_{\omega, \infty}(\mathbb{R}) \).

We note that \( C_{\omega, \infty}(\mathbb{R}, X) \) is a Banach space with the norm
\[
\|x\|_\infty = \sup_{t \in \mathbb{R}} \|x(t)\|_X.
\]
Moreover, \( C_{sl}(\mathbb{R}, X) \) and \( C_{\omega, \infty}(\mathbb{R}, X) \) form Banach algebras with pointwise multiplication if \( X \) is a Banach algebra.

**Definition 3.** Given a function \( x \in C_{\omega, \infty}(\mathbb{R}, X) \), we call the series
\[
x(t) \sim \sum_{n \in \mathbb{Z}} x_n(t)e^{i\frac{2\pi n \omega}{\omega}t}, \quad t \in \mathbb{R},
\]
its **generalized Fourier series**, where the functions \( x_n, n \in \mathbb{Z} \), are defined by the formulae
\[
x_n(t) = \frac{e^{-i\frac{2\pi n \omega}{\omega}t}}{\omega} \int_0^\omega x(t + \tau)e^{-i\frac{2\pi n \tau}{\omega}}d\tau, \quad t \in \mathbb{R}, \quad n \in \mathbb{Z},
\]
and are called **Fourier coefficients** for function \( x \). We shall say that the generalized Fourier series of function \( x \) converges absolutely if there exist functions \( y_n \in C_{sl}(\mathbb{R}, X) \), \( n \in \mathbb{Z} \), such that \( y_n - x_n \in C_0(\mathbb{R}, X) \) and \( \sum_{n \in \mathbb{Z}} \|y_n\|_\infty < \infty \).

In what follows we shall omit the word “generalized”. It is also possible that a considered Fourier series does not converge to function \( x \). In this case it is regarded as a formal series.

**Example 1.** As an example of a function in \( C_{\omega, \infty}(\mathbb{R}) \) with an absolutely convergent Fourier series we consider the function \( f : \mathbb{R} \to \mathbb{C} \) defined as
\[
f(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{n^2} \sin(\alpha_n \ln(1 + t^2)) \right) e^{i\frac{2\pi n \omega}{\omega}t}, \quad t \in \mathbb{R}, \quad \alpha_n \in \mathbb{R}.
\]
We note that the functions \( f_n, n \in \mathbb{Z} \), constructed for function \( f \) by formula (2) do not coincide with the functions \( y_n, t \mapsto \frac{1}{n^2} \sin \alpha_n \ln(1 + t^2) \), \( t \in \mathbb{R}, n \in \mathbb{Z} \), however, \( f_n - y_n \in C_b(\mathbb{R}) \).
Remark 1. If \( x \in C_\omega(R) \), then the Fourier series in Definition 3 coincides with the usual Fourier series of function \( x \).

In what follows we shall make use of the notation
\[
e_n(t) = e^{i2\pi nt}, \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}.
\]

We observe that the mapping \( x \mapsto P_n x = x_n e_n : C_{\omega, \infty}(R, X) \to C_{\omega, \infty}(R, X), \quad n \in \mathbb{Z} \), is a bounded operator obeying \( \|P_n\| \leq 1 \). Moreover, \( \text{Im}(P_n) \subset C_0(R, X) \) for the image \( \text{Im}(P_n^2 - P_n) \) of the operator \( P_n^2 - P_n \) (the proof is given in the end of Section 3), however, the operators \( P_n, n \in \mathbb{Z} \), are not projectors.

Till the end of this section the symbol \( X \) will indicate a Banach algebra.

Definition 4. We call a function \( x \in C_{b, u}(R, X) \) invertible w.r.t. subspace \( C_0(R, X) \) if there exists a number \( b \), such that \( xy - 1 \in C_0(R, X) \). We call function \( y \) inverse for \( x \) w.r.t. subspace \( C_0(R, X) \).

Remark 2. From Definition 4 it immediately follows that a function \( x \in C_{\omega, \infty}(R, X) \) is invertible w.r.t. subspace \( C_0(R, X) \) if and only if it can be represented as \( x = y + x_0 \), where \( x_0 \in C_0(R, X) \), and function \( y \in C_{\omega, \infty}(R, X) \) is so that \( \inf_{t \in \mathbb{R}} \|y(t)\|_X > 0 \). Definition 4 also implies that a function \( x \in C_{\omega, \infty}(R, X) \) is invertible w.r.t. subspace \( C_0(R, X) \) if and only if there exists a number \( A > 0 \) such that \( \inf_{|t| > A} \|x(t)\|_X > 0 \).

It is easy to see that if \( y_1, y_2 \) are inverse for \( x \in C_{b, u}(R, X) \) w.r.t. subspace \( C_0(R, X) \), then \( y_1 - y_2 \in C_0(R, X) \).

Consider a function \( a \in C_{\omega, \infty}(R, X) \) and introduce the notation \( d_n(k) = \|a_k\|_X, \quad k \in \mathbb{Z} \), where \( a_k \) is the \( k \)-th Fourier coefficient for function \( a \) defined by formula \( 2 \).

The considered function is supposed to satisfy one of the conditions in the following assumption.

Assumption 1. Function \( a \in C_{\omega, \infty}(R, X) \) satisfies one of the conditions:

1) \( \sum_{k \in \mathbb{Z}} d_n(k)\alpha(k) < \infty \), where \( \alpha : \mathbb{Z} \to \mathbb{R}_+ \) is a weight obeying the relation \( \lim_{|k| \to \infty} \frac{\ln \alpha(k)}{|k|} = 0 \);

2) \( \lim_{|k| \to \infty} d_n(k)|k|^{\gamma} = 0 \), \( \quad k \in \mathbb{Z}, \quad \gamma > 1 \);

3) \( d_n(k) \leq \text{Const} \exp(-\varepsilon|k|), \quad k \in \mathbb{Z}, \quad \varepsilon > 0 \).

In particular, the assumption holds true, if the Fourier series of function \( a \) has a finite number of non-zero Fourier coefficients, that is, there exists \( M \in \mathbb{N} \) such that \( d_n(k) = 0, \quad |k| \geq M + 1 \).

The main result of the present work is

Theorem 2. If an invertible w.r.t. subspace \( C_0(R, X) \) function \( a \in C_{\omega, \infty}(R, X) \) satisfies one of the conditions 1)–3) of Assumption 1, then its inverse \( b \) w.r.t. \( C_0(R, X) \) obeys the corresponding condition among the following ones:

1’) \( \sum_{k \in \mathbb{Z}} d_b(k)\alpha(k) < \infty \);

2’) \( \lim_{|k| \to \infty} d_b(k)|k|^{\gamma} = 0 \);

3’) \( d_b(k) \leq \text{Const} \exp(-\varepsilon_0|k|), \quad k \in \mathbb{Z}, \quad \varepsilon_0 > 0 \).

We note that quantities \( \text{Const} \) and \( \varepsilon_0 \) depend on quantities \( \text{Const} \) and \( \varepsilon \) in the conditions of Assumption 1.

2. Periodic vectors and their Fourier series

Let \( B \) be a Banach algebra with the unit and \( \omega \) is a positive number. Consider a \( \omega \)-periodic isometric strongly continuous group of operators (representation) \( T : R \to \text{End}B \) acting in \( B \).
and having the properties

\[ T(t)(ab) = T(t)a \cdot T(t)b, \]
\[ T(t)e = e, \quad t \in \mathbb{R}, \]

where \( a, b \) are arbitrary elements in \( \mathcal{B} \), and \( e \) is the unit in algebra \( \mathcal{B} \).

Thus, each of the operators \( T(t), t \in \mathbb{R} \), is a homomorphism of algebra \( \mathcal{B} \), and each function \( t \mapsto T(t)a : \mathbb{R} \rightarrow \mathcal{B}, a \in \mathcal{B} \) is a continuous \( \omega \)-periodic function.

The above properties immediately yield that if an element \( a \in \mathcal{B} \) is invertible, then

\[ e = T(t)e = T(t)(aa^{-1}) = (T(t)a)T(t)a^{-1} = (T(t)a^{-1})T(t)a, \quad a \in \mathcal{B}, \]

and hence, \( (T(t)a)^{-1} = T(t)a^{-1} \).

Consider the Fourier series (see [8])

\[ T(t)a \sim \sum_{n \in \mathbb{Z}} a_ne^{i\frac{2\pi}{\omega}nt}, \quad t \in \mathbb{R}, \]

of the function \( t \mapsto T(t)a : \mathbb{R} \rightarrow \mathcal{B}, a \in \mathcal{B} \), where the Fourier coefficients are defined by the formulae

\[ a_n = \frac{1}{\omega} \int_0^\omega T(t)ae^{-i\frac{2\pi}{\omega}nt}dt, \quad n \in \mathbb{Z}. \] (5)

We call the series \( a \sim \sum_{n \in \mathbb{Z}} a_n \) Fourier series of an element \( a \in \mathcal{B} \) and \( a_n, n \in \mathbb{Z} \), are called Fourier coefficients of this element.

If the Fourier series of an element \( a \in \mathcal{B} \) converges absolutely, i.e. the condition \( \sum_{n \in \mathbb{Z}} \|a_n\| < \infty \) holds true, then the identity \( a = \sum_{n \in \mathbb{Z}} a_n \) is valid.

**Lemma 1.** Let \( a \in \mathcal{B} \). Then \( T(\alpha)a_n = e^{i\frac{2\pi}{\omega}\alpha}a_n, n \in \mathbb{Z} \), for each \( \alpha \in \mathbb{R} \), where \( a_n, n \in \mathbb{Z} \), are the Fourier coefficients of an element \( a \). At that, the operators \( P_n \) defined by the formula

\[ P_n a = a_n = \frac{1}{\omega} \int_0^\omega T(t)ae^{-i\frac{2\pi}{\omega}nt}dt, n \in \mathbb{Z}, \]

are the projectors with \( \|P_n\| \leq 1, n \in \mathbb{Z} \).

**Proof.** We take an arbitrary element \( a \in \mathcal{B} \) and fix an arbitrary number \( \alpha \in \mathbb{R} \). Let \( a_n, n \in \mathbb{Z}, \) be the Fourier coefficient of element \( a \) defined by formula (5). Then they satisfy the following chain of identities

\[ T(\alpha)a_n = T(\alpha) \left( \frac{1}{\omega} \int_0^\omega T(t)ae^{-i\frac{2\pi}{\omega}nt}dt \right) = \frac{1}{\omega} \int_0^\omega T(\alpha)T(t)ae^{-i\frac{2\pi}{\omega}nt}dt = \]
\[ = \frac{1}{\omega} \int_0^\omega T(\alpha + t)ae^{-i\frac{2\pi}{\omega}nt}dt = e^{i\frac{2\pi}{\omega}\alpha} \int_\alpha^{\omega+\alpha} T(\tau)ae^{-i\frac{2\pi}{\omega}\tau}d\tau = \]
\[ = e^{i\frac{2\pi}{\omega}\alpha} \int_0^\omega T(\tau)ae^{-i\frac{2\pi}{\omega}nt}d\tau = e^{i\frac{2\pi}{\omega}\alpha}a_n, \quad n \in \mathbb{Z}. \]

That is, we have shown that \( T(\alpha)a_n = e^{i\frac{2\pi}{\omega}\alpha}a_n, n \in \mathbb{Z} \), for each \( \alpha \in \mathbb{R} \).

Now let us show that the operators \( P_n, n \in \mathbb{Z} \), defined by the formula \( P_n a = a_n \) are projectors, i.e. \( P_n^2 = P_n, n \in \mathbb{Z} \).
Let $a \in \mathcal{B}$. Then

$$P_n a = \frac{1}{\omega} \int_0^\omega T(t)ae^{-i\frac{2\pi n}{\omega}t}dt, \quad n \in \mathbb{Z},$$

$$P_n^2 a = P_n(P_n a) = \frac{1}{\omega} \int_0^\omega T(t)a_n e^{-i\frac{2\pi n}{\omega}t}dt = \frac{1}{\omega} \int_0^\omega a_n dt = a_n = P_n a, \quad n \in \mathbb{Z}.$$  

Let us show that $\|P_n\| \leq 1, \ n \in \mathbb{Z}$. Employing the property $\|T(t)\| = 1, \ t \in \mathbb{R}$, we obtain

$$\|P_n\| = \sup_{\|a\| \leq 1} \|P_n a\| = \sup_{\|a\| \leq 1} \frac{1}{\omega} \int_0^\omega T(t)ae^{-i\frac{2\pi n}{\omega}t}dt \leq$$

$$\leq \sup_{\|a\| \leq 1} \frac{1}{\omega} \int_0^\omega \|T(t)a\| dt \leq \sup_{\|a\| \leq 1} \frac{1}{\omega} \int_0^\omega \|T(t)||a|| dt \leq 1.$$  

The proof is complete. \(\square\)

Given an element $a \in \mathcal{B}$, we consider the operator $A \in \text{End} \mathcal{B}$ of the form

$$Ax = ax, \quad x \in \mathcal{B}.$$  

We associate with this operator a $\omega$-periodic operator-valued function $\Phi_A : \mathbb{R} \to \text{End} \mathcal{B}$ defined by the formula

$$\Phi_A(t) = T(t)AT(-t), \quad t \in \mathbb{R}.$$  

We associate with function $\Phi_A$ its Fourier series

$$\Phi_A(t) \sim \sum_{n \in \mathbb{Z}} A_n e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R},$$

where the Fourier coefficients are defined by the formulae

$$A_n = \frac{1}{\omega} \int_0^\omega T(t)AT(-t)e^{-i\frac{2\pi n}{\omega}t}dt, \quad n \in \mathbb{Z}. \quad (6)$$

We call a series $\sum_{n \in \mathbb{Z}} A_n$, Fourier series of operator $A$, and the operators $A_n$ are called Fourier coefficients of this operator. We define a two-sided number sequence $(d_A(n))$ by letting $d_A(n) = \|A_n\|, \ n \in \mathbb{Z}.$

**Lemma 2.** The Fourier coefficients $A_n$, $n \in \mathbb{Z}$, of an operator $A$ satisfy the representations $A_n x = a_n x, \ n \in \mathbb{Z}, \ x \in \mathcal{B}$. At that, $\|A_n\| = \|a_n\|, \ n \in \mathbb{Z}.$

**Proof.** Let us show that $A_n x = a_n x$ for each $x \in \mathcal{B}$.

Employing formulae (5) and (6) as well as the fact that the operators $T(t), \ t \in \mathbb{R}$, form a homomorphism of the algebra, we obtain

$$A_n x = \frac{1}{\omega} \int_0^\omega T(t)AT(-t)x e^{-i\frac{2\pi n}{\omega}t}dt = \frac{1}{\omega} \int_0^\omega T(t)(aT(-t)x)e^{-i\frac{2\pi n}{\omega}t}dt$$

$$= \frac{1}{\omega} \int_0^\omega (T(t)a T(t)(T(-t)x)e^{-i\frac{2\pi n}{\omega}t}dt = \left(\frac{1}{\omega} \int_0^\omega T(t)e^{-i\frac{2\pi n}{\omega}t}dt\right)x = a_n x.$$  

The inequality $\|A_n x\| \leq \|a_n\||x||$ holds true for each $x \in \mathcal{B}$.

Since $a_n = A_n e$ and $\|e\| = 1$, then $\|A_n\| = \|a_n\|, n \in \mathbb{Z}$. The proof is complete. \(\square\)
We observe that if the Fourier series of an operator $A$ converges absolutely, i.e.
\[
\sum_{n \in \mathbb{Z}} d_A(n) = \sum_{n \in \mathbb{Z}} \|a_n\| < \infty,
\]
then function $\Phi_A$ is continuous in the uniform operator topology.

We suppose that for the considered operator one of the conditions in the following assumption is fulfilled.

**Assumption 2.** Operator $A \in \text{End} \mathcal{B}$ satisfies one of the following conditions:
1) $\sum_{k \in \mathbb{Z}} d_A(k)\alpha(k) < \infty$, where $\alpha : \mathbb{Z} \to \mathbb{R}_+$ is a weight satisfying the relation $\lim_{|k| \to \infty} \frac{\ln \alpha(k)}{|k|} = 0$;
2) $\lim_{|k| \to \infty} d_A(k)|k|^\gamma = 0$, $k \in \mathbb{Z}$, $\gamma > 1$;
3) $d_A(k) \leq \text{Const} \exp(-\varepsilon|k|)$, $k \in \mathbb{Z}$, for some $\varepsilon > 0$.

In particular, the assumption holds true if the Fourier series of operator $A$ comprises finitely many non-zero Fourier coefficients, i.e. there exists $M \in \mathbb{N}$ such that $d_A(k) = 0$, $|k| \geq M + 1$.

In what follows we shall make use of

**Theorem 3.** Suppose that an operator $A \in \text{End} \mathcal{B}$ is invertible and satisfies one of Conditions 1)–3) of Assumption 2. Then the inverse operator $B = A^{-1} \in \text{End} \mathcal{B}$ satisfies the corresponding condition among the following ones:
1') $\sum_{k \in \mathbb{Z}} d_B(k)\alpha(k) < \infty$;
2') $\lim_{|k| \to \infty} d_B(k)|k|^\gamma = 0$;
3') $d_B(k) \leq \text{Const} \exp(-\varepsilon_0|k|)$, $k \in \mathbb{Z}$, for some $\varepsilon_0 > 0$.

This theorem follows from [9, Thm. 1].

3. Harmonic analysis of periodic at infinity functions

Throughout this section $X$ stands for a Banach algebra with unit.

It is clear that the group of shifts $S$ defined by formula (1) is not periodic in the space of periodic at infinity functions.

In what follows, by the symbol $\mathcal{B}$ we denote the factor-space $\mathcal{C}_{\omega,\infty}(\mathbb{R}, X)/\mathcal{C}_0(\mathbb{R}, X)$ which becomes an algebra if we define the multiplication as
\[
\tilde{x}\tilde{y} = \tilde{xy}, \quad \tilde{x}, \tilde{y} \in \mathcal{B}.
\] (7)

In this factor-space we construct as isometric group of operators $T : \mathbb{R} \to \text{End} \mathcal{B}$ acting by the rule
\[
T(t)\tilde{x} = \tilde{S(t)x} = S(t)x + \mathcal{C}_0(\mathbb{R}, X), \quad t \in \mathbb{R},
\] (8)

where $x$ is an element of class $\tilde{x} \in \mathcal{B}$.

Since
\[
T(\omega)\tilde{x} = \tilde{S(\omega)x} = S(\omega)x + \mathcal{C}_0(\mathbb{R}, X)
= (S(\omega)x - x) + x + \mathcal{C}_0(\mathbb{R}, X) = x + \mathcal{C}_0(\mathbb{R}, X) = \tilde{x},
\]
representation $T$ is $\omega$-periodic. Moreover, the strong continuity of presentation $S$ implies the same for representation $T$.

In terms of group $T$, the belonging of a class $\tilde{x}$ to algebra $\mathcal{B}$ means that $T(\omega)\tilde{x} = \tilde{x}$. The Fourier series of a function $x \in \mathcal{C}_{\omega,\infty}(\mathbb{R}, X)$ being an element of a class $\tilde{x}$ reads as $x(\tau) \sim$
\[ \sum_{n \in \mathbb{Z}} x_n(\tau) e^{i \frac{2\pi n}{\omega} \tau}, \] where the Fourier coefficients \( x_n, n \in \mathbb{Z} \), are determined by formula (2), while the mean \( x_0 \) is

\[ x_0(t) = \frac{1}{\omega} \int_0^\omega x(t + \tau) d\tau, \quad t \in \mathbb{R}. \]

We have

**Lemma 3.** The Fourier coefficients of a function \( x \in C_{\omega, \infty}(\mathbb{R}, X) \) possess the property \( x_n \in C_{\text{sl}}(\mathbb{R}, X), n \in \mathbb{Z} \).

**Proof.** Let us show first that mean \( x_0 \) of function \( x \in C_{\omega, \infty}(\mathbb{R}, X) \) belongs to space \( C_{\text{sl}}(\mathbb{R}, X) \). We take an arbitrary number \( \alpha \in \mathbb{R} \) and let us show that \( (S(\alpha)x_0 - x_0) \in C_0(\mathbb{R}, X) \). From Lemma 1 it follows immediately that the class \( \tilde{x}_0 \) comprising function \( x_0 \) obeys the identity \( T(\alpha)\tilde{x}_0 = \tilde{x}_0 \), i.e. \( x_0 \) satisfies \( (S(\alpha)x_0 - x_0) \in C_0(\mathbb{R}, X) \). Since number \( \alpha \in \mathbb{R} \) is arbitrary, the definition of slowly varying at infinity function yields \( x_0 \in C_{\text{sl}}(\mathbb{R}, X) \).

Now let us prove this property for the Fourier coefficients \( x_n, n \in \mathbb{Z} \), of function \( x \). Introducing the notation \( y(t) = x(t) e^{i \frac{2\pi n}{\omega} t}, t \in \mathbb{R}, n \in \mathbb{Z} \), we obtain that \( S(\omega)y - y \in C_0(\mathbb{R}, X) \), i.e. \( y \in C_{\omega, \infty}(\mathbb{R}, X) \). Then the mean of function \( y \) defined by the formula \( y_0(t) = \frac{1}{\omega} \int_0^\omega x(t + \tau) e^{i \frac{2\pi n}{\omega} (t + \tau)} d\tau, t \in \mathbb{R} \), belongs to space \( C_{\text{sl}}(\mathbb{R}, X) \). Comparing the latter formula with formula (2), we obtain that \( x_n \in C_{\text{sl}}(\mathbb{R}, X), n \in \mathbb{Z} \). The proof is complete.

Thus, we have the factor-algebra \( \mathcal{B} = C_{\omega, \infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X) \) and the \( \omega \)-periodic strongly continuous isometric group of operators (representation) \( T \) acting in this factor-algebra and defined by formula (8).

With representation \( T \) we associate its Fourier series

\[ T(t)\tilde{x} \sim \sum_{n \in \mathbb{Z}} \tilde{P}_n \tilde{x} e^{i \frac{2\pi n}{\omega} t}, \quad t \in \mathbb{R}, \quad \tilde{x} \in \mathcal{B}. \]

The Fourier coefficients of representation \( T \) read as

\[ \tilde{P}_n \tilde{x} = \frac{1}{\omega} \int_0^\omega T(t)\tilde{x} e^{-i \frac{2\pi n}{\omega} t} dt, \quad n \in \mathbb{Z}. \]

On the elements of the considered classes we have

\[ (P_n x)(\tau) = \frac{1}{\omega} \int_0^\omega (S(t)x)(\tau) e^{-i \frac{2\pi n}{\omega} t} dt = \frac{1}{\omega} \int_0^\omega x(t + \tau) e^{-i \frac{2\pi n}{\omega} \tau} dt = x_n(\tau) e^{i \frac{2\pi n}{\omega} \tau}, \]

where \( x_n, n \in \mathbb{Z} \), are the Fourier coefficients of function \( x \) defined by formula (2).

Directly from formula (5) it follows that the Fourier coefficient of representation \( T \) satisfy the identity

\[ \tilde{P}_n \tilde{x} = \tilde{x}_n, \quad n \in \mathbb{Z}. \]

Let \( x \) be an element of class \( \tilde{x} \in \mathcal{B} \). Then the latter identity means that \( \tilde{P}_n x = \tilde{x}_n \), i.e. \( P_n x - x_n \in C_0(\mathbb{R}, X), n \in \mathbb{Z} \). Since \( P_n \) are projectors, the identity \( \tilde{P}_n \tilde{x} = \tilde{x}_n \), \( n \in \mathbb{Z} \), holds true. This is why \( P_2^2 \tilde{x} = \tilde{x}_n \), i.e. \( P_2 x - x_n \in C_0(\mathbb{R}, X), n \in \mathbb{Z} \). It follows that \( P_2^2 x - P_2 x \in C_0(\mathbb{R}, X), n \in \mathbb{Z} \), i.e. \( \text{Im} (P_2^2 - P_2) \subset C_0(\mathbb{R}, X) \).

If the Fourier series of class \( \tilde{x} \in \mathcal{B} \) converges absolutely, i.e. the condition

\[ \sum_{n \in \mathbb{Z}} \|\tilde{x}_n\| < \infty \]
holds true, then from the properties of the norm in the factor-space it follows that in this case there exist elements \(y_n\) in classes \(\tilde{x}_n\) satisfying
\[
\sum_{n \in \mathbb{Z}} \|y_n\|_{\infty} < \infty.
\]

We note that function \(x \in C_{\omega, \infty}(\mathbb{R}, X)\) is invertible w.r.t. \(C_0(\mathbb{R}, X)\) if and only if the class \(\tilde{x} \in \mathcal{B}\), comprising it, is invertible. This statement is implied by Definition 4.

4. Proof of Theorem 2

In order to obtain the main results, as algebra \(\mathcal{B}\), we consider the factor-algebra \(C_{\omega, \infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)\), and as the representation \(T : \mathbb{R} \to \text{End} \mathcal{B}\), we consider the \(\omega\)-periodic group of isometric operators \(T : \mathbb{R} \to \text{End} \mathcal{B}\) defined by formula (8).

Let us show that group \(T\) possesses properties [4].

By employing formulae (7) and (8), we obtain that
\[
T(t)(\tilde{x} \tilde{y}) = T(t)(\tilde{x} \tilde{y}) = S(t)(xy) = S(t)xS(t)y + C_0(\mathbb{R}, X)
= (T(t)x)T(t)y, \quad x \in \tilde{x}, y \in \tilde{y}, \quad t \in \mathbb{R},
\]
i.e. property [4] indeed holds for group \(T\).

Consider the operator \(A \in \text{End} \mathcal{B}\)
\[
A \tilde{x} = \tilde{a} \tilde{x}, \quad \tilde{a} \in \mathcal{B}.
\]
With this operator we associate the \(\omega\)-periodic operator-valued function \(\Phi_A : \mathbb{R} \to \text{End} \mathcal{B}\) defined by the formula
\[
\Phi_A(t) = T(t)AT(-t), \quad t \in \mathbb{R}.
\]

Theorem 3 holds true for the considered operator.

Proof of Theorem 2. Consider the Banach algebra \(\mathcal{B} = C_{\omega, \infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)\) and the \(\omega\)-periodic isometric group of operators \(T : \mathbb{R} \to \text{End} \mathcal{B}\) acting in this algebra and defined by formula (8).

Given the invertible function \(a \in C_{\omega, \infty}(\mathbb{R}, X)\) introduced in the hypothesis of the theorem, we construct the class \(\tilde{a} \in \mathcal{B}\) which is invertible as well. Denoting the inverse class by the symbol \(\tilde{b}\), we obtain that \(\tilde{a} \tilde{b} = 1\).

We introduce the operator \(A \in \text{End} \mathcal{B}\) by formula (9). This is the operator of multiplication by element \(\tilde{a} \in \mathcal{B}\) and it is invertible. Then its inverse acts as
\[
B \tilde{x} = \tilde{b} \tilde{x}, \quad \tilde{b} \in \mathcal{B}.
\]

Theorem 3 also holds for operator \(A\), and hence, there exists an element \(b\) of class \(\tilde{b}\) such that \(ab - 1 \in C_0(\mathbb{R}, X)\) and it satisfies the appropriate condition in Theorem 2. The proof is complete.

Corollary 1. If a function \(a \in C_{\omega, \infty}(\mathbb{R}, X)\) is invertible w.r.t. subspace \(C_0(\mathbb{R}, X)\) and it has the absolutely convergent Fourier series, then the Fourier series of the inverse w.r.t. \(C_0(\mathbb{R}, X)\) function converges absolutely as well.

Corollary 2. If a function \(a \in C_{\omega, \infty}(\mathbb{R}, X)\) is invertible w.r.t. subspace \(C_0(\mathbb{R}, X)\) and its Fourier series converges absolutely, then there exists a function \(b \in C_{\omega, \infty}(\mathbb{R}, X)\) with an absolutely convergent Fourier series such that \(ab - 1 \in C_0(\mathbb{R}, X)\).

In conclusion we should mention that in recent paper [10] almost periodic at infinity functions were introduced. And there naturally appear the questions similar to ones studied in the present paper.
BIBLIOGRAPHY


Irina Igorevna Strukova,
Voronezh State University,
Universitetskaya sq., 1,
394006, Voronezh, Russia
E-mail: irina.k.post@yandex.ru