QUASI-ANALYTICITY CRITERIA OF SALINAS-KORENBLUM TYPE FOR GENERAL DOMAINS

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Abstract. We prove a criterion of quasi-analyticity in a boundary point of a rather general domain (not necessarily convex and simply-connected) if in a vicinity of this point the domain is close in some sense to an angle or is comparable with it.

Keywords: Carleman class, regular sequences, bilogarithmic quasi-analyticity condition.

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1. Introduction

Let \( \{M_n\}_{n=0}^{\infty} \) be a sequence of positive numbers. Some of numbers \( M_n \) can be equal to \(+\infty\), but it is assumed that there exists an infinite number of finite \( M_n \). As class \( C\{M_n\} \), we call the set of all infinitely differentiable functions \( f \) defined on the segment \( I = [a, b] \), \((-\infty \leq a < b \leq +\infty)\), for each of those there exists a constant \( K_f \) such that \[ \sup_{a< x < b} |f^{(n)}(x)| \leq K_f^n M_n \quad (n \geq 0). \]

In the general situation \( I \) can be an interval of half-interval.

In 1912 J. Hadamard posed the following question [1]: what are the numbers \( M_n \) so that for each two functions \( f \) and \( \varphi \) in class \( C\{M_n\} \), once in some point \( x_0 \) of the interval \( I = (a, b) \) for all \( n \geq 0 \)

\[ f^{(n)}(x_0) = \varphi^{(n)}(x_0), \]

it follows \( f(x) \equiv \varphi(x) \ (a < x < b) \)?

It was observed that it is true if \( M_n = n! \). The matter is that in this case, class \( C\{n!\} \) coincides with the class of real-analytic functions on the interval \( (a, b) \) [1]. Due to the additivity of classes \( C\{M_n\} \), the Hadamard problem can be reformulated as follows: what are the numbers \( M_n \) in order to class \( C\{M_n\} \) to be quasi-analytic, that is, each function \( f \in C\{M_n\} \) satisfying at some point \( x_0 \in I \)

\[ f^{(n)}(x_0) = 0 \quad (n \geq 0), \]

vanishes.

The Hadamard quasi-analyticity problem problem for the segment (interval, half-interval) \( I \) is completely solved by so-called Denjoy-Carleman theorem. One of its equivalent formulations belonging to Ostrovsky is as follows [1], [2]: class \( C\{M_n\} \) is quasi-analytic if and only if

\[ \int_{1}^{\infty} \frac{\ln T(r)}{r^2} dr = +\infty. \]
Here $T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}$ is the trace function for the sequence $\{M_n\}$.

Let $G$ be a domain in the complex plane. By $H(G, M_n)$ we denote the class of functions $f$ analytic in the domain $G$ and satisfying condition

$$\sup_{z \in G} |f^{(n)}(z)| \leq C_f M_n \quad (n \geq 0).$$

We assume that domain $G$ is so that all the derivatives $f^{(n)} (n \geq 0)$ of a function $f \in H(G, M_n)$ can be continuously extended up to the boundary of $\partial G$. In this case, class $H(G, M_n)$ is called quasi-analytic at a point $z_0 \in \partial G$, if $f \in H(G, M_n)$ and $f^{(n)}(z_0) = 0 \ (n \geq 0)$ imply $f \equiv 0$.

Let us survey briefly the results related with the quasi-analyticity problem for class $H(G, M_n)$ and let us formulate the problem we shall discuss here.

As it is known, the quasi-analyticity problem for class $H(\Delta, M_n)$ where $\Delta$ is a circle, was solved by B.I. Korenblum [5]. He proved the following statement: class $H(\Delta, M_n)$ is quasi-analytic at a boundary point if and only if

$$\int_{1}^{\infty} \frac{\ln T(r)}{r^{1+\frac{\gamma}{2}}} dr = +\infty.$$

holds true.

It should be noticed that Ostrovsky theorem is the limiting case for R. Salinas theorem (as $\gamma \to \infty$).

The quasi-analyticity problem for class $H(K, M_n)$, where $K$ is a circle, was solved by B.I. Korenblum [5]. He proved the following statement: class $H(K, M_n)$ is quasi-analytic at a boundary point if and only if

$$\int_{1}^{\infty} \frac{\ln T(r)}{r^{\frac{3}{2}}} = +\infty.$$

The criterion of quasi-analyticity of class $H(D, M_n)$ at a boundary point for an arbitrary convex bounded domain $D$ was established by R.S. Yulmukhametov in [3]. Let us describe this result.

Let $D$ be a convex bounded domain in the complex plane lying in the left half-plane and $0 \in \partial D$. In this case, the support function $h(\varphi) = \max_{\lambda \in D} \text{Re}(\lambda e^{i\varphi})$ of domain $D$ is non-negative and vanishes on some segment $[\sigma_-, \sigma_+]$ ($-\frac{\pi}{2} < \sigma_- \leq 0 \leq \sigma_+ < \frac{\pi}{2}$). Let it be the maximal segment on which $h(\varphi) = 0$. We let

$$\Delta_+(\varphi) = \sqrt{\sigma_+ - \varphi} \left( h'(\varphi) + \int_{0}^{\varphi} h(\alpha) d\alpha \right), \quad \sigma_+ \leq \varphi \leq \frac{\pi}{2};$$

$$\Delta_-(\varphi) = -\sqrt{\sigma_- - \varphi} \left( h'(\varphi) + \int_{0}^{\varphi} h(\alpha) d\alpha \right), \quad -\frac{\pi}{2} \leq \varphi \leq \sigma_-.$$

By $v(r)$ we denote the inverse to the function

$$v_1(x) = \exp \int_{x_1}^{x} \frac{(2\pi - \Delta_-^{-1}(y) + \Delta_+^{-1}(y)) dy}{(-\pi - \Delta_+^{-1}(y) - \Delta_-^{-1}(y))y}, x \to 0, x_1 > 0.$$
Theorem 1 ([3]). If \( h'(\sigma_{\pm}) = 0 \), then class \( H(D, M_n) \) is quasi-analytic at the point \( z = 0 \) if and only if
\[
\int_{1}^{\infty} \frac{\ln T(r)}{v(r)r^2} dr = +\infty.
\]

The problem arises: to find quasi-analyticity criteria for general domains (not necessary bounded, convex, and simply-connected) that depend only on a given sequence \( \{M_n\} \) so that for regular sequences they can be reformulated as bi-logarithmic Levinson condition. The present paper is devoted to studying this issue.

2. History of problem. Definitions and preliminaries

Let \( \{M_n\} \) be a sequence of positive numbers \( M_n \) satisfying condition \( M_n^{\frac{2}{n}} \to \infty \) as \( n \to \infty \). We can assume that \( M_0 = 1 \). Sequence \( \{M_n\} \) is called logarithmically convex if \( M_n^2 \leq M_{n-1}M_{n+1} \) \((n \geq 1)\). It is well know that a logarithmically convex sequence \( \{M_n\} \) is completely determined by the trace function \( T(r) \) and [1, 2]
\[
M_n = \sup_{r \geq 0} \frac{r^n}{T(r)} \quad (n \geq 0).
\]

Let us clarify the geometric meaning of logarithmic convexity of a sequence \( \{M_n\} \). In order to do it, we find the logarithms for inequalities \( M_n^2 \leq M_{n-1}M_{n+1} \), we obtain
\[
\ln M_n \leq \frac{1}{2} \ln M_{n-1} + \frac{1}{2} \ln M_{n+1} \quad (n \geq 1).
\]

Hence, we see that the logarithmical convexity of sequence \( \{M_n\} \) means that the point \( (n, \ln M_n) \) lies not higher than the segment connecting the points \((n-1, \ln M_{n-1})\) and \((n+1, \ln M_{n+1})\) \((n \geq 1)\).

By \( \{M_n^c\} \) we denote the sequence obtained from \( \{M_n\} \) as a convex regularization by logarithms (see, for instance, [1, 2, 6]).

In paper [7] the quasi-analyticity criteria were given for Carleman classes \( H(\Delta_\gamma, M_n) \) and the angle
\[
\Delta_\gamma = \{z : |\arg z| < \frac{\pi}{2\gamma}, 0 < |z| < \infty\} \quad (1 < \gamma < \infty)
\]
explicitly in terms of a given sequence \( \{M_n\} \) (or \( \{M_n^c\} \)). Namely, there was proven

Theorem 2 ([7]). Class \( H(\Delta_\gamma, M_n) \) is quasi-analytic at the point \( z = 0 \) if and only if one of following equivalent conditions

1) \[
\int_{1}^{\infty} \frac{\ln T(r)}{r^{\frac{1}{\gamma}} M_n} dr = \infty, \text{ where } T(r) = \sup_{n \geq 0} \frac{r^n}{M_n} \quad (R. \text{ Salinas criterion});
\]

2) \[
\sum_{n=0}^{\infty} \left( \frac{M_n^c}{M_n} \right)^{\frac{1}{\gamma n}} = \infty;
\]

3) \[
\sum_{n=0}^{\infty} \frac{1}{\beta_n^\gamma} = \infty, \text{ where } \beta_n = \inf_{k \geq n} M_k^{\frac{1}{\gamma}};
\]
holds true.

We proceed to considering the question on bi-logarithmic quasi-analyticity condition for the angle. Following work [5], we introduce the adjoint sequence \( \{m_n\} \), where \( m_n = \frac{M_n}{\beta_n} \). Here \( \{M_n\} \) is an arbitrary sequence of numbers. Now we assume additionally that sequence \( \{M_n\} \) obeys the following conditions,

a) \( m_n^2 \leq m_{n-1}m_{n+1} \quad (n \geq 1) \);

b) \( \sup_{n} \left( \frac{m_{n+1}}{m_n} \right)^\gamma < \infty \);
c) \( m_n^{\frac{1}{n}} \to \infty, \quad n \to \infty. \)

If conditions a)–c) hold true, sequence \( \{M_n\} \) is called regular. Condition a) is the condition of logarithmic convexity for sequence \( \{m_n\} \). We also note that condition b) implies that class \( C\{M_n\} \) is closed w.r.t. differentiation. Condition c) yields that Carleman class \( C\{M_n\} \) contains analytic function as well. For a regular sequence \( \{M_n\} \) we introduce so-called associated weight

\[
\omega(r) = \sup_{n \geq 0} \frac{r^n}{m_n}.
\]

It follows from condition a) that \( M_n^2 \leq M_{n-1}M_{n+1} \), i.e., sequence \( \{M_n\} \) is logarithmically convex (it can be checked directly). This is why in accordance with Denjoy-Carleman theorem, class \( C\{M_n\} \) is quasi-analytic if and only if at least one of the following equivalent conditions holds true.

For a regular sequence \( \{M_n\} \), as E.M. Dyn’kin showed \([8]\), condition \( 2^0 \) (and therefore, condition \( 1^0 \)) is equivalent to bi-logarithmic Levinson condition

\[
\int_0^d \ln \ln h(r) \, dr = +\infty,
\]

where \( h(r) = \omega(\frac{1}{r}) \) and quantity \( d > 0 \) is chosen so that \( h(d) \geq e \). Here

\[
h(r) = \sup_{n \geq 0} \frac{1}{m_n r^n}, \quad m_n = \frac{M_n}{n!}, \quad r > 0.
\]

It is clear that \( h(r) \) is a decaying function, \( \lim_{r \to 0} h(r) = \infty. \) Since sequence \( \{m_n\} \) is logarithmically convex, the inverse representation

\[
m_n = \sup_{r > 0} \frac{1}{r^n h(r)} \quad (n \geq 0)
\]

holds true.

We have

**Theorem 3 (\([7]\)).** Suppose a sequence \( \{M_n\} (n \geq 0) \) of positive numbers \( M_n \) is so that the changed sequence \( \{M_n^\gamma\} \), \( M_n^\gamma = M_n^\frac{\gamma}{1+\gamma} \) \((1 < \gamma < \infty)\) is regular. Then class \( H(\Delta_\gamma, M_n) \) is quasi-analytic at the point \( z = 0 \) if and only if Levinson condition

\[
\int_0^d \ln \ln h(r) \, dr = +\infty \tag{1}
\]

holds true, where

\[
h_+(r) = \sup_{n \geq 0} \frac{n!}{M_n^\frac{\gamma}{1+\gamma} r^n}, \quad 1 < \gamma < \infty.
\]

We note that Denjoy-Carleman thereom is the limiting case of conditions 1)–3) in Theorem 2. An analogue of Theorem 3 for a segment was proven earlier by E.M. Dyn’kin under a bi-logarithmic condition which can be obtained from Levinson condition (1) if one lets formally \( \gamma = \infty. \)
3. Quasi-analyticity criteria

3.1. Case of convex domain. Let $D$ be a bounded convex domain, $0 \in \partial D$, $h'(\sigma_{\pm}) = 0$. Then class $H(D, M_n)$ is quasi-analytic at the point $z = 0$ if and only if

$$\int_{1}^{\infty} \frac{\ln T(r)}{v(r)r^2} dr = +\infty.$$

The quantities $h(\varphi)$, $\sigma_{+}$, $\sigma_{-}$, $T(r)$ were defined in Introduction. This result has another more obvious formulation. In order to provide it, we introduce certain geometric characteristics of a convex domain. As it is known, the support function

$$h(\varphi) = \max_{\lambda \in \partial D} \Re(\lambda e^{i\varphi})$$

is the distance from the origin to the tangent for domain $D$ perpendicular to the direction $\{re^{-i\varphi}, r > 0\}$. We assume that the coordinate system is chosen so that the maximal segment on which $h(\varphi) = 0$ reads as $[-\sigma, \sigma]$, where $\sigma > 0$. We note that here $\sigma < \frac{\pi}{2}$. If $\sigma = \frac{\pi}{2}$, then the domain is degenerate to a segment on the negative semi-axis.

On the boundary of domain $D$ we choose the counterclockwise direction and introduce the arc length,

$$z = z(s), \quad 0 \leq s < s_0,$$

where $s_0$ is the total length of the boundary of $D$. Hence, the length for the arc of the boundary from the point $z = 0$ to the point $z(s)$ (in the chosen direction) equals $s$.

As in work [9], by $-\alpha_{-}(s)$ ($0 \leq s < s_0$) we denote the slope of the tangent to the boundary of $D$ at the point $z(s)$ w.r.t. the imaginary axis. Then function $\alpha_{-}(s)$ is well-defined everywhere on $[0, s_0]$ except a countable set of points $s$ for which $z(s)$ is the angle point. We define the function $\alpha_{-}(s)$ by the right continuity condition. By definition, $\lim_{s \to 0^{+}} \alpha_{-}(s) = -\sigma$. In the same way, the slope of the tangent at the point $z(s_0 - s)$ w.r.t. the direction of the imaginary axis is indicated by $\alpha_{+}(s)$. Then $\alpha_{+}(s)$ is positive, does not increase and $\lim_{s \to s_0^{+}} \alpha_{+}(s) = \sigma$. We let

$$\alpha(s) = \frac{\alpha_{+}(s) - \alpha_{-}(s)}{2}, \quad 0 \leq s < s_0.$$

Since $\lim_{s \to 0^{+}} \alpha(s) = \sigma < \frac{\pi}{2}$, there exists a number $\varepsilon > 0$ such that $\alpha(s) < \frac{\pi}{2}$, $0 \leq s < \varepsilon$. We define

$$R(s) = \exp \int_{s}^{\varepsilon} \frac{\pi - \alpha(t)}{\frac{\pi}{2} - \alpha(t)} d\ln t, \quad 0 \leq s < \varepsilon.$$

Let $\beta(s) = \pi - 2\alpha(s)$. Then function $\beta(s)$ is the angle between the tangents at the points $z(s)$ and $z(s_0 - s)$, domain $D$ lies in this angle and function $R(s)$ casts into the form

$$R(s) = \exp \int_{s}^{\varepsilon} \frac{\pi + \beta(t)}{\beta(t)} d\ln t, \quad 0 \leq s < \varepsilon.$$

We have

Theorem 4 ([9]). Let $D$ be a convex but not necessary bounded domain $z_0 \in \partial D$, and

$$T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}$$

is the trace function for sequence $\{M_n\}$. By $\beta(z_0, s)$ we denote the angle between the tangents to the boundary of $D$ taken at the points separated from point $z_0$ by the distance $s$ of arc of the
boundary. We let

$$ R(z_0, s) = \exp \int_{s}^{\varepsilon} \frac{\pi + \beta(z_0, x)}{\beta(z_0, x)} d\ln x, \quad 0 \leq s < \varepsilon. \tag{2} $$

Then the condition

$$ \int_{1}^{\infty} \frac{\ln T(r)}{r^{2} R^{-1}(z_0, r)} dr = \infty \tag{3} $$

is the criterion for the quasi-analyticity of class $H(D, M_n)$ at point $z_0$.

In particular, by this theorem one can easily obtain aforementioned quasi-analyticity conditions for classes $H(D, M_n)$ in the case $D$ is a circle or an angle $\pi \alpha$, $0 < \alpha \leq 1$.

Our aim is to show that if a convex domain $D$ satisfies some integral condition (depending on the geometry of the domain) at a boundary point $z_0$, then condition (3) has a simpler formulation.

Fix a point $z_0 \in \partial D$. Then the defined above angle $\beta(z_0, s)$, non-decaying, tends to $\pi \alpha$ ($0 < \alpha \leq 1$) as parameters $s$ tends to zero. Taking into consideration that $\beta(z_0, s) \equiv \pi \alpha$ for the angle, we extract the term $1 + \frac{\alpha}{\alpha}$ from the integrand in formula (2),

$$ \frac{\pi + \beta(z_0, s)}{\beta(z_0, s)} = \frac{1 + \alpha}{\alpha} + \frac{\pi \alpha - \beta(z_0, s)}{\alpha \beta(z_0, s)}. $$

Then for sufficiently small $s$ the integral $\int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{\alpha \beta(z_0, x)} \cdot \frac{dx}{x}$ differs by a small error from the quantity $\frac{1}{\pi \alpha} \int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{x} dx$. Hence, if the integrals $\int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{x} dx$ are uniformly bounded for all $s$, $0 < s < \varepsilon$, then the quasi-analyticity criterion for class $H(D, M_n)$ at point $z_0 \in \partial D$ becomes

$$ \int_{1}^{\infty} \frac{\ln T(r)}{r^{\alpha+1}} dr = +\infty. $$

Indeed, it follows from the fact that in this case

$$ R(s) = \exp \left[ \int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{x} \frac{dx}{x} \right] \cdot \exp \left[ \int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{\alpha \beta(z_0, x)} d\ln x \right], $$

and as $s \to 0$

$$ R(s) = r \sim \left( \frac{\varepsilon}{s} \right)^{\frac{\alpha+1}{\alpha}} \exp \left( \frac{c}{\pi \alpha^2} \right), $$

where

$$ c = \lim_{s \to 0} \int_{s}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{x} dx = \int_{0}^{\varepsilon} \frac{\pi \alpha - \beta(z_0, x)}{x} dx. $$

Therefore, as $r \to \infty$,

$$ R^{-1}(r) \sim \exp \left( \frac{c}{\pi \alpha^2} \cdot \frac{\alpha}{\alpha + 1} \right) \varepsilon r^{-\frac{\alpha}{\alpha+1}}, $$

and condition (3) casts into the form

$$ \int_{1}^{\infty} \frac{\ln T(r)}{r^{\frac{\alpha+2}{\alpha+1}}} dr = \int_{1}^{\infty} \frac{\ln T(r)}{r^{\frac{\alpha+2}{\alpha+1}}} dr = +\infty. $$
Thus, for convex domains, for which quantity $\beta(z_0, s)$ obeys the restriction
\[
\sup_s \int_s^\varepsilon \frac{\pi \alpha - \beta(z_0, x)}{x} \, dx < \infty, \tag{4}
\]
the quasi-analyticity criterion for class $H(D, M_n)$ at a point $z_0 \in \partial D$ coincides with Salinas quasi-analyticity criteria for the angle $\Delta_\alpha = \{ z : |\arg z| < \frac{\pi \alpha}{2} \}$ ($0 < \alpha < 1$) and Korenblyum one for half-plane $\Delta_1$.

We have

**Theorem 5.** Let $D$ be a convex but necessary bounded domain, $z_0 \in \partial D$, and
\[
T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}
\]
is the trace function for sequence $\{M_n\}$. By $\beta(z_0, s)$ we denote the angle between tangents to the boundary of $D$ taken at the points separated from point $z_0$ by the length $s$ along the boundary. Suppose that at point $z_0$, the condition
\[
\sup_s \int_s^\varepsilon \frac{\pi \alpha - \beta(z_0, x)}{x} \, dx < \infty, \quad \pi \alpha = \lim_{s \to 0} \beta(z_0, s) \quad (0 < \alpha \leq 1)
\]
holds true. Then class $H(D, M_n)$ is quasi-analytic at point $z_0$ if and only if
\[
\int_1^\infty \frac{\ln T(r)}{r^{\alpha+\frac{1}{2}}} \, dr = +\infty. \tag{5}
\]

**Remark 1.** Condition (4) holds true if, for instance,
\[
|\pi \alpha - \beta(z_0, s)| = O(s^\gamma), \quad \gamma > 0
\]
or
\[
|\pi \alpha - \beta(z_0, s)| = O\left(\frac{1}{|\ln s|^\gamma}\right), \quad \gamma > 1 \quad s \to 0.
\]

**Remark 2.** For regular sequences $\{M^{\frac{\gamma}{1+\gamma}}_n\}$, there was obtained the bi-logarithmic quasi-analyticity condition in the angle which was equivalent to condition (5) as $\alpha = \frac{1}{\gamma}$. Therefore, by Theorem 5, for convex domain with additional condition (4) at point $z_0 \in \partial D$, the bi-logarithmic quasi-analyticity condition at this point reads exactly the same as for an angle,
\[
\int_0^d \ln \ln h_\gamma(r) \, dr = +\infty, \quad h_\gamma(r) = \sup_{n \geq 0} \frac{n!}{r^{\gamma(n+\gamma)}M^{\frac{\gamma}{1+\gamma}}_n}, \quad 1 < \gamma < \infty. \tag{6}
\]

Theorem 5 imply several corollaries.

**Corollary 1.** Let $\Delta_\alpha = \{ z : |\pi - \arg z| < \frac{\pi \alpha}{2} \}$ be the angle $\pi \alpha$ ($0 < \alpha < 1$) with vertex at the point $z = 0$. Then, obviously, $\beta(s) \equiv \pi \alpha$, and Condition (4) holds true.

If we let $\alpha = \frac{1}{\gamma}$, then condition (5) coincides with R. Salinas quasi-analyticity criterion for an angle
\[
\Delta_\gamma = \{ z : |\arg z| < \frac{\pi}{2\gamma}, 0 < |z| < \infty \} \quad (1 < \gamma < \infty).
\]

**Corollary 2.** Let $K = \{ z : |z+R| < R \}$ be a circle. It can be checked that in this case
\[
\beta(s) = \pi - 2\frac{s}{R},
\]
and \( \beta(s) \uparrow \pi \) \((\alpha = 1) \) \( s \to 0 \). Since \( \frac{\pi - \beta(x)}{x} = \frac{2}{R} \), condition (4) holds true at each point of \( \partial K \), while relation (5) in this case \((\alpha = 1)\) becomes Korenblyum criterion.

### 3.2. Special domains.

Consider the domains of special form, lunes \( K^\alpha \). As a lune \( K^\alpha \), following work \([10]\), we treat the intersection of exterior or interior for two circles of arbitrary but the same radius so that their circumferences pass the origin \( O \) and intersect by the angle \( \pi \alpha \) \((0 < \alpha < 2) \). As \( K^1 \), we treat the exterior or interior for a circumference passing point \( O \).

Let us show that for the lune \( K^\alpha \) obtained as the intersection of two interiors for two circles, condition (4) holds true. In order to do it, we shall make use of the following

**Lemma 1.** Let us draw the tangent at the point \( A \) to a circumference of an arbitrary radius \( R \) passing through point \( O \) with the center located below axis \( Ox \). Let also \( \beta_1 \) \((0 < \beta_1 < \frac{\pi}{2})\) be the angle between the tangent and the negative direction of axis \( Ox \) and \( \beta_1 \to \gamma \) as \( A \to O \). Then

\[
\gamma - \beta_1 = \frac{AO}{R},
\]

where \( AO \) is the length of arc of the circumference between the points \( A \) and \( O \).

Indeed, we observe that \((\pi - \gamma) + \beta_1 = \pi - \alpha \). It yields \( \gamma - \beta_1 = \alpha \). Taking into consideration that \( \alpha = \frac{AO}{R} \), we obtain the desired identity \( \gamma - \beta_1 = \frac{AO}{R} \).

Let \( K^\alpha \) be the lune formed by the intersection of the interior of two circles. Obviously, it is a convex set. We shall assume that \( K^\alpha \) is located in the left half-plane and is symmetric w.r.t. axis \( Ox \). Then by Lemma 1 we obtain that

\[
\pi \alpha - \beta(s) = \frac{2}{R} s.
\]

Hence,

\[
\int_s^\varepsilon \frac{\pi \alpha - \beta(x)}{x} dx = \frac{2}{R} (\varepsilon - s) \quad (0 < s < \varepsilon),
\]

and condition (4) holds true for \( K^\alpha \).

Finally, we formulate the last corollary.

**Corollary 3.** For a convex lune \( K^\alpha \) \((0 < \alpha < 2)\), condition (4) holds true everywhere. For lunes \( K^\alpha \) \((0 < \alpha < 2)\), the quasi-analyticity criterion for at point \( O \) coincides with R. Salinas criterion for the angle

\[
\Delta_\alpha = \left\{ z : |\pi - \arg z| < \frac{\pi \alpha}{2} \right\}.
\]

By Theorem 5 one can also get quasi-analyticity criteria for classes \( H(G, M_n) \) for non-convex domains \( G \) satisfying certain additional restrictions.

Let \( G \) be a domain in the complex plain not containing infinity. We shall say that domain \( G \) satisfies condition (A), if its boundary \( C \) consists of a finite number of piecewise-smooth closed simple curves \( c_1, c_2, \ldots, c_n \), each of which has a piecewise-continuous curvature and contains at most finite number of angle points and all interior angles (w.r.t. domain \( G \)) are not equal to 0 or \( 2\pi \). We denote the interior angle between one-sided tangents to \( C \) at a point \( z \) by \( \pi \alpha(z) \).

Let \( \alpha = \min_{z \in C} \alpha(z) > 0 \). Then a domain \( G \) satisfying condition (A) possesses the feature \([10]\): for each point \( z \in \partial G \), there exist lunes \( K_1^\alpha(z) \) and \( K_2^\alpha(z) \) such that

\[
K_1^\alpha(z) \subset G \subset K_2^\alpha(z).
\]

Here \( K_1^\alpha(z) \) is a convex lune formed by the intersection of interiors, while \( K_2^\alpha(z) \) is a lune formed by intersection of exterior for two circles of the same but sufficiently small radius such that their circumferences pass point \( z \).
Classes \( H(K_1^{\alpha(z)}, M_n) \) and \( H(K_2^{\alpha(z)}, M_n) \) are quasi-analytic or not at a point \( z \in C \) simultaneously \([10]\). Therefore, taking into account Corollary 3 and applying Theorem 5, we obtain: all three classes \( H(G, M_n), H(K_1^{\alpha(z)}, M_n), \) and \( H(K_2^{\alpha(z)}, M_n) \) are quasi-analytic at a point \( z \in C \) if and only if
\[
\int_1^\infty \frac{\ln T(r)}{r^{\alpha(z)/2}} dr = +\infty. \tag{7}
\]

We note that if a point \( z \in C \) is a point of smoothness for the boundary of domain \( G \) (i.e., \( \alpha(z) \equiv 1 \)), the quasi-analyticity criterion for class \( H(G, M_n) \) at this point reads as follows,
\[
\int_1^\infty \frac{\ln T(r)}{r^2} dr = +\infty.
\]

If we take into consideration Remark 2, for regular sequences \( \{M_n^{1/\gamma}\} \), condition (7) is equivalent to bi-logarithmic condition (6) as \( \gamma = \frac{1}{\alpha} \).

We note that the quasi-analyticity criterion for class \( H(G, M_n) \), where \( G \) is a domain satisfying condition \( A \), was proven in a different way in work \([10]\).

4. Existence criterion for regular minorant of non-quasi-analyticity

Let \( \{M_n\} \) be a regular sequence, \( \omega(r) = \max_{n \geq 0} r^n m_n \left( m_n = \frac{M_n}{n!} \right) \) is the associated weight \([8]\). Then sequence \( \{M_n\} \) is completely determined by function \( \omega(r) \),
\[
M_n = n! \sup_{r > 0} \frac{r^n}{\omega(r)}. \tag{9}
\]

As it was said in Section 2, in this case, the condition
\[
\sum_{n=0}^\infty \frac{M_n}{M_{n+1}} < \infty \tag{8}
\]
can be reformulated in terms of bi-logarithmic Levinson condition
\[
\int_0^d \ln \ln H(r) dr < \infty,
\]
where \( H(r) = \omega(\frac{1}{r}) \) and \( d > 0 \) is so that \( H(d) > e \).

We shall sequence \( \{M_n\} \) weakly regular if it obeys conditions a), b) in the definition of regular sequence \( \{M_n\} \) (see Section 2). It happens that for weakly regular sequences, condition (8) has another interpretation.

Lemma 2. Suppose the sequence \( \{M_n\} \) is weakly regular. Condition (8) holds true if and only if there exists a positive continuous on \( \mathbb{R}_+ \) function \( R = R(t) \) such that \( R(t) \downarrow 0, tR(t) \downarrow 0 \) as \( t \to \infty \) and
\[
1) \frac{1}{M_n^{1/\gamma}} \leq R(n); \quad 2) \int_1^\infty R(t) dt < \infty.
\]

Proof. Sufficiency is almost obvious. Indeed, since \( M_n^{1/\gamma} \uparrow \infty \) as \( n \to \infty \) (it follows from the logarithmic convexity of sequence \( \{M_n\} \) and property c)), according to Denjoy-Carleman theorem, condition can be written as \([2]\)
\[
\sum_{n=1}^\infty \frac{1}{M_n^{\gamma}} < \infty. \tag{9}
\]
This is why the sufficiency of lemma follows from conditions 1), 2) and properties of function $R = R(t)$.

Necessity. Letting $r(n) = M_n^{-\frac{1}{n}}$, we have

$$r(n)n = \frac{n}{M_n^{\frac{1}{n}}} = \frac{1}{m_n^{\frac{1}{n}}} n \left(\frac{1}{n!}\right)^{\frac{1}{n}}.$$ 

By Stirling formula \[11\],

$$n! = \sqrt{2\pi n} \ n^n e^{-n} e^{\theta(n)} \quad |\theta(n)| \leq \frac{1}{12n},$$

it implies

$$r(n)n = \frac{1}{m_n^{\frac{1}{n}}} \frac{e^{1-\theta(n)}}{(2\pi n)^{\frac{1}{2}}} \leq e^{\frac{13}{12}} \frac{1}{m_n^{\frac{1}{n}}}.$$ 

If we denote by $R(n)n$ the right hand side of (10), we see that $R(n)n \downarrow 0$ as $n \to \infty$. Then as $n \to \infty$,

$$R(n) = e^{\frac{13}{12}} \frac{1}{n} \left(\frac{n!}{M_n}\right)^{\frac{1}{n}} \leq e^{\frac{13}{8}} (2\pi n)^{\frac{1}{2n}} \frac{1}{M_n^{\frac{1}{n}}} = O \left(\frac{1}{M_n^{\frac{1}{n}}}\right).$$

Therefore, it follows from condition (9) that $\sum_{n=1}^{\infty} R(n) < \infty$. Hence,

$$\frac{1}{M_n^{\frac{1}{n}}} \leq R(n), \quad \sum_{n=1}^{\infty} R(n) < \infty, \quad R(n) \downarrow 0, \ R(n)n \downarrow 0 \ n \to \infty.$$ 

The desired function is obviously $R = R(t)$ which is linear for $t \in (n, n+1)$ and it takes values $R(n)$ and $R(n+1)$ at the endpoints of the interval $(n, n+1)$.

Lemma 2 is supplemented by

**Lemma 3.** Let $\{M_n\}$ $(M_n > 0)$ be an arbitrary sequence such that there exists a continuous function $r = r(t)$ on $\mathbb{R}_+$, $r(t) \downarrow 0$, $r(t)t \downarrow 0$ as $t \to \infty$ and

$$\frac{1}{M_n^{\frac{1}{n}}} \leq r(n), \quad \int_1^{\infty} r(t)dt < \infty.$$ 

Then there exists a weakly regular sequence $\{M_n^*\}$ such that

$$M_n^* \leq M_n, \quad \sum_{n=1}^{\infty} (M_n^*)^{-\frac{1}{n}} < \infty.$$ 

**Proof.** We have

$$D_n = \left(\frac{1}{r(n)}\right)^n \leq M_n \quad (n \geq 1).$$

The sequence $\{\frac{D_n}{n!}\}$ is not necessary logarithmically convex. This is why we replace it by a minorant possessing the required properties.

Bearing in mind Stirling formula, we have

$$\frac{D_n}{n!} = \frac{1}{n^n \Delta_n} \left(\frac{1}{r(n)}\right)^n,$$

where $\Delta_n = e^{-n} \sqrt{2\pi n} \ e^{\theta(n)} \ (|\theta(n)| \leq \frac{1}{12n})$. Since it is obvious that

$$\Delta_n \leq \sqrt{2\pi} \ \exp \left(\frac{1}{12n} - n + \frac{1}{2} \ln n\right) \leq \sqrt{2\pi} < e,$$
then
\[
\frac{D_n}{n!} \geq \frac{1}{e} \left( \frac{1}{nr(n)} \right)^n > \left( \frac{1}{enr(n)} \right)^n \quad (n \geq 1).
\]

If we let
\[
m^*_n = \frac{M^*_n}{n!} = \left( \frac{1}{enr(n)} \right)^n,
\]
then \(M^*_n \leq D_n \leq M_n\). Since \(nr(n) \downarrow 0\) as \(n \to \infty\), then \((m^*_n)^{\frac{1}{n}} \uparrow \infty\) as \(n \to \infty\). We see that sequence \(\{M^*_n\}\) is weakly regular.

Let us make sure that \(\sum_{n=1}^{\infty} \frac{1}{(M^*_n)^{\frac{1}{n}}} < \infty\).

Indeed,
\[
M^*_n = n! \left( \frac{1}{enr(n)} \right)^n = \sqrt{2\pi n} e^{-2n+\theta(n)} \left( \frac{1}{r(n)} \right)^n.
\]

It yields
\[
\left( \frac{1}{M^*_n} \right)^{\frac{1}{n}} \leq e^{\frac{2n}{n^2}} r(n) \quad (n \geq 1),
\]
and condition (11) is implied by the convergence of the series \(\sum_{n=1}^{\infty} r(n)\).

**Remark 3.** Under the hypothesis of Lemma 3, without loss of generality, one can assume that \(t^2r(t) \uparrow \infty\) as \(t \to \infty\). It implied by the following statement [12].

Let \(r = r(t)\) be a positive continuous on \(\mathbb{R}_+\) function, \(tr(t) \downarrow 0\) as \(t \to \infty\), and
\[
\int_1^{\infty} r(t)dt < \infty.
\]

Then for each \(\varepsilon > 0\) there exists a function \(r_1 = r_1(t)\) satisfying conditions

1. \(r(t) \leq r_1(t)\);
2. \(t r_1(t) \downarrow 0, t^{1+\varepsilon} r_1(t) \uparrow \) as \(t \to \infty\);
3. \(\int_1^{\infty} r_1(t)dt < \infty\).

By Remark 3, sequence \(\{M^*_n\}\) constructed in Lemma 3 satisfies also regularity condition b). Thus, under the hypothesis of Lemma 3 there exists a regular sequence \(\{M^*_n\}\) such that
\[
M^*_n \leq M_n, \quad \sum_{n=1}^{\infty} \left( \frac{1}{M^*_n} \right)^{\frac{1}{n}} < \infty.
\]

Let us make sure that sequence \(\{M^*_n\}\) satisfies condition b). Indeed, we have
\[
a_n = \left( \frac{m^*_{n+1}}{m^*_n} \right)^{\frac{1}{n}} \leq \sqrt{n} \frac{1}{(n+1) r(n+1)} \frac{n r(n)}{(n+1)(n+1) r(n+1)}.
\]

But
\[
\frac{n r(n)}{(n+1) r(n+1)} \leq (n+1)^2 \frac{r(n+1)}{n+1} \frac{1}{n} = \frac{n+1}{n} \leq 2.
\]

It yields
\[
a_n \leq 2^n \sqrt{n+1} \leq 2 \left( \frac{n+1}{4r(2)} \right)^{\frac{1}{n}} \leq C < \infty,
\]
and

$$\sup_n \left( \frac{m_{n+1}^*}{m_n^*} \right)^{\frac{1}{n}} < \infty.$$ 

Hence, we have proven

**Theorem 6.** Let $M_n > 0$. There exists a regular sequence $\{M_n^*\}$ such that

$$M_n^* \leq M_n, \quad \sum_{n=1}^{\infty} \frac{M_n^*}{M_{n+1}^*} < \infty$$

if and only if there exists a positive continuous on $\mathbb{R}_+$ function $r = r(t)$, $tr(t) \downarrow 0$, $t^2r(t) \uparrow$ as $t \to \infty$ such that

1) $\frac{1}{M_n^*} \leq r(n) \quad (n \geq 1);$ 

2) $\int_1^{\infty} r(t)dt < \infty.$

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**BIBLIOGRAPHY**


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